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Research Article

Cliques and Clique Covers in Interval-Valued Fuzzy Graphs

Napur Patra[1], Tarasankar Pramanik[2], Madhumangal Pal[3,*], Sukumar Mondal[1]

[1]*Department of Mathematics, Raja N. L. Khan Women's College, Midnapore 721102, India*
[2]*Department of Mathematics, Khanpur Gangche High School, Midnapore 721201, India*
[3]*Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore 721102, India*

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ABSTRACT

Finding cliques and clique covers in graphs are one of the most needful tasks. In this paper, interval-valued fuzzy cliques (IVFQs) and interval-valued fuzzy clique covers (IVFQCs) of an interval-valued fuzzy graph (IVFG) are introduced by introducing the fuzziness because, the crisp graphs has some limitations in real world due to uncertainty of vagueness. Here, the concept of cliques and clique covers are slightly modified so that every IVFQ is complete. Also, a clique cover of a crisp graph always covers all the edges and vertices of the graph whereas, the IVFQCs obtained by fuzzify to the clique covers does not satisfy the property. Hence, the definition is modified and studied some theorems on it. To better understand the useability of this work a model application is stated in this paper.

1. INTRODUCTION

In our real life situations, we see that some objects are related by some relations. For example, in a city, places are connected by trum ways. There may have a problem to construct the transportation roads so that there is minimum number of ways to move from one place to another place. These type of problems can easily be handled by considering the objects as nodes (or, vertices) and trum ways are the links (or, edges). This one-to-one representation is called the graph. The concept of graph theory was first introduced by Euler in his paper "Seven bridges of Königsberg" (1736). A suitable mathematical model is needed when fuzziness arises in the kind of objects or that in the relationship among the objects. Rosenfeld [1] first developed the crisp graph to fuzzy graph by introducing fuzzy relation in fuzzy sets which was first introduced by Zadeh [2]. Since then, researchers are delve into field of fuzzy graph theory and many of the real phenomena has been expressed in terms of fuzzy model (see [3]). Thus the field of fuzzy graph theory is flourishing it can handle the vaguenesses in real-world. Several real-world problems like human cardiac function, fuzzy neural network, routing problem, traffic light problem, time table scheduling, etc. can be nicely expressed using fuzzy graph model.

After Rosenfeld, fuzzy graph theory developed with many variations in fuzziness. In 1971, Zadeh [4] introduced the concept of interval-valued fuzzy sets (IVFSs) to generalize the fuzzy sets [2] in which the membership function describes to return interval numbers instead of classical numbers. It is more strong enough to consider the uncertainty cases than the traditional fuzzy sets as interval numbers are considered instead of classical numbers. The interval-valued fuzzy graph (IVFG) theory studies the generalized class of fuzzy graphs with IVFSs with interval-valued fuzzy relations. Therefore, it has more area of applications such as fuzzy control, approximate reasoning, medical diagnosis, intelligent control, multivated logic, etc. Talebi and Rashmanlou [5] studied isomorphism on IVFG. Several theorems and the properties of Complete IVFGs are studied by Rashmanlou and Jun in their literature [6]. Pal and Rashmanlou [7] have studied another variation of IVFG namely, irregular IVFG in which the adjacent vertices have distinct degrees. Antipodal IVFG is an another classification of IVFG which is introduced by Rashmanlou and Pal [8]. They also have defined a new type of IVFG – Balanced IVFG in the literature [9]. Bipolar fuzzy graphs are studied by Rashmanlou *et al.* [10]. Pramanik *et al.* [11] have extended the fuzzy competition graph to a bipolar fuzzy competition graph so that it can solve several real-world problems. Samanta *et al.* [12] have represented and analyzed the competitions among the participants in social networks. Pramanik *et al.* [13] have introduced the fuzzy ϕ-tolerance competition graphs. In 2016, concept of planarity is first introduced in IVFG by Pramanik *et al.* [14]. In this year, they also have extended the idea of fuzzy ϕ-tolerance competition graphs in IVFGs [15]. They also have considered certain threshold in each of fuzzy vertices and introduced interval-valued fuzzy threshold graph [16]. Bipolar fuzzy planar graphs have been extensively studied by Pramanik *et al.* [17] in 2018. They have shown its uses in image shrinking with an arbitrary graph model. In 2020, fuzzy competition graphs have been extended by Pramanik *et al.* [11] and given an idea of application to manufacturing industries. In literature [18], a new type of measurements in IVFGs

*Corresponding author. Email: mmpalvu@gmail.com

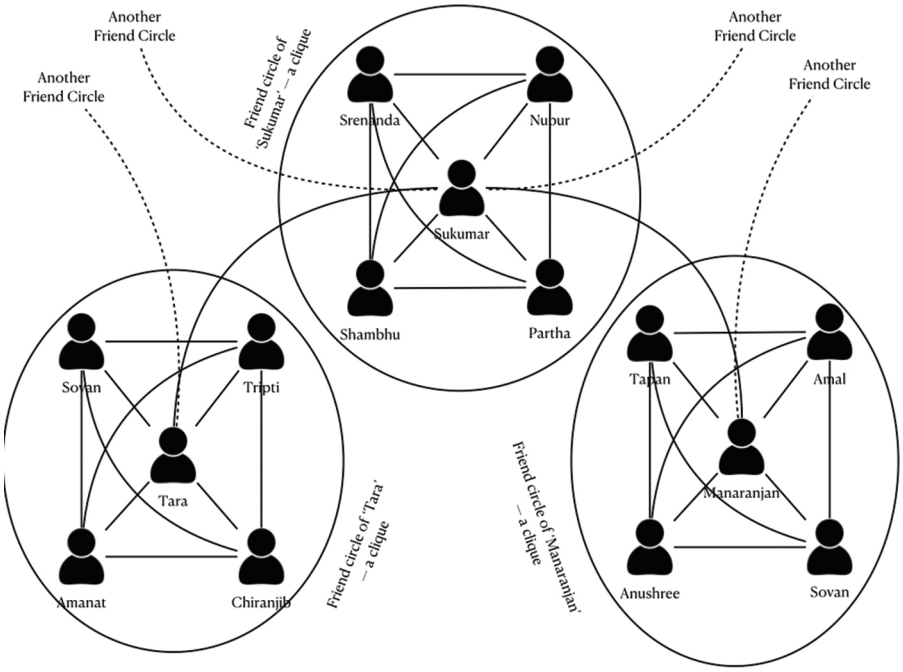

Figure 1 An example of communication network consisting of cliques.

are introduced by Pramanik *et al.* Several algorithmic approach to find the fuzzy shortest path in an interval-valued fuzzy hypergraphs is presented by Pramanik and Pal [19]. Also, the all-pairs shortest path problem for general network is investigated in [20]. For further definitions, terminologies and applications the reader may read the newly published book by Pal *et al.* [21]. For more details of fuzzy graphs, look in [1,22–29].

In today's communication networks, one person has friends (or buddies) and it is assumed that each friend in his friend list is known to each other. This type of network in graph modelling is called the *clique* (See Figure 1). The *maximal clique* is a clique with no proper subset which is also a clique. The maximal clique with maximum number of nodes (vertices) is the *maximum clique*. In many problems such as circuit design, transprotation, human brain analysis, artificial intelligence etc., it is an important task to find the set of maximum number of nodes (vertices) each of which are related to each other. Motivating from this idea, IVFQs and IVFQCs of IVFGs are studied in this article. It is also known that, if all the vertices of any subgraph of a graph forms a clique, then the subgraph is complete. But this conceptual phenomena is not intact in fuzzy graph according to the definition of fuzzy cliques introduced by Nair and Cheng [30]. In this paper, we have modified the definition of IVFQ and IVFQC given by Nair and Cheng and found a new dimension of work. We have built a connection between crisp concept and interval-valued fuzzy concept. The definition of fuzzy clique is modified so that every complete fuzzy subgraph is a fuzzy clique and then the fuzzy cliques and fuzzy clique covers are generalized for IVFGs.

Definition of the problem

In this paper, IVFQs and IVFQCs of an IVFG are introduced here. The concept of cliques and clique covers are modified to show that every IVFQ is complete. Also, a clique cover of a crisp graph always covers all the edges and vertices of the graph whereas, the IVFQCs obtained. Hence, the definition is modified and studied some theorems on it. Contribution of different authors towards the development of the fuzzy cliques of fuzzy graphs is shown in Table 1.

Table 1 Contribution of different authors towards fuzzy cliques of fuzzy graphs.

Authors	Year	Contributions
Kauffman [31]	1973	Fuzzy graphs and its several properties are introduced.
Rosenfeld [1]	1975	Notion of fuzzy graphs introduced by Kauffman [31] are modified. He added a constraint that edge membership value is less than minimum of vertex membership values.
Nair and Cheng [30]	2001	Defined fuzzy cliques.
Akram and Dudek *et al.* [32]	2011	Introduction of IVFGs.
Anjali and Mathew [22]	2015	Blocks in fuzzy graphs are discussed.
This paper	–	Defined fuzzy cliques in an IVFGs.

IVFG, interval-valued fuzzy graph.

2. PRELIMINARIES

A *graph* $Z = (U, T)$ includes a set denoted by U, or by $U(Z)$ and a collection T, or $T(Z)$, of un-ordered pair (u, v) of elements from the set U. Each element of U is called a *vertex*, and each element of T is called an *edge*.

An edge that connects vertices v_i and v_j in U is denoted by (v_i, v_j). Two vertices v_i and v_j in U are *adjacent* if $(v_i, v_j) \in T$.

A *walk* in Z is a sequence of vertices and edges of the form $v_1, (v_1, v_2), v_2, (v_2, v_3), \ldots, (v_{n-1}, v_n), v_n$ which is denoted by $v_1 v_2 v_3 \cdots v_n$. A *path* is a walk where none of the vertices is repeated. The walk where $v_1 = v_n (n \geq 3)$ and $v_i \neq v_j$ for any $i, j \in \underline{n-1}$ is called *cycle*. The *length* of a path or a cycle is the number of its edges.

In a crisp graph J, a set of pairwise adjacent vertices is a *clique*. A clique is said to be a maximal clique if the set consisting of the clique is not a subset of any other clique set.

A *clique cover* of a graph J is a clique induced by all the vertices of the graph J. The minimum number of cliques in a graph J required to cover the graph J is called the *clique cover number* and is denoted by $cc(J)$.

The *union* of two graphs $Z = (U_Z, T_Z)$ and $H = (U_H, T_H)$ is the graph $Z \cup H = (U_Z \cup U_H, T_Z \cup T_H)$.

A *fuzzy set* P on a set L is defined by a function $P : L \to [0, 1]$ and the function is called membership function. The mapping $R : L \times M \to [0, 1]$ is said to be fuzzy relation. For any two fuzzy sets Q and R on L, Q is said to be the fuzzy subset of R i.e., Q is included in R, denoted by $Q \subseteq R$, if and only if $Q(p) \leq R(p)$ for all $p \in P$.

A *family* of fuzzy sets is denoted by $\mathcal{F}(L)$ defined on L and $\mathcal{F}(L \times M)$ be the family of fuzzy relations defined on $L \times M$.

Let us consider a fuzzy set σ and a fuzzy relation μ be such that $\mu \leq \sigma(u) \wedge \sigma(v)$ for all $u, v \in U$. Then the tuple $J = (U, \sigma, \mu)$ is called the *fuzzy graph*. The fuzzy set σ is called the *fuzzy vertex set* of J and the fuzzy relation μ is called the *fuzzy edge set* of J.

When the fuzzy relation μ is symmetric then the fuzzy graph is said to be undirected fuzzy graph otherwise the fuzzy graph is said to be directed.

Throughout this paper, undirected fuzzy graphs are considered and also there is no loops i.e., $\mu(u, u) = 0$ for all $u \in U$.

The crisp graph $J^* = (U, \sigma^*, \mu^*)$, where $\sigma^* = \{u \in U : \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in U \times U : \mu(u, v) > 0\}$.

Let us consider a mapping $\mu_{\mathcal{Q}} : L \to [0, 1] \times [0, 1]$ then the fuzzy set $\mu_{\mathcal{Q}}$ on L is said to be the IVFS if and only if $\mu^-_{\mathcal{Q}}(u) \leq \mu^+_{\mathcal{Q}}(u)$ where $\mu_{\mathcal{Q}}(u) = [\mu^-_{\mathcal{Q}}(u), \mu^+_{\mathcal{Q}}(u)]$ for all $u \in L$.

Let $\mathcal{F} = \left\{\mathcal{Q}_1, \mathcal{Q}_2, \cdots, \mathcal{Q}_n\right\}$ be a finite family of interval-valued fuzzy subsets on a set L. The *fuzzy intersection* of two IVFSs $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is an IVFS defined by

$$\mathcal{Q}_1 \cap \mathcal{Q}_2 = \left\{ \left(u, \left[\min\left\{\mu^-_{\mathcal{Q}_1}(u), \mu^-_{\mathcal{Q}_2}(u)\right\}, \min\left\{\mu^+_{\mathcal{Q}_1}(u), \mu^+_{\mathcal{Q}_2}(u)\right\} \right] \right) : u \in L \right\}.$$

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$$\mathcal{Q}_1 \cap \mathcal{Q}_2 = \left\{ \left(u, \left[\max\left\{\mu^-_{\mathcal{Q}_1}(u), \mu^-_{\mathcal{Q}_2}(u)\right\}, \max\left\{\mu^+_{\mathcal{Q}_1}(u), \mu^+_{\mathcal{Q}_2}(u)\right\} \right] \right) : u \in L \right\}.$$

Let $\mathcal{Q} = \left\{[\mu^-_{\mathcal{Q}}(u), \mu^+_{\mathcal{Q}}(u)] : u \in U\right\}$ be a fuzzy set and $\mathcal{R} = \left\{[\mu^-_{\mathcal{R}}(u, v), \mu^+_{\mathcal{R}}(u, v)] : (u, v) \in U \times U\right\}$ defined on U together forms an *IVGF*. Here the fuzzy set $\mathcal{Q}$ is said to be the *interval-valued fuzzy vertex set* (IVFVS) and the fuzzy set $\mathcal{R}$ is said to be the *interval-valued fuzzy edge set* (IVFES) of the IVFG. An edge (u, v), $u, v \in U$ in an IVFG is said to be *interval-valued strong* if $I^-_{(u,v)} = \frac{\mu^-_{\mathcal{R}}(u,v)}{\min\left\{\mu^-_{\mathcal{Q}}(u), \mu^-_{\mathcal{Q}}(v)\right\}} \geq 0.5$ and $I^+_{(u,v)} = \frac{\mu^+_{\mathcal{R}}(u,v)}{\min\left\{\mu^+_{\mathcal{Q}}(u), \mu^+_{\mathcal{Q}}(v)\right\}} \geq 0.5$ otherwise the edge is called *interval-valued weak*. The *strength* of an edge (u, v) is assumed by $I_{(u,v)} = [I^-_{(u,v)}, I^+_{(u,v)}]$. An edge is said to be *nontrivial* if $I^-_{(u,v)} > 0$. An IVFG $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$ is said to be an interval-valued fuzzy strong graph if and only if $I^-_{(u,v)} = \frac{\mu^-_{\mathcal{R}}(u,v)}{\min\left\{\mu^-_{\mathcal{Q}}(u), \mu^-_{\mathcal{Q}}(v)\right\}} \geq 0.5$ and $I^+_{(u,v)} = \frac{\mu^+_{\mathcal{R}}(u,v)}{\min\left\{\mu^+_{\mathcal{Q}}(u), \mu^+_{\mathcal{Q}}(v)\right\}} \geq 0.5, \forall (u, v) \in U \times U$.

The *underlying graph* of the IVFG $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$ is the crisp graph $\mathcal{Z}^* = (U, \mathcal{Q}^*, \mathcal{R}^*)$, where $\mathcal{Q}^* = \left\{u \in U : \mu^-_{\mathcal{Q}}(u) > 0\right\}$ and $\mathcal{R}^* = \left\{(u, v) \in U \times U : \mu^-_{\mathcal{R}}(u, v) > 0\right\}$. An interval-valued fuzzy digraph (IVFDG) $\overrightarrow{\mathcal{Z}} = (U, \mathcal{Q}, \overrightarrow{\mathcal{R}})$ is an IVFG where the fuzzy relation $\overrightarrow{\mathcal{R}}$ is antisymmetric.

An IVFG $Z = (U, \mathcal{Q}, \mathcal{R})$ is said to be *complete IVFG* if $\mu^-_{\mathcal{R}}(u, v) = \min\left\{\mu^-_{\mathcal{Q}}(u), \mu^-_{\mathcal{Q}}(v)\right\}$ and $\mu^+_{\mathcal{R}}(u, v) = \min\left\{\mu^+_{\mathcal{Q}}(u), \mu^+_{\mathcal{Q}}(v)\right\}, \forall u, v \in U$. An IVFG is said to be *bipartite* if the vertex set U can be partitioned into two sets U_1 and U_2 such that $\mu^+_{\mathcal{R}}(u, v) = 0$ if $u, v \in U_1$ or $u, v \in U_2$ and $\mu^+_{\mathcal{R}}(v_1, v_2) > 0$ if $v_1 \in U_1$ (or U_2) and $v_2 \in U_2$ (or U_1).

An IVFG $\mathcal{H} = (U', \mathcal{Q}', \mathcal{R}')$ is called an *interval-valued fuzzy subgraph* (IVFSG) of the IVFG $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$ induced by U' if $U' \subseteq U$, $\mu^-_{\mathcal{Q}'}(u) = \mu^-_{\mathcal{Q}}(u)$ and $\mu^+_{\mathcal{Q}'}(u) = \mu^+_{\mathcal{Q}}(u)$ for all $u \in U'$ and $\mu^-_{\mathcal{R}'}(u, v) = \mu^-_{\mathcal{R}}(u, v)$ and $\mu^+_{\mathcal{R}'}(u, v) = \mu^+_{\mathcal{R}}(u, v)$ for all $u, v \in U'$.

An IVFSG $\mathcal{H} = (U', \mathcal{Q}', \mathcal{R}')$ induced by $\mathcal{Q}'$ is the maximal IVFSG of $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$ which has the IVFVS $\mathcal{Q}'$ and the IVFES $\mathcal{R}'$ be such that $\mu^-_{\mathcal{R}'}(u, v) = \mu^-_{\mathcal{Q}'}(u) \wedge \mu^-_{\mathcal{Q}'}(v) \wedge \mu^-_{\mathcal{R}}(u, v)$ and $\mu^+_{\mathcal{R}'}(u, v) = \mu^+_{\mathcal{Q}'}(u) \wedge \mu^+_{\mathcal{Q}'}(v) \wedge \mu^+_{\mathcal{R}}(u, v)$ for all $u, v \in U$.

An IVFG is called an *interval-valued fuzzy cycle* (IVFC) if and only if it contains more than one weakest edge (i.e., there is no unique $(u, v) \in \mathcal{R}^*$ such that $\mu^-_{\mathcal{R}}(u, v) = \wedge\left\{\mu^-_{\mathcal{R}}(u, v) \mid (u, v) \in \mathcal{R}^*\right\}$) and is a cycle.

An IVFSG IVFSG $\mathcal{H} = \left(U', \mathcal{Q}', \mathcal{R}'\right)$ is $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$ is an *IVFQ* if $\mathcal{H}^*$ is a clique and each cycle in $\mathcal{H}$ is an IVFC.

2.1. Some Definitions

Definition 1. (Interval-valued c-strong edge) Let c be a real number. An edge (u, v), $u, v \in U$ in an IVFG is said to be interval-valued c-strong if

$$I^-_{(u,v)} = \frac{\mu^-_{\mathcal{R}}(u, v)}{\min\left\{\mu^-_{\mathcal{Q}}(u), \mu^-_{\mathcal{Q}}(v)\right\}} \geq c \text{ and } I^+_{(u,v)} = \frac{\mu^+_{\mathcal{R}}(u, v)}{\min\left\{\mu^+_{\mathcal{Q}}(u), \mu^+_{\mathcal{Q}}(v)\right\}} \geq c.$$

Since, in an IVFG, $\mu^-_{\mathcal{R}}(u, v) \leq \min\left\{\mu^-_{\mathcal{Q}}(u), \mu^-_{\mathcal{Q}}(v)\right\}$ and $\mu^+_{\mathcal{R}}(u, v) \leq \min\left\{\mu^+_{\mathcal{Q}}(u), \mu^+_{\mathcal{Q}}(v)\right\}$ then, $\frac{\mu^-_{\mathcal{R}}(u,v)}{\min\left\{\mu^-_{\mathcal{Q}}(u), \mu^-_{\mathcal{Q}}(v)\right\}} \leq 1$ and $\frac{\mu^+_{\mathcal{R}}(u,v)}{\min\left\{\mu^+_{\mathcal{Q}}(u), \mu^+_{\mathcal{Q}}(v)\right\}} \leq 1$. Therefore, c can take at most the value 1.

Definition 2. (Perfect interval-valued fuzzy strong graph) If in an IVFG every edge is interval-valued 1-strong then the graph is called the perfect interval-valued fuzzy strong graph.

Table 2 Membership values for the fuzzy edges of the graph $\mathscr{Z} = (U, \mathcal{Q}, \mathcal{R})$ of Example 1.

$e \in U \times U$	(v_1, v_2)	(v_1, v_3)	(v_1, v_4)	(v_1, v_5)	(v_2, v_3)	(v_2, v_4)	(v_2, v_5)
$\mu_{\mathcal{R}}(e)$	$[0.5, 0.6]$	$[0.5, 0.6]$	$[0.6, 0.7]$	$[0.5, 0.6]$	$[0.8, 0.9]$	$[0.6, 0.7]$	$[0.5, 0.6]$
$e \in U \times U$	(v_3, v_4)	(v_3, v_5)	(v_4, v_5)				
$\mu_{\mathcal{R}}(e)$	$[0.7, 0.8]$	$[0.5, 0.6]$	$[0.5, 0.6]$				

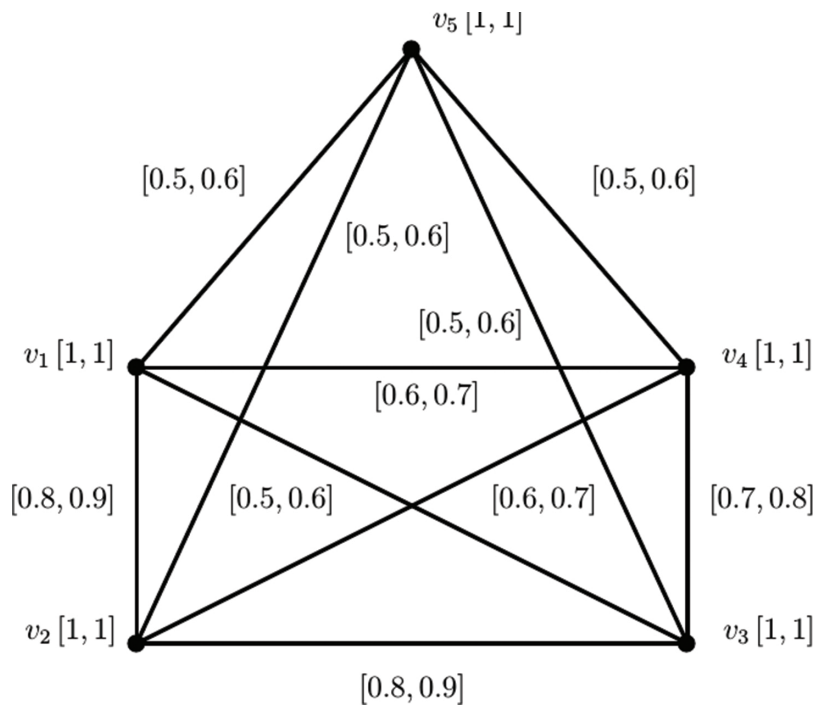


Figure 2 An example of a fuzzy graph $\mathscr{Z} = (U, \mathcal{Q}, \mathcal{R})$.

Definition 3. (t-cut graph of an IVFG) Let $\mathscr{Z} = (U, \mathcal{Q}, \mathcal{R})$ be an IVFG. Then for any threshold $t \in [0, 1]$, the t-cut graph of the IVFG $\mathscr{Z}$ is a crisp graph $\mathscr{Z}_t = (\mathcal{Q}_t, \mathcal{R}_t)$ where $\mathcal{Q}_t = \{u \in U : \mu^-_{\mathcal{Q}}(u) \geq t\}$ is the vertex set and $\mathcal{R}_t = \{(u, v) \in U \times U : \mu^-_{\mathcal{R}}(u, v) \geq t\}$ is the edge set of $\mathscr{Z}_t$.

2.2. IVFQs in IVFGs

In graph theory, a clique induces a complete subgraph. However, the IVFSG induced by an IVFQ may not be complete.

Example 1.

Consider the IVFG $\mathscr{Z} = (U, \mathcal{Q}, \mathcal{R})$ where $U = \{v_1, v_2, v_3, v_4, v_5\}$ be the vertex set and $\mathcal{Q}$ being the IVFS on U with $\mu_{\mathcal{Q}}(v_1) = \mu_{\mathcal{Q}}(v_2) = \mu_{\mathcal{Q}}(v_3) = \mu_{\mathcal{Q}}(v_4) = \mu_{\mathcal{Q}}(v_5) = [1, 1]$ and $\mathcal{R}$ being the interval-valued fuzzy relation on the set $U \times U$ whose definition is given in the Table 2:

The diagrammatical representation of this graph is shown in Figure 2. The IVFSG induced by an interval-valued fuzzy subset $\mathcal{Q}' = \{v_1[0.7, 0.8], v_2[1, 1], v_3[0.7, 0.8], v_4[0.8, 0.9], v_5[0.6, 0.7]\}$. The diagrammatical representation of this graph is shown in Figure 3. Obviously, this is not a complete IVFSG.

Now, definition of IVFQ should be so modified that the IVFSG induced by each fuzzy clique is complete. This complete IVFQ is known as *complete IVFQ*.

Definition 4. (Complete IVFQ) In an IVFG $\mathscr{Z} = (U, \mathcal{Q}, \mathcal{R})$, a subset $\mathcal{Q}'$ of $\mathcal{Q}$ is called a complete IVFQ if the fuzzy subgraph induced by $\mathcal{Q}'$is a complete IVFG.

Now, we see an example of a complete IVFQ as a subgraph of the graph shown in Figure 2.

Example 2.

Consider an IVFSG of the graph in Example 1 induced by $\mathcal{Q}' = \{v_1[0.7, 0.8], v_2[0.8, 0.9], v_3[0.5, 0.6], v_4[0.6, 0.7], v_5[0.5, 0.6]\}$ which is shown in Figure 4. This graph is a complete IVFG and hence the interval-valued fuzzy subset $\mathcal{Q}'$ of $\mathcal{Q}$ is a complete IVFQ.

Theorem 1. *Each complete IVFSG is an IVFQ.*

Proof. Let $\mathcal{H} = (U, \mathcal{Q}', \mathcal{R}')$ be a complete IVFSG of the graph $\mathscr{Z} = (U, \mathcal{Q}, \mathcal{R})$. Since $\mathcal{H}$ is complete then, $\mu^-_{\mathcal{R}}(u, v) = \min\{\mu^-_{\mathcal{Q}}(u), \mu^-_{\mathcal{Q}}(v)\}$ and $\mu^+_{\mathcal{R}}(u, v) = \min\{\mu^+_{\mathcal{Q}}(u), \mu^+_{\mathcal{Q}}(v)\}$, $\forall u, v \in U$. This shows that if $\mu^-_{\mathcal{Q}}(u), \mu^-_{\mathcal{Q}}(v), \mu^+_{\mathcal{Q}}(u), \mu^+_{\mathcal{Q}}(v) > 0$ then $\mu^-_{\mathcal{R}}(u, v), \mu^+_{\mathcal{R}}(u, v) > 0$. Then the crisp graph $\mathcal{H}^*$ is complete and

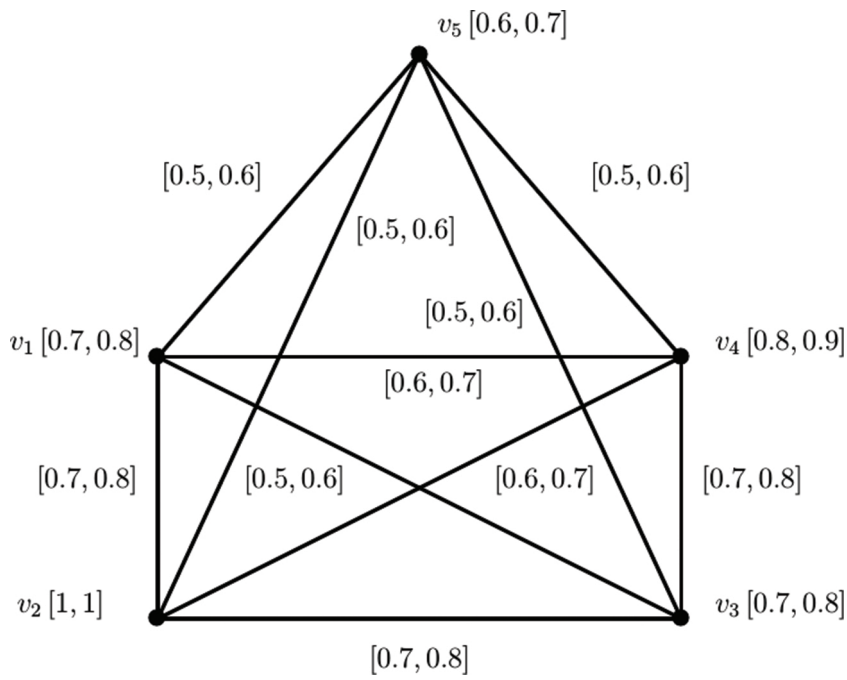


Figure 3 An interval-valued fuzzy subgraph (IVFSG) which is fuzzy clique but not complete.

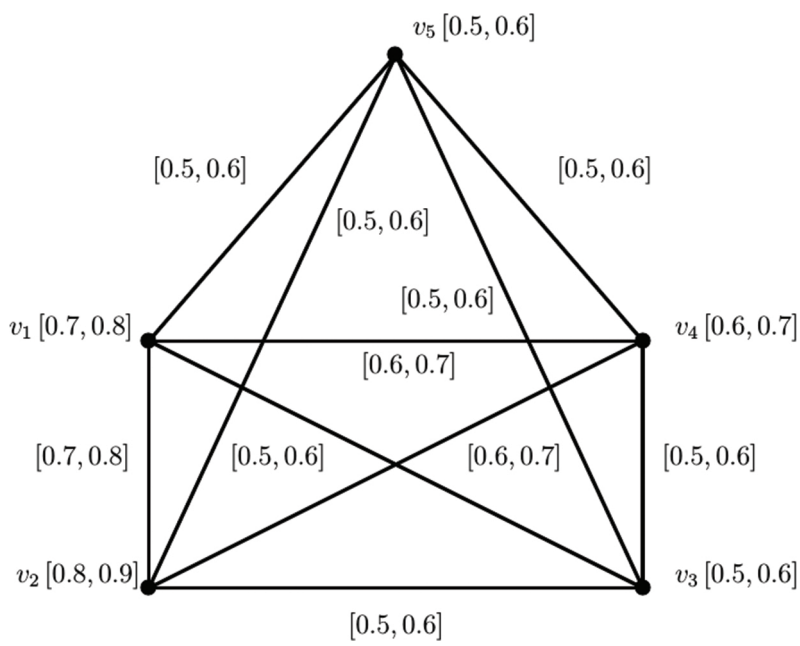


Figure 4 An example of a complete interval-valued fuzzy cliques (IVFQ) of the graph of Figure 2.

therefore, for any cycle $v_1v_2v_3v_1$ ($v_1, v_2, v_3 \in \mathcal{Q}^*$) of length 3 in $\mathcal{H}$,

$$\mu^-_{\mathcal{R}}(v_1, v_2) = \min\left\{\mu^-_{\mathcal{Q}}(v_1), \mu^-_{\mathcal{Q}}(v_2)\right\} \tag{1}$$

$$\mu^-_{\mathcal{R}}(v_2, v_3) = \min\left\{\mu^-_{\mathcal{Q}}(v_2), \mu^-_{\mathcal{Q}}(v_3)\right\} \tag{2}$$

$$\mu^-_{\mathcal{R}}(v_3, v_1) = \min\left\{\mu^-_{\mathcal{Q}}(v_3), \mu^-_{\mathcal{Q}}(v_1)\right\} \tag{3}$$

Now, without any loss of generality, suppose that $\mu^-_{\mathcal{Q}}(v_1) = \min\left\{\mu^-_{\mathcal{Q}}(v_1), \mu^-_{\mathcal{Q}}(v_2), \mu^-_{\mathcal{Q}}(v_3)\right\}$

Then from (1), (3) we have, $\mu^-_{\mathcal{R}}(v_1, v_2) = \mu^-_{\mathcal{Q}}(v_1) = \min\left\{\mu^-_{\mathcal{Q}}(v_3), \mu^-_{\mathcal{Q}}(v_1)\right\}$. This shows that the graph $\mathcal{H}$ has more than one weakest edges. Therefore, every cycle of length 3 is an IVFC. Hence, $\mathcal{H}$ is an IVFQ. ■

Corollary 1. The IVFSG induced by a complete IVFQ is an IVFQ.

Proof. By the definition of complete IVFQ we have the IVFSG induced by a complete IVFQ is complete. Then by Theorem 1, it is an IVFQ. ■

Theorem 2. *If IVFQ is a perfect interval-valued fuzzy strong then it is complete.*

Proof. Let $\mathcal{H} = (U, \mathcal{Q}, \mathcal{R})$ be an IVFQ as well as perfect interval-valued fuzzy strong graph. Since, $\mathcal{H}$ is an IVFQ then, $\mathcal{H}^*$ is a clique. Then for any $u, v \in \mathcal{Q}^*$, $(u, v) \in \mathcal{R}^*$. Again $\mathcal{H}$ is perfect interval-valued fuzzy strong then it follows that $\mu^-_{\mathcal{R}}(u, v) = \min\left\{\mu^-_{\mathcal{Q}}(u), \mu^-_{\mathcal{Q}}(v)\right\}$, $\forall (u, v) \in \mathcal{R}^*$. Therefore, $\mu^-_{\mathcal{R}}(u, v) = \min\left\{\mu^-_{\mathcal{Q}}(u), \mu^-_{\mathcal{Q}}(v)\right\}, \forall u, v \in \mathcal{Q}^*$. Hence, the IVFQ is complete. ■

Theorem 3. *In an IVFG $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$, a subset $\mathcal{Q}'$ of $\mathcal{Q}$ is a complete IVFQ if and only if $\mu^-_{\mathcal{R}}(u, v) \geq \min\left\{\mu^-_{\mathcal{Q}'}(u), \mu^-_{\mathcal{Q}'}(v)\right\}$ and $\mu^+_{\mathcal{R}}(u, v) \geq \min\left\{\mu^+_{\mathcal{Q}'}(u), \mu^+_{\mathcal{Q}'}(v)\right\} \forall u, v \in \mathcal{Q}^*$ and $u \neq v$.*

Proof. First consider that $\mathcal{Q}'$ is a complete IVFQ in $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$ and $\mathcal{H} = (U, \mathcal{Q}', \mathcal{R}')$ is the IVFSG induced by $\mathcal{Q}'$. By Corollary 1, $\mathcal{H}$ is a complete IVFSG. Then, we have $\min\left\{\mu^-_{\mathcal{Q}'}(u), \mu^-_{\mathcal{Q}'}(v)\right\} = \mu^-_{\mathcal{R}'}(u, v) = \min\left\{\mu^-_{\mathcal{Q}'}(u), \mu^-_{\mathcal{Q}'}(v), \mu^-_{\mathcal{R}}(u, v)\right\} \leq \mu^-_{\mathcal{R}}(u, v)$ for all $u, v \in \mathcal{Q}^*$ and $\min\left\{\mu^+_{\mathcal{Q}'}(u), \mu^+_{\mathcal{Q}'}(v)\right\} = \mu^+_{\mathcal{R}'}(u, v) = \min\left\{\mu^+_{\mathcal{Q}'}(u), \mu^+_{\mathcal{Q}'}(v), \mu^+_{\mathcal{R}}(u, v)\right\} \leq \mu^+_{\mathcal{R}}(u, v)$ for all $u, v \in \mathcal{Q}^*$ and $u \neq v$.

Conversely, consider that $\mathcal{Q}'$ is a subset of $\mathcal{Q}$ such that $\mu^-_{\mathcal{R}}(u, v) \geq \min\left\{\mu^-_{\mathcal{Q}'}(u), \mu^-_{\mathcal{Q}'}(v)\right\}$ and $\mu^+_{\mathcal{R}}(u, v) \geq \min\left\{\mu^+_{\mathcal{Q}'}(u), \mu^+_{\mathcal{Q}'}(v)\right\} \forall u, v \in \mathcal{Q}^*$ and $u \neq v$. Since $\mathcal{H} = (U, \mathcal{Q}', \mathcal{R}')$ is the IVFSG induced by $\mathcal{Q}'$, we have $\mu^-_{\mathcal{R}'}(u, v) = \min\left\{\mu^-_{\mathcal{Q}'}(u), \mu^-_{\mathcal{Q}'}(v), \mu^-_{\mathcal{R}}(u, v)\right\} = \min\left\{\mu^-_{\mathcal{Q}'}(u), \right\} \mu^-_{\mathcal{Q}'}(v)$ and $\mu^+_{\mathcal{R}'}(u, v) = \min\left\{\mu^+_{\mathcal{Q}'}(u), \mu^+_{\mathcal{Q}'}(v), \mu^+_{\mathcal{R}}(u, v)\right\} = \min\left\{\mu^+_{\mathcal{Q}'}(u), \mu^+_{\mathcal{Q}'}(v)\right\} \forall u, v \in \mathcal{Q}^*$ and $u \neq v$. Hence, $\mathcal{H}$ is complete IVFSG of the graph $\mathcal{Z}$. ■

The following corollaries are two consequences of Theorem 3.

Corollary 2. $\mathcal{Q}'$ be a complete IVFQ in $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$. Then for each $t \in (0, 1]$, $\mathcal{Q}'_t$ is a clique in $\mathcal{Z}_t$.

Proof. Since, $\mathcal{Q}'$ is a complete IVFQ in $\mathcal{Z}$, then from Theorem 3, we have $\mu^-_{\mathcal{R}}(u, v) \geq \min\left\{\mu^-_{\mathcal{Q}'}(u), \mu^-_{\mathcal{Q}'}(v)\right\}$ and $\mu^+_{\mathcal{R}}(u, v) \geq \min\left\{\mu^+_{\mathcal{Q}'}(u), \mu^+_{\mathcal{Q}'}(v)\right\} \forall u, v \in \mathcal{Q}^*$ and $u \neq v$. For each $t \in (0, 1]$, $\mathcal{Q}'_t = \left\{u \in U : \mu^-_{\mathcal{Q}'}(u) \geq t\right\}$ which implies that $\mu^-_{\mathcal{R}}(u, v) \geq \min\left\{\mu^-_{\mathcal{Q}'}(u), \mu^-_{\mathcal{Q}'}(v)\right\} \geq t$. Therefore $(u, v) \in \mathcal{R}_t$. Hence $\mathcal{Q}'_t$ is a clique in $\mathcal{Z}_t$. ■

Corollary 3. Let $\mathcal{Q}'$ be a complete IVFQ in $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$. Then each nonempty subset $\mathcal{Q}''$ of $\mathcal{Q}'$ is a complete IVFQ in $\mathcal{Z}$.

Proof. This is immediate from Theorem 3 since, $\mu^-_{\mathcal{Q}''}(u) \leq \mu^-_{\mathcal{Q}'}(u)$ and $\mu^+_{\mathcal{Q}''}(u) \leq \mu^+_{\mathcal{Q}'}(u) \forall u \in U$. ■

In the following, we shall give another characterization of a complete IVFQ. For an IVFG $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$, define an $n \times n$ interval-valued fuzzy matrix $M_{\mathcal{Z}}$ by

$$(M_{\mathcal{Z}})_{ij} = \begin{cases} \left[\mu^-_{\mathcal{Q}}(v_i), \mu^+_{\mathcal{Q}}(v_i)\right], & i = j \\ \left[\mu^-_{\mathcal{R}}(v_i, v_j), \mu^+_{\mathcal{R}}(v_i, v_j)\right], & i \neq j. \end{cases}$$

For a subset $\mathcal{Q}'$ of $\mathcal{Q}$ of the IVFG $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$, define an $n \times 1$ interval-valued fuzzy vector $U_{\mathcal{Q}}$ by

$$\left(U_{\mathcal{Q}}\right) = \begin{cases} [\mu^-_{\mathcal{Q}'}(v_i), \mu^+_{\mathcal{Q}'}(v_i)], & v_i \in (\mathcal{Q}')^*, \\ [0, 0], & v_i \notin (\mathcal{Q}')^* \end{cases}$$

Let $$\chi_{\mathcal{Z}} = \left\{ \mathcal{L} = \begin{pmatrix} [u_1, v_1] \\ [u_2, v_2] \\ \vdots \\ [u_n, v_n] \end{pmatrix} : u_i, v_i \in [0, 1] (i = 1, 2, \ldots, n) \text{ and } \mathcal{L} \odot \mathcal{L}^T \leq M_{\mathcal{Z}} \right\}.$$

Then we have the follwing theorem.

Theorem 4. *If $\mathcal{Q}'$ be a complete IVFQ of an IVFG $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$, then $U_{\mathcal{Q}'} \in \chi_{\mathcal{Z}}$. Conversely, for any $\mathcal{L} = \begin{pmatrix} [u_1, v_1] \\ [u_2, v_2] \\ \vdots \\ [u_n, v_n] \end{pmatrix} \in \chi_{\mathcal{Z}}$, the IVFS $\mathcal{Q}'$ with $\mu^-_{\mathcal{Q}'}(v_i) = u_i$ and $\mu^+_{\mathcal{Q}'}(v_i) = v_i$ for all $i \in \underline{n}$ is a complete IVFQ.*

Proof. Let $\mathcal{Q}'$ be a complete IVFQ in $\mathcal{Z}$. Then from Theorem 3 it follows that $(U_{\mathcal{Q}'} \odot U^T_{\mathcal{Q}'})_{ij} = \min\left\{(U_{\mathcal{Q}'})_i, (U_{\mathcal{Q}'})_j\right\} = [\min\left\{\mu^-_{\mathcal{Q}'}(v_i), \mu^-_{\mathcal{Q}'}(v_j)\right\}, \min\left\{\mu^+_{\mathcal{Q}'}(v_i), \mu^+_{\mathcal{Q}'}(v_j)\right\}] \leq [\mu^-_{\mathcal{R}}(v_i, v_j), \mu^+_{\mathcal{R}}(v_i, v_j)]$ $(M_{\mathcal{Z}})_{ij}$ for any $i, j \in \underline{n}$ with $i \neq j$ and $(U_{\mathcal{Q}'} \odot U^T_{\mathcal{Q}'})_{ii} = \min\left\{(U_{\mathcal{Q}'})_i, (U_{\mathcal{Q}'})_i\right\} = [\min\left\{\mu^-_{\mathcal{Q}'}(v_i), \mu^-_{\mathcal{Q}'}(v_i)\right\}, \min\left\{\mu^+_{\mathcal{Q}'}(v_i), \mu^+_{\mathcal{Q}'}(v_i)\right\}] \leq [\mu^-_{\mathcal{R}}(v_i, v_i), \mu^+_{\mathcal{R}}(v_i, v_i)] = (M_{\mathcal{Z}})_{ii}$ for any $i \in \underline{n}$. Therefore, $U_{\mathcal{Q}'} \in \chi_{\mathcal{Z}}$.

Conversely, let $\mathcal{L} = \begin{pmatrix} [u_1, v_1] \\ [u_2, v_2] \\ \vdots \\ [u_n, v_n] \end{pmatrix} \in \chi_{\mathcal{Z}}$ and an IVFS $\mathcal{Q}'$ be such that $\mu^-_{\mathcal{Q}'}(v_i) = u_i$ and $\mu^+_{\mathcal{Q}'}(v_i) = v_i \forall i \in \underline{n}$. Since, $\mu_{\mathcal{Q}'}(v_i) = [u_i, v_i] \leq (M_{\mathcal{Z}})_{ii} = \mu_{\mathcal{Q}}(v_i)$ for all $i \in \underline{n}$, we have $\mathcal{Q}' \subseteq \mathcal{Q}$. Then from Theorem 3, $\mathcal{Q}'$ is a complete IVFQ. Hence the result follows. ■

The following example illustrates Theorem 4.

Example 3.

Let us consider the IVFG $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$ and the complete IVFQ $\mathcal{Q}'$ in Example 2. Then we have

$$M_{\mathcal{Z}} = \begin{pmatrix} [1,1] & [0.8,0.9] & [0.5,0.6] & [0.6,0.7] & [0.5,0.6] \\ [0.8,0.9] & [1,1] & [0.8,0.9] & [0.6,0.7] & [0.5,0.6] \\ [0.5,0.6] & [0.8,0.9] & [1,1] & [0.7,0.8] & [0.5,0.6] \\ [0.6,0.7] & [0.6,0.7] & [0.7,0.8] & [1,1] & [0.5,0.6] \\ [0.5,0.6] & [0.5,0.6] & [0.5,0.6] & [0.5,0.6] & [1,1] \end{pmatrix}$$

and $U_{\mathcal{Q}'} = \begin{pmatrix} [0.7, 0.8] \\ [0.8, 0.9] \\ [0.5, 0.6] \\ [0.6, 0.7] \\ [0.5, 0.6] \end{pmatrix}$. Then it can be easily verified that

$$U_{\mathcal{Q}'} \odot (U_{\mathcal{Q}'})^T = \begin{pmatrix} [0.7,0.8] & [0.7,0.8] & [0.5,0.6] & [0.6,0.7] & [0.5,0.6] \\ [0.7,0.8] & [0.8,0.9] & [0.5,0.6] & [0.6,0.7] & [0.5,0.6] \\ [0.5,0.6] & [0.5,0.6] & [0.5,0.6] & [0.5,0.6] & [0.5,0.6] \\ [0.6,0.7] & [0.6,0.7] & [0.5,0.6] & [0.6,0.7] & [0.5,0.6] \\ [0.5,0.6] & [0.5,0.6] & [0.5,0.6] & [0.5,0.6] & [0.5,0.6] \end{pmatrix} \leq M_{\mathcal{Z}}.$$

This shows that $U_{\mathcal{Q}'} \in \chi_{\mathcal{Z}}$.

Corollary 3 states that every nonempty subset of a complete IVFQ is complete IVFQ. This gives a clue of having maximal and maximum complete IVFQ. Now, we give the definitions of maximal and maximum complete IVFQ.

Definition 5. (Maximal and maximum complete IVFQ) A complete IVFQ $\mathcal{Q}''$ in an IVFG $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$ is said to be maximal if there is no complete IVFQ $\mathcal{Q}'$ in $\mathcal{Z}$ such that $\mathcal{Q}'' \subset \mathcal{Q}'$. A maximal complete IVFQ $\mathcal{Q}''$ is maximum if it posses the largest possible cardinality of the crisp set $(\mathcal{Q}'')^*$.

Using Theorem 4, we can characterize the maximal complete IVFQs.

Theorem 5. *In an IVFG $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$, an interval-valued fuzzy subset $\mathcal{Q}''$ of $\mathcal{Q}$ is a maximal complete IVFQ if and only if $U_{\mathcal{Q}''}$ is a maximal element in $\chi_{\mathcal{Z}}$.*

Proof. By Theorem 4, $\mathcal{Q}''$ is a complete IVFQ if and only if $U_{\mathcal{Q}''} \in \chi_{\mathcal{Z}}$. Furthermore, if $\mathcal{Q}''$ is maximal, then $U_{\mathcal{Q}''}$ is maximal in $\chi_{\mathcal{Z}}$ and vice-versa. Hence, the result follows. ■

Theorem 6. *Let $\mathcal{Q}'$ be a maximal complete IVFQ in $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$. Then $\bigwedge_{u \in (\mathcal{Q}')^*} \mu^-_{\mathcal{Q}'}(u) = \bigwedge_{v,z \in (\mathcal{Q}')^*} \mu^-_{\mathcal{R}}(v,z)$.*

Proof. Let us consider that the set $\mathcal{Q}'$ is a complete IVFQ in the IVFG $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$. Then by Theorem 3, we have $\mu^-_{\mathcal{R}}(u,v) \geq \min\left\{\mu^-_{\mathcal{Q}'}(u), \mu^-_{\mathcal{Q}'}(v)\right\}$ and $\mu^+_{\mathcal{R}}(u,v) \geq \min\left\{\mu^+_{\mathcal{Q}'}(u), \mu^+_{\mathcal{Q}'}(v)\right\}$ for any $u, v \in (\mathcal{Q}')^*$. Therefore, $\bigwedge_{u \in (\mathcal{Q}')^*} \mu^-_{\mathcal{Q}'}(u) \leq \bigwedge_{v,z \in (\mathcal{Q}')^*} \mu^-_{\mathcal{R}}(v,z)$. Let $\mu^-_{\mathcal{Q}'}\left(v_k\right) = \bigwedge_{u \in (\mathcal{Q}')^*} \mu^-_{\mathcal{Q}'}(u)$ and $\mu^-_{\mathcal{R}}\left(v_i, v_j\right) = \bigwedge_{v,z \in (\mathcal{Q}')^*} \mu^-_{\mathcal{R}}(v,z)$ and $\mu^+_{\mathcal{Q}'}\left(v_k\right) = \bigwedge_{u \in (\mathcal{Q}')^*} \mu^+_{\mathcal{Q}'}(u)$ and $\mu^+_{\mathcal{R}}\left(v_i, v_j\right) = \bigwedge_{v,z \in (\mathcal{Q}')^*} \mu^+_{\mathcal{R}}(v,z)$. Now, we prove that $\mu^-_{\mathcal{Q}'}(v_k) = \mu^-_{\mathcal{R}}(v_i, v_j)$ and the similar proof can be shown for $\mu^+_{\mathcal{Q}'}(v_k) = \mu^+_{\mathcal{R}}(v_i, v_j)$. If possible let $\mu^-_{\mathcal{Q}'}(v_k) \neq \mu^-_{\mathcal{R}}(v_i, v_j)$. Then, $\mu^-_{\mathcal{Q}'}(v_k) < \mu^-_{\mathcal{R}}(v_i, v_j)$ as the other case $\mu^-_{\mathcal{Q}'}(v_k) > \mu^-_{\mathcal{R}}(v_i, v_j)$ does not hold in the fuzzy graph. Now, define an IVFS $\mathcal{Q}''$ such that $\mu^-_{\mathcal{Q}''}(u) = \mu^-_{\mathcal{Q}'}(u)$, $\forall u (\neq v_k) \in U$ and $\mu^-_{\mathcal{Q}''}(v_k) = \mu^-_{\mathcal{R}}(v_i, v_j)$. Then it is obvious that $\mathcal{Q}' \subset \mathcal{Q}'' \subseteq \mathcal{Q}$, since, $\mu^-_{\mathcal{Q}''}(v_k) = \mu_{\mathcal{R}}(v_i, v_j) \bigwedge_{v,z \in (\mathcal{Q}')^*} \mu^-_{\mathcal{R}}(v,z) \leq \mu^-_{\mathcal{R}}(v_k, z) \leq \mu^-_{\mathcal{Q}}(v_k) \wedge \mu^-_{\mathcal{Q}}(z) \leq \mu^-_{\mathcal{Q}}(v_k)$ for any $z \in (\mathcal{Q}')^*$. Now, it is to be shown that $\mathcal{Q}''$ is a complete IVFQ, i.e., $\mu^-_{\mathcal{Q}''}(v) \wedge \mu^-_{\mathcal{Q}''}(z) \leq \mu^-_{\mathcal{R}}(v,z)$ for any $v, z \in (\mathcal{Q}')^*$. For all $v, z \in U$ with $v \neq v_k, z \neq v_k$, $\mu^-_{\mathcal{Q}''}(v) \wedge \mu^-_{\mathcal{Q}''}(z) = \mu^-_{\mathcal{Q}'}(v) \wedge \mu^-_{\mathcal{Q}'}(z) \leq \mu^-_{\mathcal{R}}(v,z)$. Otherwise, without any loss of generality, let, $v_k = v$, then $\mu^-_{\mathcal{Q}''}(v) \wedge \mu^-_{\mathcal{Q}''}(z) = \mu^-_{\mathcal{Q}'}(v_k) \wedge \mu^-_{\mathcal{Q}'}(z)\ \mu^-_{\mathcal{R}}(v_i, v_j) \wedge \mu^-_{\mathcal{Q}'}(z) \leq \mu^-_{\mathcal{R}}(v_i, v_j) \bigwedge_{u,v \in (\mathcal{Q}')^*} \mu^-_{\mathcal{R}}(u,v) \leq \mu^-_{\mathcal{R}}(v,z)$. Therefore, $\mathcal{Q}''$ is a complete IVFQ, which contradicts the assumption that $\mathcal{Q}'$ is maximal. Thus $\bigwedge_{u \in (\mathcal{Q}')^*} \mu^-_{\mathcal{Q}'}(u) = \bigwedge_{v,z \in (\mathcal{Q}')^*} \mu^-_{\mathcal{R}}(v,z)$ and hence the result follows. ■

Corollary 4. Let $\mathcal{Q}'$ be a maximal complete IVFQ in the IVFG $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$ and $\mathcal{H} = (U, \mathcal{Q}', \mathcal{R}')$ be the IVFSG induced by $\mathcal{Q}'$ and $\bigwedge_{u,v \in (\mathcal{Q}')^*} \mu_{\mathcal{R}}(u,v) = \mu_{\mathcal{R}}\left(v_i, v_j\right)$. Then $\mu_{\mathcal{R}'}(v_i, v_j) = \mu_{\mathcal{R}}(v_i, v_j)$.

Proof. Since, $\mathcal{H}$ is an IVFSG then it is obvious that, $\mu^-_{\mathcal{R}'}(v_i, v_j) \leq \mu^-_{\mathcal{R}}(v_i, v_j)$ and $\mu^+_{\mathcal{R}'}(v_i, v_j) \leq \mu^+_{\mathcal{R}}(v_i, v_j)$. We show that, $\mu^-_{\mathcal{R}'}(v_i, v_j) \not< \mu^-_{\mathcal{R}}(v_i, v_j)$ and $\mu^+_{\mathcal{R}'}(v_i, v_j) \not< \mu^+_{\mathcal{R}}(v_i, v_j)$. If possible let, $\mu^-_{\mathcal{R}'}(v_i, v_j) < \mu^-_{\mathcal{R}}(v_i, v_j)$, then $\mu^-_{\mathcal{R}'}(v_i, v_j) = \mu^-_{\mathcal{Q}'}(v_i) \wedge \mu^-_{\mathcal{Q}'}(v_j) < \mu^-_{\mathcal{R}}(v_i, v_j)$. Therefore, $\bigwedge_{u \in (\mathcal{Q}')^*} \mu^-_{\mathcal{Q}'}(u) \leq \mu^-_{\mathcal{Q}'}\left(v_i\right) \wedge \mu^-_{\mathcal{Q}'}\left(v_j\right) < \mu_{\mathcal{R}}\left(v_i, v_j\right) = \bigwedge_{u,v \in (\mathcal{Q}')^*} \mu^-_{\mathcal{R}}(u,v)$, which contradicts Theorem 6. Therefore, $\mu^-_{\mathcal{R}'}(v_i, v_j) = \mu^-_{\mathcal{R}}(v_i, v_j)$ and similarly it can be shown that $\mu^+_{\mathcal{R}'}(v_i, v_j) = \mu^+_{\mathcal{R}}(v_i, v_j)$. Hence $\mu_{\mathcal{R}'}(v_i, v_j) = \mu_{\mathcal{R}}(v_i, v_j)$. ■

Theorem 7. *Let $\mathcal{Q}'$ be a maximal complete IVFQ in $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$. Then there is at least one $u \in (\mathcal{Q}')^*$ such that $\mu_{\mathcal{Q}'}(u) = \mu_{\mathcal{Q}}(u)$.* ■

Proof. Let $\mathcal{Q}'$ be a maximal complete IVFQ in $\mathcal{Z} = (U, \mathcal{Q}, \mathcal{R})$. Obviously, $\mu^-_{\mathcal{Q}'}(u) \leq \mu^-_{\mathcal{Q}}(u)$ and $\mu^+_{\mathcal{Q}'}(u) \leq \mu^+_{\mathcal{Q}}(u)$ for all $u \in (\mathcal{Q}')^*$. It is to prove that, $\exists u \in (\mathcal{Q}')^*$ such that $\mu^-_{\mathcal{Q}'}(u) = \mu^-_{\mathcal{Q}}(u)$ and $\mu^+_{\mathcal{Q}'}(u) = \mu^+_{\mathcal{Q}}(u)$. If possible let, $\mu^-_{\mathcal{Q}'}(u) < \mu^-_{\mathcal{Q}}(u)$ for all $u \in (\mathcal{Q}')^*$. Define two crisp sets U_1 and U_2 be such that $U_1 = \{u \in \left(\mathcal{Q}'\right)^* : \mu^-_{\mathcal{Q}'}(u) \leq \bigwedge_{v \in (\mathcal{Q}')^*} \mu^-_{\mathcal{R}}(u,v)\}$ and $U_2 = \{u \in \left(\mathcal{Q}'\right)^* : \mu^-_{\mathcal{Q}'}(u) > \bigwedge_{v \in (\mathcal{Q}')^*} \mu^-_{\mathcal{R}}(u,v)\}$. Then $(\mathcal{Q}')^* = U_1 \cup U_2$. Then consider the following two cases:

Case-I. In this case, let us consider, $U_1 = (\mathcal{Q}')^*$ and take an arbitrary element $u_0 \in U_1$ and construct an IVFS $\mathcal{Q}''$ such that $\mu^-_{\mathcal{Q}''}(u) = \mu^-_{\mathcal{Q}'}(u)$ and $\mu^+_{\mathcal{Q}''}(u) = \mu^+_{\mathcal{Q}'}(u)$, whenever $u \neq u_0$ and for $u = u_0$, $\mu^-_{\mathcal{Q}''}(u_0) = \mu^-_{\mathcal{Q}}(u_0)$ and $\mu^+_{\mathcal{Q}''}(u_0) = \mu^+_{\mathcal{Q}}(u_0)$, then $\mathcal{Q}' \subset \mathcal{Q}''$. Now,

$$\begin{aligned} \mu^-_{\mathcal{Q}''}(u) \wedge \mu^-_{\mathcal{Q}''}\left(u_0\right) = \mu^-_{\mathcal{Q}'}(u) \wedge \mu^-_{\mathcal{Q}''}\left(u_0\right) &\leq \bigwedge_{v \in (\mathcal{Q}')^*} \mu^-_{\mathcal{R}}(u,v) \wedge \mu^-_{\mathcal{Q}}\left(u_0\right) \\ &\leq \mu_{\mathcal{R}}\left(u, u_0\right) \wedge \mu^-_{\mathcal{Q}}\left(u_0\right) \\ &\leq \mu^-_{\mathcal{R}}\left(u, u_0\right) \\ &\quad \text{for any } u \in U_1 - \left\{u_0\right\}. \end{aligned}$$

and $\mu^-_{\mathcal{Q}''}(u) \wedge \mu^-_{\mathcal{Q}''}(v) = \mu^-_{\mathcal{Q}''}(u) \wedge \mu^-_{\mathcal{Q}'}(v) \leq \mu^-_{\mathcal{R}}(u,v)$ for any $u, v \in U_1 - \left\{u_0\right\}$. Therefore, by Theorem 3, $\mathcal{Q}''$ is a complete IVFQ and which contradicts the maximality of $\mathcal{Q}'$ since, $\mathcal{Q}' \subset \mathcal{Q}''$.

Case-II. Let us consider $U_1 \neq (\mathcal{Q}')^*$, i.e., $U_2 \neq \phi$. Then take an arbitrary u_0 of U_2 be such that $\mu^-_{\mathcal{Q}'}(u_0) = \max\left\{\mu^-_{\mathcal{Q}'}(u) : u \in U_2\right\}$ and define an IVFS $\mathcal{Q}''$ by

$$\mu^-_{\mathcal{Q}''}(u) = \begin{cases} \mu^-_{\mathcal{Q}'}(u), & u \neq u_0 \\ \mu^-_{\mathcal{Q}}(u), & u = u_0 \end{cases}$$

$$\mu^+_{\mathcal{Q}''}(u) = \begin{cases} \mu^+_{\mathcal{Q}'}(u), & u \neq u_0 \\ \mu^+_{\mathcal{Q}}(u), & u = u_0 \end{cases}$$

Then $\mathcal{Q}' \subset \mathcal{Q}''$. Now,

$$\begin{aligned} \mu^-_{\mathcal{Q}''}(u) \wedge \mu^-_{\mathcal{Q}''}\left(u_0\right) = \mu^-_{\mathcal{Q}'}(u) \wedge \mu^-_{\mathcal{Q}}\left(u_0\right) &\leq \bigwedge_{v \in (\mathcal{Q}')^*} \mu^-_{\mathcal{R}}(u,v) \wedge \mu^-_{\mathcal{Q}}\left(u_0\right) \\ &\leq \mu^-_{\mathcal{R}}\left(u, u_0\right) \wedge \mu^-_{\mathcal{Q}}\left(u_0\right) \\ &\leq \mu^-_{\mathcal{R}}\left(u, u_0\right) \text{ for any } u \in U_1. \end{aligned}$$

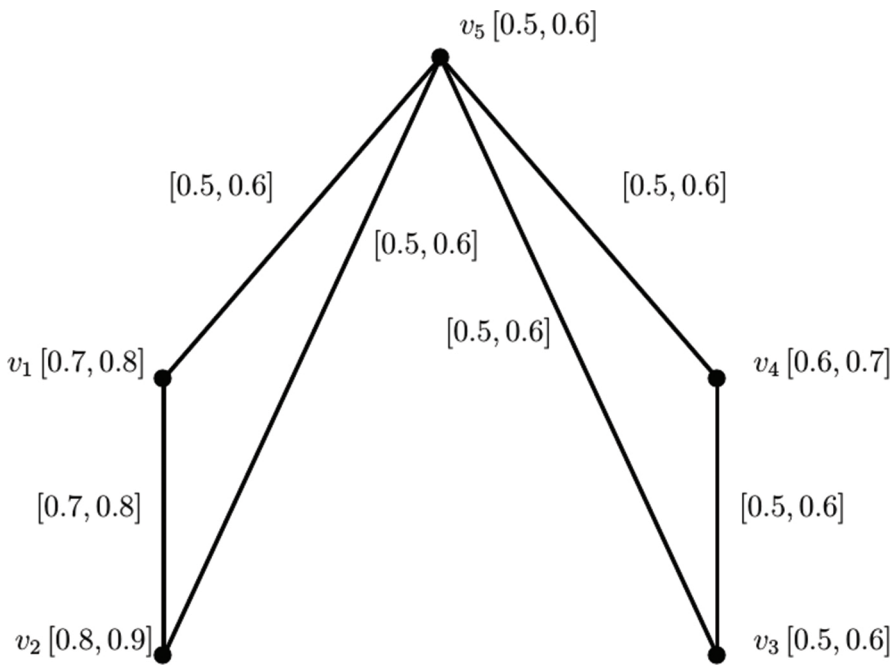


Figure 5 An interval-valued fuzzy graph (IVFG) $\mathscr{Z} = (U, \mathscr{Q}, \mathcal{R})$.

and $\mu^-_{\mathscr{Q}''}(u) \wedge \mu^-_{\mathscr{Q}'}(u_0) = \mu^-_{\mathscr{Q}'}(u) \wedge \mu^-_{\mathscr{Q}}(u_0) \leq \mu^-_{\mathscr{Q}'}(u) \wedge \mu^-_{\mathscr{Q}'}(u_0) \leq \mu^-_{\mathcal{R}}(u, u_0)$ when $u \in U_2$ and $\mu^-_{\mathscr{Q}''}(u) \wedge \mu^-_{\mathscr{Q}''}(v) = \mu^-_{\mathscr{Q}'}(u) \wedge \mu^-_{\mathscr{Q}'}(v) \leq \mu^-_{\mathcal{R}}(u, v)$ for any $u, v \in (\mathscr{Q}')^* - \{u_0\}$. Therefore, by Theorem 3, $\mathscr{Q}''$ is a complete IVFQ and which contradicts the maximality of $\mathscr{Q}'$ since, $\mathscr{Q}' \subset \mathscr{Q}''$.

Therefore, considering the Cases I and II, it can be concluded that there is at least one $u \in (\mathscr{Q}')^*$ such that $\mu^-_{\mathscr{Q}'}(u) = \mu^-_{\mathscr{Q}}(u)$. Making similar arguments, it can be also shown that there is at least one $u \in (\mathscr{Q}')^*$ such that $\mu^+_{\mathscr{Q}'}(u) = \mu^+_{\mathscr{Q}}(u)$. Hence, for at least one $u \in (\mathscr{Q}')^*$, $\mu_{\mathscr{Q}'}(u) = \mu_{\mathscr{Q}}(u)$. ■

3. IVFQCs IN IVFGs

In this section, complete IVFQCs and minimum complete IVFQCs are discussed, and an algorithm to find a minimum complete IVFQC of a given IVFG is provided.

Definition 6. (Complete IVFQ edge cover) A complete IVFQ edge cover for an IVFG $\mathscr{Z} = (U, \mathscr{Q}, \mathcal{R})$ is an IVFS $\mathscr{C}$ of complete IVFQs that includes all of the interval-valued fuzzy edges in $\mathscr{Z}$.

In crisp graphs, if all the edges of the crisp graph J is adjacent to at least one of the vertices of the clique set then the clique is said to be the clique cover of the crisp graph. However, according to Definition 6, an IVFQ edge cover for an IVFG $\mathscr{Z}$ may not cover some interval-valued fuzzy vertices in $\mathscr{Z}$. We can verify this through the following example.

Example 4.

Consider an IVFG $\mathscr{Z} = (U, \mathscr{Q}, \mathcal{R})$ which is shown in Figure 5.

Now, $\mathscr{Q}' = \{v_1[0.7, 0.8], v_2[0.7, 0.8], v_5[0.5, 0.6]\}$ and $\mathscr{Q}'' = \{v_3[0.5, 0.6], v_4[0.6, 0.7], v_5[0.5, 0.6]\}$ are two complete IVFQs shown in Figures 6(a) and 6(b). Here we see that the interval-valued fuzzy vertex $v_2[0.8, 0.9]$ is not covered by the complete IVFQ set $\{\mathscr{Q}', \mathscr{Q}''\}$.

Motivating from this criticism, we define the IVFQC for an IVFG which covers all of the interval-valued fuzzy edges and interval-valued fuzzy vertices.

Definition 7. (IVFQC) An IVFQC for an IVFG $\mathscr{Z} = (U, \mathscr{Q}, \mathcal{R})$ is a set $\mathscr{C}$ of complete IVFQs such that $\mathscr{Z}$ can be decomposed as the union of all IVFSGs induced by the IVFQs in $\mathscr{C}$. The fuzzy clique cover number of $\mathscr{Z}$ is denoted by $cc(\mathscr{Z})$ is the minimum cardinality of an IVFQC of $\mathscr{Z}$. A minimium IVFQC is a complete IVFQC $\mathscr{C}$ such that $|\mathscr{C}| = cc(\mathscr{Z})$.

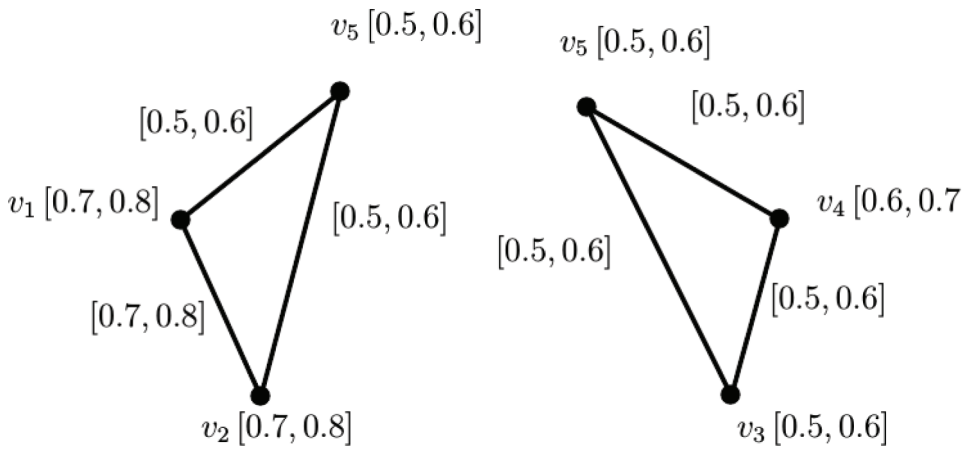
Before going to characterize the minimum IVFQC, we first give the following lemma.

Lemma 8. For a complete IVFQ $\mathscr{Q}'$ in an IVFG $\mathscr{Z} = (U, \mathscr{Q}, \mathcal{R})$, the composition $U_{\mathscr{Q}'} \odot (U_{\mathscr{Q}'})^T$ represents the IVFSG induced by $\mathscr{Q}'$.

Proof. Let $\mathscr{Q}'$ be a complete IVFQ in an IVFG $\mathscr{Z}$ and $\mathcal{H} = (U, \mathscr{Q}', \mathcal{R}')$ be the IVFSG induced by $\mathscr{Q}'$. Define $n \times n$ interval-valued fuzzy matrix $M_{\mathcal{H}}$ as $(M_{\mathcal{H}})_{ij} = \mu_{\mathcal{R}'}(v_i, v_j)$ whenever $i \neq j$, and $(M_{\mathcal{H}})_{ii} = \mu_{\mathscr{Q}'}(v_i)$. Then, for any $i, j \in \underline{n}$, we have $(U_{\mathscr{Q}'} \odot (U_{\mathscr{Q}'})^T)_{ij} = \min\{(U_{\mathscr{Q}'})_i, (U_{\mathscr{Q}'})_j\} = \min\{\mu_{\mathscr{Q}'}(v_i), \mu_{\mathscr{Q}'}(v_j)\} = \mu_{\mathcal{R}'(v_i, v_j)} = (M_{\mathcal{H}})_{ij}$ with $i \neq j$, and $(U_{\mathscr{Q}'} \odot (U_{\mathscr{Q}'})^T)_{ii} = \min\{(U_{\mathscr{Q}'})_i, (U_{\mathscr{Q}'})_i = \min\{\mu_{\mathscr{Q}'}(v_i), \mu_{\mathscr{Q}'}(v_i)\} = \mu_{\mathcal{R}'(v_i, v_i)} = (M_{\mathcal{H}})_{ii}$. Therefore, $U_{\mathscr{Q}'} \odot (U_{\mathscr{Q}'})^T$ represents $\mathcal{H}$. ■

Theorem 9. *For an IVFG $\mathscr{Z}$, if $\{\mathscr{Q}', \mathscr{Q}'', \ldots, \mathscr{Q}^{(k)}\}$ where $k \in \underline{m}$ is an IVFQC of $\mathscr{Z}$, then the $n \times m$ matrix $M^r_{\mathscr{Z}}$ with $(M^r_{\mathscr{Z}})_{ik} = \mu_{\mathscr{Q}^{(k)}}(v_i)$ for all $k \in \underline{m}$ and $v_i \in U$, $i \in \underline{n}$ is a realizing interval-valued fuzzy matrix of $M_{\mathscr{Z}}$.*

Proof. For an IVFQC $\{\mathscr{Q}', \mathscr{Q}'', \ldots, \mathscr{Q}^{(k)}\}$ where $k \in \underline{m}$ an $n \times m$ matrix $M^r_{\mathscr{Z}}$ with $(M^r_{\mathscr{Z}})_{ik} = \mu_{\mathscr{Q}^{(k)}}(v_i)$ for all $k \in \underline{m}$ and $v_i \in U$, $i \in \underline{n}$



(a) A complete IVFQ $\mathcal{Q}'$ (b) A complete IVFQ $\mathcal{Q}''$

Figure 6 Example of complete interval-valued fuzzy clique (IVFQ) edge cover of $\mathcal{Z}$ shown in Figure 5.

is given. Then by 8, $M^r_{\mathcal{Z}} \odot (M^r_{\mathcal{Z}})^T$ represents the IVFG $\mathcal{Z}$. Hence, the interval-valued fuzzy matrix $M^r_{\mathcal{Z}}$ is a realizing fuzzy matrix of $M_{\mathcal{Z}}$. ■

Theorem 10. *For an IVFG $\mathcal{Z}$, if $M^r_{\mathcal{Z}}$ is an $n \times m$ realizing interval-valued fuzzy matrix of $M^r_{\mathcal{Z}}$, then $\{\mathcal{Q}', \mathcal{Q}'', \ldots, \mathcal{Q}^{(k)}\}$ for each $k \in \underline{m}$ with $(M^r_{\mathcal{Z}})_{ik} = \mu_{\mathcal{Q}^{(k)}}(v_i)$ for all $k \in \underline{m}$ and $v_i \in U$, $i \in \underline{n}$ is an IVFQC of $\mathcal{Z}$.*

Proof. From the definition of $M_{\mathcal{Z}}$ and inerval-valued fuzzy graph, we have $(M_{\mathcal{Z}})_{ii} \geq (M_{\mathcal{Z}})_{ij} = (M_{\mathcal{Z}})_{ji}$ for all $i, j \in \underline{n}$. Then it is obvious that $M_{\mathcal{Z}}$ is a realizable. Let $M^r_{\mathcal{Z}}$ be an $n \times m$ realizing interval-valued fuzzy matrix of $M_{\mathcal{Z}}$. Construct a set $\{\mathcal{Q}', \mathcal{Q}'', \ldots, \mathcal{Q}^{(k)}\}$ where $k \in \underline{m}$ of IVFSs such that $(M^r_{\mathcal{Z}})_{ik} = \mu_{\mathcal{Q}^{(k)}}(v_i)$ for all $k \in \underline{m}$ and $v_i \in U$, $i \in \underline{n}$. Then by Theorem 4, $\mathcal{Q}^{(k)}$ is an IVFQ for any $k \in \underline{m}$. Then from Lemma 8 it follows that union of all IVFSGs induced by $\mathcal{Q}^{(k)}$, $k \in \underline{m}$ is $\mathcal{Z}$. Therefore, $\{\mathcal{Q}', \mathcal{Q}'', \ldots, \mathcal{Q}^{(k)}\}$ for each $k \in \underline{m}$ is an IVFQC of $\mathcal{Z}$. ■

Theorem 11. *For an IVFG $\mathcal{Z}$, $\{\mathcal{Q}', \mathcal{Q}'', \ldots, \mathcal{Q}^{(k)}\}$ for each $k \in \underline{cc(\mathcal{Z})}$ is a minimum IVFQC of $\mathcal{Z}$ if and only if $M^r_{\mathcal{Z}}$ is a $n \times c(M_{\mathcal{Z}})$ realizing fuzzy matrix of $M_{\mathcal{Z}}$, where $(M^r_{\mathcal{Z}})_{ik} = \mu_{\mathcal{Q}^{(k)}}(v_i)$ for all $k \in \underline{cc(\mathcal{Z})} = \underline{c(M_{\mathcal{Z}})}$, $v_i \in U$, $i \in \underline{n}$.*

Proof. It follows from Theorems 9 and 10. ■

4. APPLICATION OF IVFQC IN MOBILE NETWORKING COMMUNICATION

Today's world advances with wireless technology as much as possible. One of the great uses of wireless technology is in mobile networking. In mobile networking, the communications are done through some cell towers which receives the signals from the base station and send the signals to the mobile devices. These cell towers have some specific range within which it can serve better to reach the signals to the right receiver. Now, the wireless communication companies wants to set up minimum number of cell towers with maximum strength to cover all the region of consideration. This problem can be solved by setting up a model for IVFG where, each cell towers are taken as vertices and with the strength as the fuzzy values and edges are the connection between them and assigning the fuzzy values are their connection strength.

Suppose there are six cell towers with their strength and connections are given as shown in Figure 7. Obvioulsy, this is an IVFG. Now, to cover all the receiving devices by minimum number of cell towers is same as minimizing the number of cell towers which can cover all the towers and the connections between them. And this is equivalently, finding the minimum IVFQC of the IVFG. It is easy to find that, the minimum IVFQC the graph shown in Figure is $\{\mathcal{Q}', \mathcal{Q}''\}$ where, $\mathcal{Q}' = \{v_1[0.7, 0.8], v_2[0.6, 0.8], v_6[0.5, 0.7]\}$ and $\mathcal{Q}'' = \{v_3[0.6, 0.7], v_4[0.5, 0.6], v_5[0.7, 0.8]\}$. This shows that only two cell towers are required to send the signals to all receiving devices within the specified region.

5. CONCLUSION

Fuzzy cliques are the most important mathematical model to describe and analyze the relationship networks where one has to deal with the inter-relationships among some objects, like—human, stars, countries, etc. IFVQs are better capable over fuzzy cliques to deal such problems. The definition of IVFQ given by Nair and Cheng does not confront with the classical graph theory in the sense that "each subgraph induces by a clique is complete." For this reason, we have modified the definition of IVFQ so that each IVFSG induces by an IVFQ is complete. Since, the communication is an important criterion for modern civilization, the study of IVFQs and IVFQCs is more demanding among researchers. The theorems developed in this paper can be applied in several network models such as—setting up wireless cell towers considering several parameters, installation of satellites, development of data searching algorithm, development of social networking sites, etc. Farther studies of this concept can be developed in future so that the theory can also be applied in the real-world situation where the parameters related to objects are self-contradictory or has negative implications.

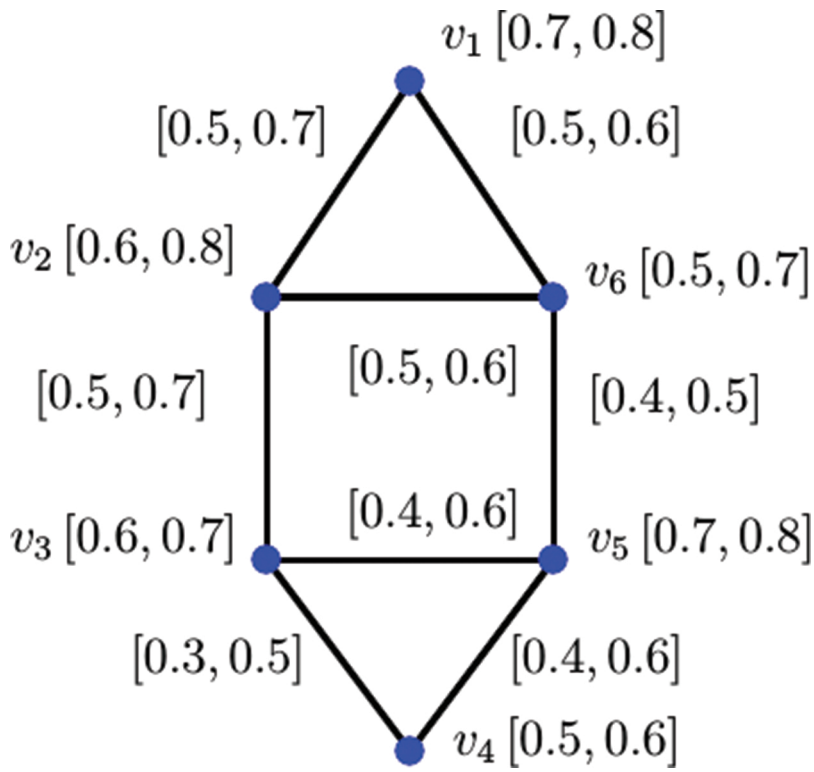


Figure 7 An interval-valued fuzzy graph (IVFG) model of a mobile communication.

CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interest.

AUTHORS' CONTRIBUTIONS

All authors are equally contributed in the paper.

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