

# On Finite 3-Component Mixture of Rayleigh Distributions: A Classical Look

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## ABSTRACT

In this study, we have discussed various statistical properties for 3-component mixture of Rayleigh distributions. Here initially, the main properties of mixture distributions are presented and analyzed. Second, some of the famous entropies, measures of inequality are also discussed. Also, the statistical properties of the density functions of  $r^{\text{th}}$ -,  $1^{\text{st}}$ - and  $n^{\text{th}}$ -order statistics are derived. Moreover, the parameters estimation of the considered mixture model under the maximum likelihood (ML) estimation is also performed using censored and complete data scheme. Finally, the results on ML estimation are also computed via Monte Carlo simulation study and as well as by using a real-life data set.

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## 1. INTRODUCTION

Mixture of distributions is arising naturally and discussed where a statistical population has more than two sub populations. In many applicable areas, the mixture representations have been compensated excessive care. However, the mixture of distribution that rises with a grouping of different distributions is said to be a mixture component, and the probabilities (weights) which are related to each of the component are known as the mixture weights.

In different practical situations, several authors have worked on the mixture modeling. Mostly, in the biology the direct applications of the mixture models are discussed by Bhattacharya [1] and by Gregor [2], in the medicine are presented by Chivers [3] and by Burekhardt [4], in the social sciences are observed by Harris [5], in an economics are mentioned by Jedidi *et al.* [6], in the reliability and survival are analyzed by Sultan *et al.* [7], in the life testing are suggested by Shawky and Bakoban [8], in the industrial engineering are observed by Ali *et al.* [9].

This study plan to develop a 3-component mixture of Rayleigh distributions for an effective modeling of time-to-failure data. Let consider a variable of interest  $y$  which follows a mixture distribution having  $q$  components and its density function is  $f(y) = \sum_{m=1}^q w_m f_m(y)$ , where

$w_m$  ( $m = 1, 2, \dots, q$ ) is  $m^{\text{th}}$  mixing proportions such that  $w_q = 1 - \sum_{m=1}^{q-1} w_m$  and  $f_m(y)$  is  $m^{\text{th}}$  component density function. The pdf and cdf of 3-component mixture of Rayleigh distributions for the unknown mixing proportions  $w_1$  and  $w_2$  is defined as follows:

$$f(y; \Omega) = w_1 f_1(y; \theta_1) + w_2 f_2(y; \theta_2) + w_3 f_3(y; \theta_3), \quad w_1, w_2 \geq 0, w_1 + w_2 \leq 1, \quad (1)$$

$$1 - R(y; \Omega) = w_1 (1 - R(y; \theta_1)) + w_2 (1 - R(y; \theta_2)) + (1 - w_1 - w_2) (1 - R(y; \theta_3)), \quad (2)$$

where  $\Omega = (\theta_1, \theta_2, \theta_3, w_1, w_2)$ .

$$f_m(y; \theta_m) = \frac{y}{\theta_m^2} \exp\left(-\frac{y^2}{2\theta_m^2}\right), \quad 0 < y < \infty, \theta_m > 0, m = 1, 2, 3,$$

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The cdf  $F_m(y; \Omega_m) = 1 - R_m(y; \theta_m)$  of the  $m^{\text{th}}$  component density is given by

$$1 - R_m(y; \theta_m) = 1 - \exp\left(-\frac{y^2}{2\theta_m^2}\right), 0 < y < \infty, \theta_m > 0. \quad (3)$$

## 2. STATISTICAL PROPERTIES

Here, we derive computable representations of some statistical properties associated with the 3-component mixture of Rayleigh distributions having pdf given in (1).

### 2.1. $r^{\text{th}}$ Moments about Origin

The  $r^{\text{th}}$  moments about origin of a 3-component mixture of Rayleigh distributions for a random variable  $Y$  is given as follows:

$$E(Y^r) = w_1 2^{\frac{r}{2}} \theta_1^r \left[ \frac{r}{2} + 1 \right] + w_2 2^{\frac{r}{2}} \theta_2^r \left[ \frac{r}{2} + 1 \right] + (1 - w_1 - w_2) 2^{\frac{r}{2}} \theta_3^r \left[ \frac{r}{2} + 1 \right]. \quad (4)$$

### 2.2. Mean and Variance

The mean and variance of the considered mixture distributions are as follows:

$$E(Y) = w_1 \theta_1 \sqrt{\frac{\pi}{2}} + w_2 \theta_2 \sqrt{\frac{\pi}{2}} + (1 - w_1 - w_2) \theta_3 \sqrt{\frac{\pi}{2}}, \quad (5)$$

$$\text{Var}(Y) = 2w_1 \theta_1^2 + 2w_2 \theta_2^2 + 2(1 - w_1 - w_2) \theta_3^2 - \left[ w_1 \theta_1 \sqrt{\frac{\pi}{2}} + w_2 \theta_2 \sqrt{\frac{\pi}{2}} + (1 - w_1 - w_2) \theta_3 \sqrt{\frac{\pi}{2}} \right]^2. \quad (6)$$

### 2.3. $h^{\text{th}}$ -Order Negative Moments

If we replaced  $(-h)$  instead of  $r$  in Equation (4), then it is converted into  $h^{\text{th}}$ -order negative moments and it is defined as follows:

$$E(Y^{(-h)}) = 2^{\frac{(-h)}{2}} w_1 \theta_1^{(-h)} \left[ \frac{(-h)}{2} + 1 \right] + 2^{\frac{(-h)}{2}} w_2 \theta_2^{(-h)} \left[ \frac{(-h)}{2} + 1 \right] + 2^{\frac{(-h)}{2}} (1 - w_1 - w_2) \theta_3^{(-h)} \left[ \frac{(-h)}{2} + 1 \right]. \quad (7)$$

### 2.4. Factorial Moments

If we replaced  $(\alpha - \mu)$  instead of  $r$  in Equation (4), then it is converted into factorial moments and it is defined as follows:

$$E(Y^{(\alpha-\mu)}) = 2^{\frac{(\alpha-\mu)}{2}} \left\{ w_1 \theta_1^{(\alpha-\mu)} \left[ \frac{(\alpha-\mu)}{2} + 1 \right] + w_2 \theta_2^{(\alpha-\mu)} \left[ \frac{(\alpha-\mu)}{2} + 1 \right] + (1 - w_1 - w_2) \theta_3^{(\alpha-\mu)} \left[ \frac{(\alpha-\mu)}{2} + 1 \right] \right\}. \quad (8)$$

### 2.5. Quantile Function

The quantile function for a  $Y$  random variable is as follows:

$$y_q = w_1 \theta_1 \sqrt{-2 \ln(1-p)} + w_2 \theta_2 \sqrt{-2 \ln(1-p)} + (1 - w_1 - w_2) \theta_3 \sqrt{-2 \ln(1-p)}. \quad (9)$$

### 2.6. Median

To solve the below mentioned Equation (10) for  $y$ , we obtain the median as follows:

$$p_1 \exp\left(-\frac{y^2}{2\theta_1^2}\right) + p_2 \exp\left(-\frac{y^2}{2\theta_2^2}\right) + (1 - p_1 - p_2) \exp\left(-\frac{y^2}{2\theta_3^2}\right) = \frac{1}{2}. \quad (10)$$

## 2.7. Mode

To resolve the below mentioned Equation (11) for  $y$ , we obtain the mode as follows:

$$\sum_{m=1}^3 p_1 \frac{1}{\theta_m^2} \exp\left(-\frac{y^2}{2\theta_m^2}\right) - \sum_{m=1}^3 p_1 \frac{1}{\theta_m^4} y^2 \exp\left(-\frac{y^2}{2\theta_m^2}\right) = 0. \quad (11)$$

The numerical results of mean, median, mode, variance and coefficient of skewness for various choices of  $\theta_1, \theta_2, \theta_3, p_1, p_2$  are offered in Table 1.

It is revealed that the distribution of the 3-component mixture of Rayleigh distributions is positively skewed because the value of mean is greater than median and  $SK > 0$  (cf. Table 1). As the values of  $\theta_1, \theta_2, \theta_3$  increases the variance of the distribution is also increased for fixed value of  $p_1$  and  $p_2$  (cf. Table 1). Moreover, there is also an increase in the variance values as the value of  $p_1$  and  $p_2$  is also increased for fixed values of  $\theta_1, \theta_2, \theta_3$  (cf. Table 1).

## 3. RELIABILITY PROPERTIES OF THE PROPOSED MIXTURE DISTRIBUTIONS

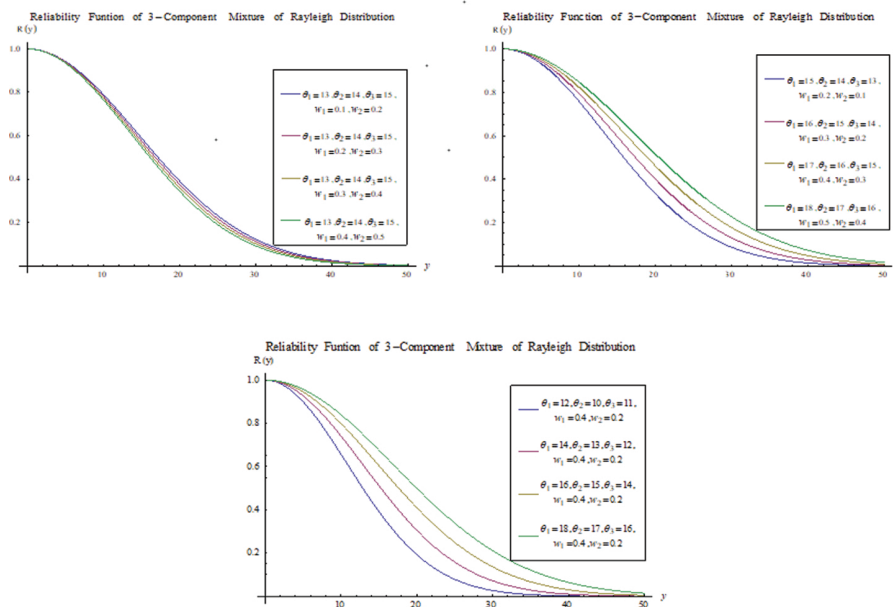
### 3.1. Reliability Function

The reliability function of the proposed mixture distributions is given as follows:

$$R(y; \Omega) = w_1 R_1(y; \theta_1) + w_2 R_2(y; \theta_2) + (1 - w_1 - w_2) R_3(y; \theta_3), \quad (12)$$

$$R(y; \Omega) = w_1 \exp\left(-\frac{y^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{y^2}{2\theta_2^2}\right) + (1 - w_1 - w_2) \exp\left(-\frac{y^2}{2\theta_3^2}\right). \quad (13)$$

The graphical presentation of the reliability function for proposed mixture distributions is as follows:



### 3.2. Hazard Rate Function

The hazard rate function for proposed mixture distributions is as follows:

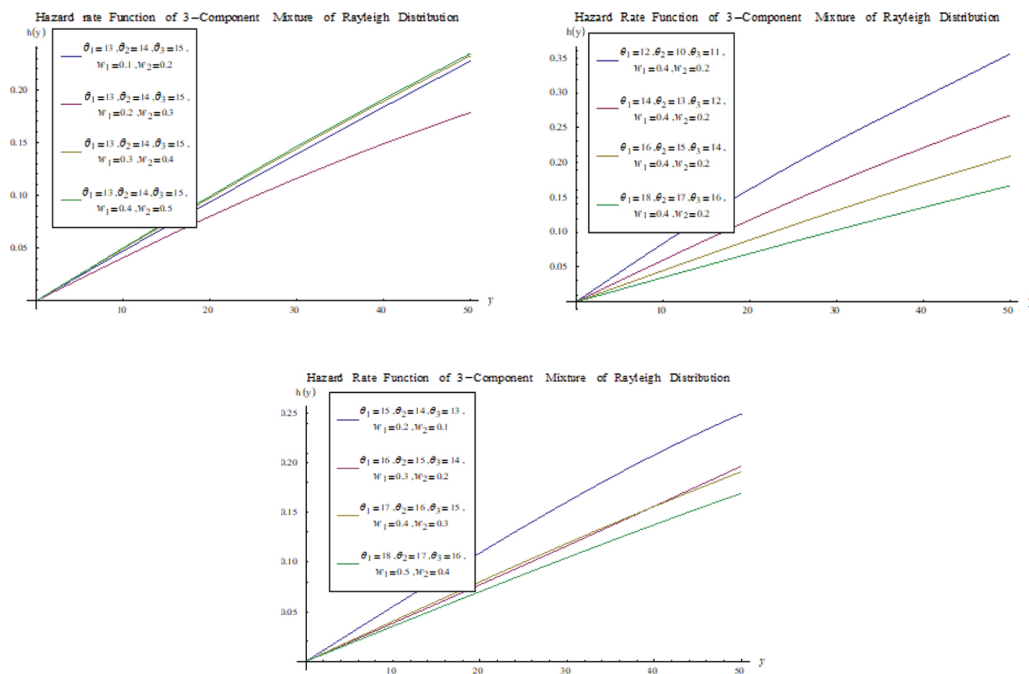
$$h(y; \Omega) = \frac{f(y; \Omega)}{R(y; \Omega)} = \frac{w_1 f_1(y; \theta_1) + w_2 f_2(y; \theta_2) + (1 - w_1 - w_2) f_3(y; \theta_3)}{w_1 R_1(y; \theta_1) + w_2 R_2(y; \theta_2) + (1 - w_1 - w_2) R_3(y; \theta_3)}, \quad (14)$$

**Table 1** Mean, variance, median, mode and skewness.

$\theta_1, \theta_2, \theta_3, p_1, p_2$	Mean	Variance	Median	Mode	Skewness
13, 14, 15, 0.1, 0.2	18.2984	91.6779	17.1902	14.5176	0.075906
13, 14, 15, 0.2, 0.3	17.9224	88.0297	16.837	14.1885	0.0759013
13, 14, 15, 0.3, 0.4	17.5464	84.3814	16.4837	13.8928	0.0759134
13, 14, 15, 0.4, 0.5	17.1704	80.7332	16.1305	13.6268	0.0759086
15, 14, 13, 0.2, 0.1	16.9197	78.5014	15.895	13.3907	0.0759137
16, 15, 14, 0.3, 0.2	18.549	94.339	17.4257	14.6759	0.0759014
17, 16, 15, 0.4, 0.3	20.1784	111.55	18.9563	15.9914	0.0759112
18, 17, 16, 0.5, 0.4	21.8077	130.135	20.4869	17.3345	0.0759137
12, 11, 10, 0.4, 0.2	13.7865	52.277	12.9515	10.8186	0.0759095
14, 13, 12, 0.4, 0.2	16.2931	72.8788	15.3063	12.8464	0.195787
16, 15, 14, 0.4, 0.2	18.7997	96.9142	17.6612	14.8668	0.179804
18, 17, 16, 0.4, 0.2	21.3063	124.383	20.016	16.8825	0.167591

$$h(y; \Omega) = \frac{f(y; \Omega)}{R(y; \Omega)} = \frac{w_1 \frac{y}{\theta_1^2} \exp\left(-\frac{y^2}{2\theta_1^2}\right) + w_2 \frac{y}{\theta_2^2} \exp\left(-\frac{y^2}{2\theta_2^2}\right) + (1 - w_1 - w_2) \frac{y}{\theta_3^2} \exp\left(-\frac{y^2}{2\theta_3^2}\right)}{w_1 \exp\left(-\frac{y^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{y^2}{2\theta_2^2}\right) + (1 - w_1 - w_2) \exp\left(-\frac{y^2}{2\theta_3^2}\right)}. \quad (15)$$

The graphical performance of the hazard rate function for proposed mixture distributions is given as follows:



### 3.3. Cumulative Hazard Rate Function

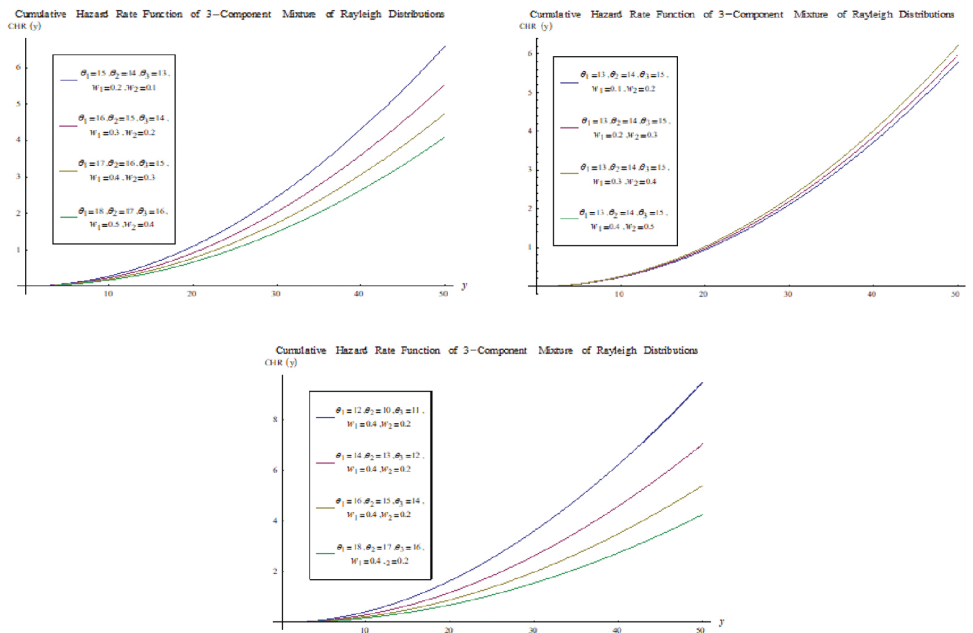
The cumulative hazard rate (CHR) function for proposed mixture distributions is defined as follows:

$$H(y; \Omega) = \int_0^y h(y; \Omega) dy = -\ln R(y; \Omega), \quad (16)$$

$$H(y; \Omega) = -\ln \left\{ w_1 \exp\left(-\frac{y^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{y^2}{2\theta_2^2}\right) + (1 - w_1 - w_2) \exp\left(-\frac{y^2}{2\theta_3^2}\right) \right\}. \quad (17)$$

The graphical performance of the CHR function for proposed mixture distributions is shown as follows:





### 3.4. Mean Residual Life Function

The mean residual life (MRL) function for proposed mixture distributions is written as follows:

$$M(y; \Omega) = \frac{1}{R(y; \Omega)} \int_y^{\infty} (x - y) f(x; \Omega) dx, \quad (18)$$

$$M(y; \Omega) = \frac{1}{R(y; \Omega)} \left\{ E(Y) - \int_0^y u f(y; \Omega) du \right\} - y, \quad (19)$$

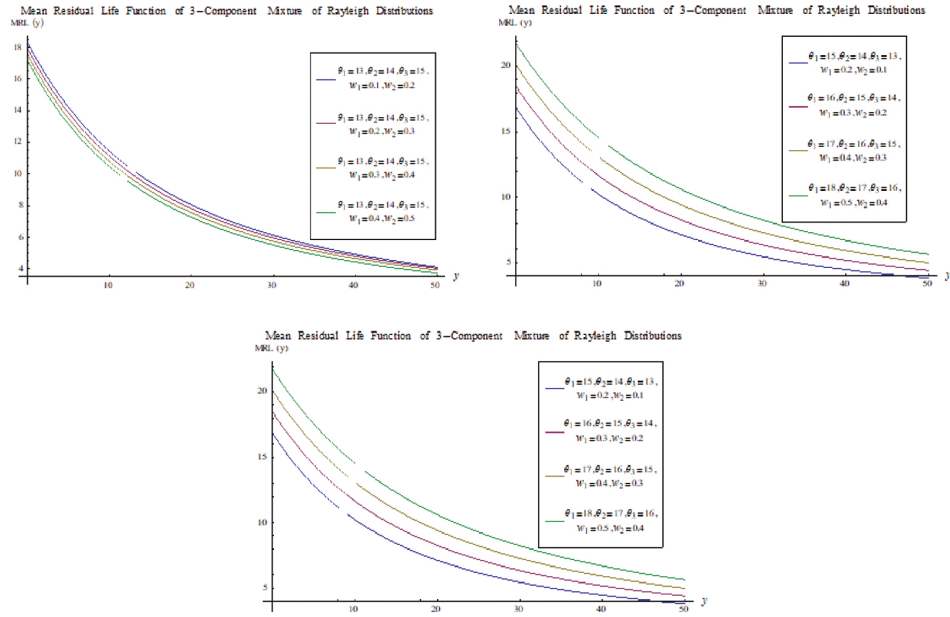
$$M(y; \Omega) = \frac{\left\{ w_1 \theta_1 \sqrt{\frac{\pi}{2}} + w_2 \theta_2 \sqrt{\frac{\pi}{2}} + (1 - w_1 - w_2) \theta_3 \sqrt{\frac{\pi}{2}} \right\} - \left\{ w_1 \theta_1 \sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\frac{y}{\sqrt{2}\theta_1}\right) + w_2 \theta_2 \sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\frac{y}{\sqrt{2}\theta_2}\right) + (1 - w_1 - w_2) \theta_3 \sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\frac{y}{\sqrt{2}\theta_3}\right) \right\}}{w_1 \exp\left(-\frac{y^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{y^2}{2\theta_2^2}\right) + (1 - w_1 - w_2) \exp\left(-\frac{y^2}{2\theta_3^2}\right)}. \quad (20)$$

The graphical performance of the MRL function for proposed mixture distributions is shown as follows:

### 3.5. Mean Waiting Time Function

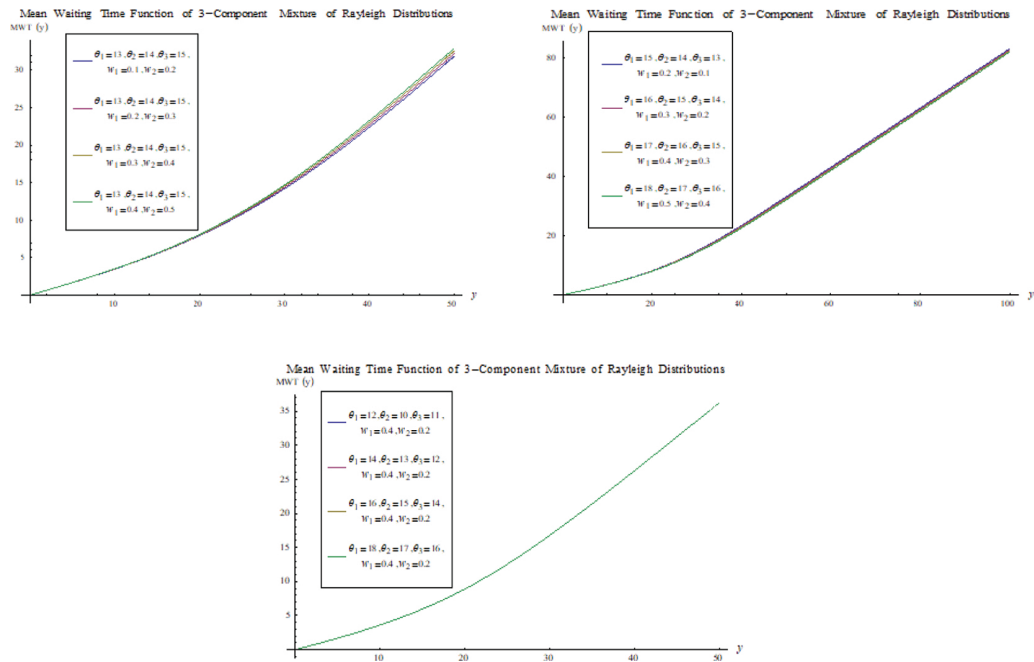
The mean waiting time (MWT) function for proposed mixture distributions is defined as follows:

$$\bar{\omega}(y; \Omega) = y - \left\{ \frac{1}{1 - R(y; \Omega)} \int_0^y u f(y; \Omega) du \right\}, \quad (21)$$



$$\bar{\omega}(y; \Omega) = \frac{y - w_1 \theta_1 \sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\frac{y}{\sqrt{2}\theta_1}\right) - w_2 \theta_2 \sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\frac{y}{\sqrt{2}\theta_2}\right) - (1 - w_1 - w_2) \theta_3 \sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\frac{y}{\sqrt{2}\theta_3}\right)}{1 - w_1 \exp\left(-\frac{y^2}{2\theta_1^2}\right) - w_2 \exp\left(-\frac{y^2}{2\theta_2^2}\right) - (1 - w_1 - w_2) \exp\left(-\frac{y^2}{2\theta_3^2}\right)} \quad (22)$$

The graphical presentation of the MWT function for proposed mixture distributions is shown as follows:



#### 4. STATISTICAL FUNCTIONS

The different statistical functions such as moment generating function ( $M_y(t)$ ), characteristic function ( $\phi_y(t)$ ), probability generating function ( $G(\alpha)$ ) and factorial moment generating function ( $H_0(\delta)$ ) for proposed mixture distributions are given below, respectively:

$$M_y(t) = \left[ w_1 + t\theta_1 \sqrt{\frac{\pi}{2}} \exp\left(\frac{\theta_1^2 t^2}{2}\right) \left\{ \operatorname{erf} \frac{t\theta_1}{\sqrt{2}} + 1 \right\} \right] + \left[ w_2 + t\theta_2 \sqrt{\frac{\pi}{2}} \exp\left(\frac{\theta_2^2 t^2}{2}\right) \left\{ \operatorname{erf} \frac{t\theta_2}{\sqrt{2}} + 1 \right\} \right] + \left[ (1 - w_1 - w_2) + t\theta_3 \sqrt{\frac{\pi}{2}} \exp\left(\frac{\theta_3^2 t^2}{2}\right) \left\{ \operatorname{erf} \frac{t\theta_3}{\sqrt{2}} + 1 \right\} \right]. \quad (23)$$

$$\phi_y(t) = \left[ w_1 + it\theta_1 \sqrt{\frac{\pi}{2}} \exp\left(-\frac{\theta_1^2 t^2}{2}\right) \left\{ \operatorname{erf} \frac{it\theta_1}{\sqrt{2}} + 1 \right\} \right] + \left[ w_2 + it\theta_2 \sqrt{\frac{\pi}{2}} \exp\left(-\frac{\theta_2^2 t^2}{2}\right) \left\{ \operatorname{erf} \frac{it\theta_2}{\sqrt{2}} + 1 \right\} \right] + \left[ (1 - w_1 - w_2) + it\theta_3 \sqrt{\frac{\pi}{2}} \exp\left(-\frac{\theta_3^2 t^2}{2}\right) \left\{ \operatorname{erf} \frac{it\theta_3}{\sqrt{2}} + 1 \right\} \right]. \quad (24)$$

$$G(\alpha) = \left[ w_1 + \ln \alpha \theta_1 \sqrt{\frac{\pi}{2}} \exp\left(\frac{\theta_1^2 (\ln \alpha)^2}{2}\right) \left\{ \operatorname{erf} \frac{\ln \alpha \theta_1}{\sqrt{2}} + 1 \right\} \right] + \left[ w_2 + \ln \alpha \theta_2 \sqrt{\frac{\pi}{2}} \exp\left(\frac{\theta_2^2 (\ln \alpha)^2}{2}\right) \left\{ \operatorname{erf} \frac{\ln \alpha \theta_2}{\sqrt{2}} + 1 \right\} \right] + \left[ (1 - w_1 - w_2) + \ln \alpha \theta_3 \sqrt{\frac{\pi}{2}} \exp\left(\frac{\theta_3^2 (\ln \alpha)^2}{2}\right) \left\{ \operatorname{erf} \frac{\ln \alpha \theta_3}{\sqrt{2}} + 1 \right\} \right]. \quad (25)$$

$$H_0(\delta) = \left[ w_1 + \ln(1 + \delta) \theta_1 \sqrt{\frac{\pi}{2}} \exp\left(\frac{\theta_1^2 (\ln(1 + \delta))^2}{2}\right) \left\{ \operatorname{erf} \frac{\ln(1 + \delta) \theta_1}{\sqrt{2}} + 1 \right\} \right] + \left[ w_2 + \ln(1 + \delta) \theta_2 \sqrt{\frac{\pi}{2}} \exp\left(\frac{\theta_2^2 (\ln(1 + \delta))^2}{2}\right) \left\{ \operatorname{erf} \frac{\ln(1 + \delta) \theta_2}{\sqrt{2}} + 1 \right\} \right] + \left[ (1 - w_1 - w_2) + \ln(1 + \delta) \theta_3 \sqrt{\frac{\pi}{2}} \exp\left(\frac{\theta_3^2 (\ln(1 + \delta))^2}{2}\right) \left\{ \operatorname{erf} \frac{\ln(1 + \delta) \theta_3}{\sqrt{2}} + 1 \right\} \right]. \quad (26)$$

#### 5. MEASURES OF INEQUALITY

In this section, we have considered various measures of inequality such as Gini index ( $G$ ), Lorenz curve ( $L(p)$ ), Bonferroni curve ( $BC(p)$ ) and Zenga index ( $\xi$ ) for proposed mixture distribution. The mathematical expressions of these inequalities are as follows:

$$G = \left( \sqrt{\frac{\pi}{2}} \right)^{-1} \left\{ w_1 \theta_1 + w_2 \theta_2 + (1 - w_1 - w_2) \theta_3 \right\}^{-1} \left[ \sqrt{\frac{\pi}{2}} \left\{ w_1 \theta_1 + w_2 \theta_2 + (1 - w_1 - w_2) \theta_3 \right\} - \left\{ \sqrt{\frac{\pi}{2}} \left( w_1^2 \theta_1 + w_2^2 \theta_2 + (1 - w_1 - w_2)^2 \theta_3 \right) + \sqrt{\pi} (4w_1 w_2 \theta_1 \theta_2 + 4w_2 (1 - w_1 - w_2) \theta_2 \theta_3 + 4w_1 (1 - w_1 - w_2) \theta_1 \theta_3) \right\} \right] \quad (27)$$

$$L(p) = \left\{ w_1 \theta_1 \sqrt{\frac{\pi}{2}} + w_2 \theta_2 \sqrt{\frac{\pi}{2}} + (1 - w_1 - w_2) \theta_3 \sqrt{\frac{\pi}{2}} \right\}^{-1} \left\{ w_1 \theta_1 \sqrt{\frac{\pi}{2}} \operatorname{erf} \left( \frac{y}{\sqrt{2} \theta_1} \right) - w_1 y \exp \left( -\frac{y^2}{2\theta_1^2} \right) + w_2 \theta_2 \sqrt{\frac{\pi}{2}} \operatorname{erf} \left( \frac{y}{\sqrt{2} \theta_2} \right) - w_2 y \exp \left( -\frac{y^2}{2\theta_2^2} \right) + (1 - w_1 - w_2) \theta_3 \sqrt{\frac{\pi}{2}} \operatorname{erf} \left( \frac{y}{\sqrt{2} \theta_3} \right) - (1 - w_1 - w_2) y \exp \left( -\frac{y^2}{2\theta_3^2} \right) \right\}. \quad (28)$$

$$BC(p) = \frac{L(p)}{1 - R(y; \mathbf{\Omega})}, \quad (29)$$

where

$$L(p) = \left\{ w_1 \theta_1 \sqrt{\frac{\pi}{2}} + w_2 \theta_2 \sqrt{\frac{\pi}{2}} + (1 - w_1 - w_2) \theta_3 \sqrt{\frac{\pi}{2}} \right\}^{-1} \left\{ w_1 \theta_1 \sqrt{\frac{\pi}{2}} \operatorname{erf} \left( \frac{y}{\sqrt{2} \theta_1} \right) - w_1 y \exp \left( -\frac{y^2}{2\theta_1^2} \right) + w_2 \theta_2 \sqrt{\frac{\pi}{2}} \operatorname{erf} \left( \frac{y}{\sqrt{2} \theta_2} \right) - w_2 y \exp \left( -\frac{y^2}{2\theta_2^2} \right) + (1 - w_1 - w_2) \theta_3 \sqrt{\frac{\pi}{2}} \operatorname{erf} \left( \frac{y}{\sqrt{2} \theta_3} \right) - (1 - w_1 - w_2) y \exp \left( -\frac{y^2}{2\theta_3^2} \right) \right\}, \quad (30)$$

and

$$1 - R(y; \mathbf{\Omega}) = \left\{ 1 - w_1 \exp \left( -\frac{y^2}{2\theta_1^2} \right) - w_2 \exp \left( -\frac{y^2}{2\theta_2^2} \right) - (1 - w_1 - w_2) \exp \left( -\frac{y^2}{2\theta_3^2} \right) \right\}. \quad (31)$$

$$\xi = 1 - \frac{\bar{\eta}_{(y)}}{\eta_{(y)}^+}, \quad (32)$$

where

$$\bar{\eta}_{(y)} = \frac{1}{1 - R(y; \mathbf{\Omega})} \int_0^y y f(y; \mathbf{\Omega}) dy \text{ and } \eta_{(y)}^+ = \frac{1}{R(y; \mathbf{\Omega})} \left\{ y - \int_0^y y f(y; \mathbf{\Omega}) dy \right\}.$$

After substitution the values in (36), the simplified Zenga index is as follows:

$$\xi = 1 - \frac{\left\{ w_1 \theta_1 \sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\frac{y}{\sqrt{2}\theta_1}\right) - w_1 y \exp\left(-\frac{y^2}{2\theta_1^2}\right) + w_2 \theta_2 \sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\frac{y}{\sqrt{2}\theta_2}\right) - w_2 y \exp\left(-\frac{y^2}{2\theta_2^2}\right) + (1 - w_1 - w_2) \theta_3 \sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\frac{y}{\sqrt{2}\theta_3}\right) \right.}{\left. \mu - \left\{ w_1 \theta_1 \sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\frac{y}{\sqrt{2}\theta_1}\right) - w_1 y \exp\left(-\frac{y^2}{2\theta_1^2}\right) + w_2 \theta_2 \sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\frac{y}{\sqrt{2}\theta_2}\right) - w_2 y \exp\left(-\frac{y^2}{2\theta_2^2}\right) + (1 - w_1 - w_2) \theta_3 \sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\frac{y}{\sqrt{2}\theta_3}\right) \right. \right.} \\ \left. \left. - (1 - w_1 - w_2) y \exp\left(-\frac{y^2}{2\theta_3^2}\right) \right\} \right] \left[ \frac{\left\{ w_1 \exp\left(-\frac{y^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{y^2}{2\theta_2^2}\right) + (1 - w_1 - w_2) \exp\left(-\frac{y^2}{2\theta_3^2}\right) \right\}}{\left\{ 1 - w_1 \exp\left(-\frac{y^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{y^2}{2\theta_2^2}\right) + (1 - w_1 - w_2) \exp\left(-\frac{y^2}{2\theta_3^2}\right) \right\}} \right]. \quad (33)$$

## 6. ENTROPIES

The entropy is used as measure of uncertainty in various applied fields of science and engineering. The various type of entropy such as Shannon's entropy ( $S(y; \mathbf{\Omega})$ ), Rényi entropy ( $L_R(v)$ ) and  $\beta$ -entropy ( $L_\beta(\pi)$ ) are considered in this study. The mathematical expression of these entropies is given below:

$$S(y; \mathbf{\Omega}) = - \int_0^\infty \left[ \left\{ w_1 \frac{y}{\theta_1^2} \exp\left(-\frac{y^2}{2\theta_1^2}\right) + w_2 \frac{y}{\theta_2^2} \exp\left(-\frac{y^2}{2\theta_2^2}\right) + (1 - w_1 - w_2) \frac{y}{\theta_3^2} \exp\left(-\frac{y^2}{2\theta_3^2}\right) \right\} \right. \\ \left. \log \left\{ w_1 \frac{y}{\theta_1^2} \exp\left(-\frac{y^2}{2\theta_1^2}\right) + w_2 \frac{y}{\theta_2^2} \exp\left(-\frac{y^2}{2\theta_2^2}\right) + (1 - w_1 - w_2) \frac{y}{\theta_3^2} \exp\left(-\frac{y^2}{2\theta_3^2}\right) \right\} \right] dy. \quad (34)$$

$$L_R(v) = \frac{1}{(1-v)} \log \left[ \sum_{j=0}^v \sum_{k=0}^j \binom{v}{j} \binom{j}{k} y^v w_1^{v-j} w_2^{j-k} (1 - w_1 - w_2)^k \theta_1^{-2(v-j)} \theta_2^{-2(j-k)} \theta_3^{-2k} \left\{ 2 \left( \frac{1}{2\theta_1^2} (v-j) + \frac{1}{2\theta_2^2} (j-k) + \frac{1}{2\theta_3^2} k \right)^{-1} \right\} \right] \quad (35)$$

where  $v > 0$  and  $v \neq 1$ .

$$L_\beta(\pi) = \frac{1}{(\pi-1)} \left[ \sum_{j=0}^\pi \sum_{k=0}^j \binom{\pi}{j} \binom{j}{k} y^\pi w_1^{\pi-j} w_2^{j-k} (1 - w_1 - w_2)^k \theta_1^{-2(\pi-j)} \theta_2^{-2(j-k)} \theta_3^{-2k} \left\{ 2 \left( \frac{1}{2\theta_1^2} (\pi-j) + \frac{1}{2\theta_2^2} (j-k) + \frac{1}{2\theta_3^2} k \right)^{-1} \right\} \right] \quad (36)$$

where  $\pi > 0$  and  $\pi \neq 1$ .

## 7. DISTRIBUTIONS OF ORDER STATISTICS

In this section, we have provided the different function of order statistics such as  $k^{\text{th}}$ ,  $1^{\text{st}}$  and  $n^{\text{th}}$  for proposed mixture distribution. The mean and variance of considered order statistics are also presented.

## 7.1. The $k^{\text{th}}$ -Order Statistic

The pdf of  $k^{\text{th}}$ -order statistic for proposed mixture distribution is as follows:

$$h(y_{k:n}; \Omega) = \frac{n!}{(k-1)!(n-k)!} \{1 - R(y; \Omega)\}^{k-1} \{R(y; \Omega)\}^{n-k} f(y; \Omega), \quad (37)$$

After substitution in Equation (37), we get

$$\begin{aligned} h(y_{k:n}; \Omega) &= \frac{n!}{(k-1)!(n-k)!} \sum_{j=0}^{k-1} \sum_{l=0}^j \sum_{r=0}^l (-1)^j \binom{k-1}{j} \binom{j}{l} \binom{l}{r} \exp \left\{ -(\psi_{01}^{-2}) \frac{y_k^2}{2} \right\} w_1^{\alpha_{01}-1} w_2^{\beta_{01}-1} (1-w_1-w_2)^{\gamma_{01}-1} \\ &\quad \sum_{e=0}^{n-k} \sum_{f=0}^e \binom{n-k}{e} \binom{e}{f} \exp \left\{ -(\psi_{02}^{-2}) \frac{y_k^2}{2} \right\} w_1^{\alpha_{02}-1} w_2^{\beta_{02}-1} (1-w_1-w_2)^{\gamma_{02}-1} \\ &\quad \left\{ w_1 \frac{y_k}{\theta_1^2} \exp \left( -\frac{y_k^2}{2\theta_1^2} \right) + w_2 \frac{y_k}{\theta_2^2} \exp \left( -\frac{y_k^2}{2\theta_2^2} \right) + (1-w_1-w_2) \frac{y_k}{\theta_3^2} \exp \left( -\frac{y_k^2}{2\theta_3^2} \right) \right\}. \end{aligned} \quad (38)$$

where

$$\begin{aligned} \psi_{01}^{-2} &= \frac{1}{\theta_1^2} (j-l) + \frac{1}{\theta_2^2} (l-r) + \frac{1}{\theta_3^2} (r), \alpha_{01} = j-l+1, \beta_{01} = l-r+1, \gamma_{01} = l+1, \\ \psi_{02}^{-2} &= \frac{1}{\theta_1^2} (n-k-e) + \frac{1}{\theta_2^2} (e-f) + \frac{1}{\theta_3^2} (f), \alpha_{02} = n-k-e+1, \beta_{02} = e-f+1, \gamma_{02} = f+1. \end{aligned}$$

## 7.2. The 1<sup>st</sup>-Order Statistic

Placing  $k = 1$  in Equation (38), we have obtained the pdf of 1<sup>st</sup>-order statistic for proposed mixture distribution as follows:

$$h(y_{1:n}; \Omega) = ny_1 \left[ \sum_{g=1}^3 \frac{1}{\theta_g^2} \sum_{e=0}^{n-1} \sum_{f=0}^e \binom{n-1}{e} \binom{e}{f} \exp \left\{ -(\psi_{1g}^{-2}) \frac{y_1^2}{2} \right\} w_1^{\alpha_{1g}-1} w_2^{\beta_{1g}-1} (1-w_1-w_2)^{\gamma_{1g}-1} \right]. \quad (39)$$

where

$$\begin{aligned} \psi_{11}^{-2} &= \frac{1}{\theta_1^2} (n-e) + \frac{1}{\theta_2^2} (e-f) + \frac{1}{\theta_3^2} (f), \alpha_{11} = n+1-e, \beta_{11} = e-f+1, \gamma_{11} = f+1, \\ \psi_{12}^{-2} &= \frac{1}{\theta_1^2} (n-1-e) + \frac{1}{\theta_2^2} (e-f+1) + \frac{1}{\theta_3^2} (f), \alpha_{12} = n-e, \beta_{12} = e-f+2, \gamma_{12} = f+1, \\ \psi_{13}^{-2} &= \frac{1}{\theta_1^2} (n-1-e) + \frac{1}{\theta_2^2} (e-f) + \frac{1}{\theta_3^2} (f+1), \alpha_{13} = n-e, \beta_{13} = e-f+1, \gamma_{13} = f+2. \end{aligned}$$

## 7.3. The $n^{\text{th}}$ -Order Statistic

Replacing  $k = n$  in Equation (38), we have found the pdf of  $n^{\text{th}}$ -order statistic for proposed mixture distribution as follows:

$$h(y_{n:n}; \Omega) = ny_n \left[ \sum_{g=1}^3 \frac{1}{\theta_g^2} \sum_{j=0}^{n-1} \sum_{l=0}^j \sum_{r=0}^l (-1)^j \binom{n-1}{j} \binom{j}{l} \binom{l}{r} \exp \left\{ -(\psi_{2g}^{-2}) \frac{y_n^2}{2} \right\} w_1^{\alpha_{2g}-1} w_2^{\beta_{2g}-1} (1-w_1-w_2)^{\gamma_{2g}-1} \right], \quad (40)$$

where

$$\begin{aligned} \psi_{21}^{-2} &= \frac{1}{\theta_1^2} (j-l+1) + \frac{1}{\theta_2^2} (l-r) + \frac{1}{\theta_3^2} (r), \alpha_{21} = j-l+2, \beta_{21} = l-r+1, \gamma_{21} = r+1, \\ \psi_{22}^{-2} &= \frac{1}{\theta_1^2} (j-l) + \frac{1}{\theta_2^2} (l-r+1) + \frac{1}{\theta_3^2} (r), \alpha_{22} = j-l+1, \beta_{22} = l-r+2, \gamma_{22} = r+1, \\ \psi_{23}^{-2} &= \frac{1}{\theta_1^2} (j-l) + \frac{1}{\theta_2^2} (l-r) + \frac{1}{\theta_3^2} (r+1), \alpha_{23} = j-l+1, \beta_{23} = l-r+1, \gamma_{23} = r+2. \end{aligned}$$

## 7.4. $r^{\text{th}}$ Moments about Origin of 1<sup>st</sup>- Order Statistic

The  $r^{\text{th}}$  moment of 1<sup>st</sup>-order statistic about the origin is obtained as follows:

$$E(Y_1^r) = n \left[ \sum_{g=1}^3 \frac{1}{\theta_g^2} \sum_{e=0}^{n-1} \sum_{f=0}^e \binom{n-1}{e} \binom{e}{f} w_1^{\alpha_{1g-1}} w_2^{\beta_{1g-1}} (1-w_1-w_2)^{\gamma_{1g-1}} 2^{\left(\frac{r-1}{2}\right)} \left| \frac{r-1}{2} + 1 \right| (\psi_{1w}^{-2})^{\frac{r-1}{2}+1} \right]. \quad (41)$$

The mean and variance of 1<sup>st</sup>-order statistic are obtained as follows:

$$E(Y_1) = n \left[ \sum_{g=1}^3 \frac{1}{\theta_g^2} \sum_{e=0}^{n-1} \sum_{f=0}^e \binom{n-1}{e} \binom{e}{f} w_1^{\alpha_{1g-1}} w_2^{\beta_{1g-1}} (1-w_1-w_2)^{\gamma_{1g-1}} (\psi_{1g}^{-2}) \right]. \quad (42)$$

$$\begin{aligned} \text{Var}(Y_1) = n & \left[ \sqrt{\frac{\pi}{2}} \sum_{g=1}^3 \frac{1}{\theta_g^2} \sum_{e=0}^{n-1} \sum_{f=0}^e \binom{n-1}{e} \binom{e}{f} w_1^{\alpha_{1g-1}} w_2^{\beta_{1g-1}} (1-w_1-w_2)^{\gamma_{1g-1}} (\psi_{1g}^{-2})^{\left(\frac{3}{2}\right)} \right] \\ & - \left[ n \left\{ \sum_{g=1}^3 \frac{1}{\theta_g^2} \sum_{e=0}^{n-1} \sum_{f=0}^e \binom{n-1}{e} \binom{e}{f} w_1^{\alpha_{1g-1}} w_2^{\beta_{1g-1}} (1-w_1-w_2)^{\gamma_{1g-1}} (\psi_{1g}^{-2}) \right\} \right]^2. \end{aligned} \quad (43)$$

## 7.5. $r^{\text{th}}$ Moments about Origin of $n^{\text{th}}$ - Order Statistic

The  $r^{\text{th}}$  moment of  $n^{\text{th}}$ -order statistic about the origin is obtained as follows:

$$E(Y_n^r) = n \left[ \sum_{g=1}^3 \frac{1}{\theta_g^2} \sum_{j=0}^{n-1} \sum_{l=0}^j \sum_{r=0}^l (-1)^j \binom{n-1}{j} \binom{j}{l} \binom{l}{r} w_1^{\alpha_{2g-1}} w_2^{\beta_{2g-1}} (1-w_1-w_2)^{\gamma_{2g-1}} 2^{\left(\frac{r-1}{2}\right)} \left| \frac{r-1}{2} + 1 \right| (\psi_{1w}^{-2})^{\frac{r-1}{2}+1} \right]. \quad (44)$$

The mean and variance of the  $n^{\text{th}}$  order statistic are obtained as follows:

$$E(Y_n) = n \left[ \sum_{g=1}^3 \frac{1}{\theta_g^2} \sum_{j=0}^{n-1} \sum_{l=0}^j \sum_{r=0}^l (-1)^j \binom{n-1}{j} \binom{j}{l} \binom{l}{r} w_1^{\alpha_{2g-1}} w_2^{\beta_{2g-1}} (1-w_1-w_2)^{\gamma_{2g-1}} (\psi_{2g}^{-2}) \right]. \quad (45)$$

$$\begin{aligned} \text{Var}(Y_n) = n & \left[ \sqrt{\frac{\pi}{2}} \sum_{g=1}^3 \frac{1}{\theta_g^2} \sum_{j=0}^{n-1} \sum_{l=0}^j \sum_{r=0}^l (-1)^j \binom{n-1}{j} \binom{j}{l} \binom{l}{r} w_1^{\alpha_{2g-1}} w_2^{\beta_{2g-1}} (1-w_1-w_2)^{\gamma_{2g-1}} (\psi_{2g}^{-2})^{\left(\frac{3}{2}\right)} \right] \\ & - \left[ n \left\{ \sum_{g=1}^3 \frac{1}{\theta_g^2} \sum_{j=0}^{n-1} \sum_{l=0}^j \sum_{r=0}^l (-1)^j \binom{n-1}{j} \binom{j}{l} \binom{l}{r} w_1^{\alpha_{2g-1}} w_2^{\beta_{2g-1}} (1-w_1-w_2)^{\gamma_{2g-1}} (\psi_{2g}^{-2}) \right\} \right]^2. \end{aligned} \quad (46)$$

## 8. ESTIMATION OF PARAMETERS

The ML method is used to estimate the unknown parameters of the proposed mixture distribution under type-I censored and complete sampling situations. The ML estimation under type-I censored and complete sampling situations are given below:

### 8.1. The Likelihood Function

The likelihood function for suggested mixture distribution under type-I censoring is written as follows:

$$L(\Omega|y) \propto \left\{ \prod_{h=1}^{s_1} w_1 f_1(y_{1h}) \right\} \left\{ \prod_{h=1}^{s_2} w_2 f_2(y_{2h}) \right\} \left\{ \prod_{h=1}^{s_3} (1-w_1-w_2) f_3(y_{3h}) \right\} \{R(t)\}^{n-s}, \quad (47)$$

where  $t$  represents the test termination time,  $s_1$ ,  $s_2$  and  $s_3$  denote the failures which are belonging to the subpopulations I, II and III,  $s = s_1 + s_2 + s_3$  indicates the uncensored observation and  $n - s$  denotes the censored observation.

After simplifying the Equation (47), we get

$$L(\mathbf{\Omega}|y) \propto \left\{ \frac{w_1^{s_1} w_2^{s_2} (1 - w_1 - w_2)^{s_3}}{\theta_1^{2s_1} \theta_2^{2s_2} \theta_3^{2s_3}} \exp\left(-\frac{1}{2\theta_1^2} \sum_{h=1}^{s_1} y_{1h}^2\right) \exp\left(-\frac{1}{2\theta_2^2} \sum_{h=1}^{s_2} y_{2h}^2\right) \exp\left(-\frac{1}{2\theta_3^2} \sum_{h=1}^{s_3} y_{3h}^2\right) \right\} \\ \left\{ w_1 \exp\left(-\frac{t^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{t^2}{2\theta_2^2}\right) + (1 - w_1 - w_2) \exp\left(-\frac{t^2}{2\theta_3^2}\right) \right\}^{n-s}. \quad (48)$$

## 8.2. ML Estimators and Their Variances for Censored Data

The ML estimators for  $\mathbf{\Omega} = (\theta_1, \theta_2, \theta_3, w_1, w_2)$  of a 3-component mixture of Rayleigh distributions are obtained by solving the following nonlinear system of Equations (49–53).

$$\frac{\partial \ln L(\mathbf{\Omega}|y)}{\partial \theta_1} = -\frac{2s_1}{\theta_1} + \frac{\sum_{h=1}^{s_1} y_{1h}^2}{\theta_1^3} - \frac{(n-s) w_1 t^2 \exp\left(-\frac{t^2}{2\theta_1^2}\right)}{\theta_1^3 \left\{ w_1 \exp\left(-\frac{t^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{t^2}{2\theta_2^2}\right) + (1 - w_1 - w_2) \exp\left(-\frac{t^2}{2\theta_3^2}\right) \right\}} = 0. \quad (49)$$

$$\frac{\partial \ln L(\mathbf{\Omega}|y)}{\partial \theta_2} = -\frac{2s_2}{\theta_2} + \frac{\sum_{h=1}^{s_2} y_{2h}^2}{\theta_2^3} - \frac{(n-s) w_2 t^2 \exp\left(-\frac{t^2}{2\theta_2^2}\right)}{\theta_2^3 \left\{ w_1 \exp\left(-\frac{t^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{t^2}{2\theta_2^2}\right) + (1 - w_1 - w_2) \exp\left(-\frac{t^2}{2\theta_3^2}\right) \right\}} = 0. \quad (50)$$

$$\frac{\partial \ln L(\mathbf{\Omega}|y)}{\partial \theta_3} = -\frac{2s_3}{\theta_3} + \frac{\sum_{h=1}^{s_3} y_{3h}^2}{\theta_3^3} - \frac{(n-s) (1 - w_1 - w_2) t^2 \exp\left(-\frac{t^2}{2\theta_3^2}\right)}{\theta_3^3 \left\{ w_1 \exp\left(-\frac{t^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{t^2}{2\theta_2^2}\right) + (1 - w_1 - w_2) \exp\left(-\frac{t^2}{2\theta_3^2}\right) \right\}} = 0. \quad (51)$$

$$\frac{\partial \ln L(\mathbf{\Omega}|y)}{\partial w_1} = \frac{s_1}{w_1} - \frac{s_3}{(1 - w_1 - w_2)} + \frac{(n-s) \left\{ \exp\left(-\frac{t^2}{2\theta_1^2}\right) - \exp\left(-\frac{t^2}{2\theta_3^2}\right) \right\}}{\left\{ w_1 \exp\left(-\frac{t^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{t^2}{2\theta_2^2}\right) + (1 - w_1 - w_2) \exp\left(-\frac{t^2}{2\theta_3^2}\right) \right\}} = 0. \quad (52)$$

$$\frac{\partial \ln L(\mathbf{\Omega}|y)}{\partial w_2} = \frac{s_2}{w_2} - \frac{s_3}{(1 - w_1 - w_2)} + \frac{(n-s) \left\{ \exp\left(-\frac{t^2}{2\theta_2^2}\right) - \exp\left(-\frac{t^2}{2\theta_3^2}\right) \right\}}{\left\{ w_1 \exp\left(-\frac{t^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{t^2}{2\theta_2^2}\right) + (1 - w_1 - w_2) \exp\left(-\frac{t^2}{2\theta_3^2}\right) \right\}} = 0. \quad (53)$$

It is tough to find out the closed form for the ML estimators. The normal equations do not have explicit solutions and they have to be obtained numerically. The Mathematica (Wolfram [10]) software is used to find the ML estimates (MLEs) of  $\theta_1, \theta_2, \theta_3, w_1$  and  $w_2$ .

The simplest large sample approach is to assume that the MLE  $(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2)$  are approximately multivariate normal distribution with mean  $\mathbf{\Omega} = (\theta_1, \theta_2, \theta_3, w_1, w_2)$  and covariance matrix  $I^{-1}(\mathbf{\Omega})$ , where  $I^{-1}(\mathbf{\Omega})$  is the inverse of the observed information matrix. Therefore, ML variances (MLVs) are on the main diagonal of an inverted information matrix as given below:

$$I^{-1} = -E \begin{pmatrix} \frac{\partial^2 l}{\partial \theta_1^2} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial \theta_1 \partial \theta_2} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial \theta_1 \partial \theta_3} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial \theta_1 \partial w_1} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial \theta_1 \partial w_2} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} \\ \frac{\partial^2 l}{\partial \theta_2 \partial \theta_1} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial \theta_2^2} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial \theta_2 \partial \theta_3} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial \theta_2 \partial w_1} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial \theta_2 \partial w_2} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} \\ \frac{\partial^2 l}{\partial \theta_3 \partial \theta_1} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial \theta_3 \partial \theta_2} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial \theta_3^2} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial \theta_3 \partial w_1} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial \theta_3 \partial w_2} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} \\ \frac{\partial^2 l}{\partial w_1 \partial \theta_1} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial w_1 \partial \theta_2} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial w_1 \partial \theta_3} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial w_1^2} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial w_1 \partial w_2} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} \\ \frac{\partial^2 l}{\partial w_2 \partial \theta_1} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial w_2 \partial \theta_2} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial w_2 \partial \theta_3} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial w_2 \partial w_1} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} & \frac{\partial^2 l}{\partial w_2^2} \Big|_{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{w}_1, \hat{w}_2} \end{pmatrix}^{-1}$$

$$I^{-1} = \begin{pmatrix} \text{Var} \left( \hat{\theta}_1 \right) & \text{Cov} \left( \hat{\theta}_1, \hat{\theta}_2 \right) & \text{Cov} \left( \hat{\theta}_1, \hat{\theta}_3 \right) & \text{Cov} \left( \hat{\theta}_1, \hat{w}_1 \right) & \text{Cov} \left( \hat{\theta}_1, \hat{w}_2 \right) \\ \text{Cov} \left( \hat{\theta}_2, \hat{\theta}_1 \right) & \text{Var} \left( \hat{\theta}_2 \right) & \text{Cov} \left( \hat{\theta}_2, \hat{\theta}_3 \right) & \text{Cov} \left( \hat{\theta}_2, \hat{w}_1 \right) & \text{Cov} \left( \hat{\theta}_2, \hat{w}_2 \right) \\ \text{Cov} \left( \hat{\theta}_3, \hat{\theta}_1 \right) & \text{Cov} \left( \hat{\theta}_3, \hat{\theta}_2 \right) & \text{Var} \left( \hat{\theta}_3 \right) & \text{Cov} \left( \hat{\theta}_3, \hat{w}_1 \right) & \text{Cov} \left( \hat{\theta}_3, \hat{w}_2 \right) \\ \text{Cov} \left( \hat{w}_1, \hat{\theta}_1 \right) & \text{Cov} \left( \hat{w}_1, \hat{\theta}_2 \right) & \text{Cov} \left( \hat{w}_1, \hat{\theta}_3 \right) & \text{Var} \left( \hat{w}_1 \right) & \text{Cov} \left( \hat{w}_1, \hat{w}_2 \right) \\ \text{Cov} \left( \hat{w}_2, \hat{\theta}_1 \right) & \text{Cov} \left( \hat{w}_2, \hat{\theta}_2 \right) & \text{Cov} \left( \hat{w}_2, \hat{\theta}_3 \right) & \text{Cov} \left( \hat{w}_2, \hat{w}_1 \right) & \text{Var} \left( \hat{w}_2 \right) \end{pmatrix}$$

where

$$\frac{\partial^2 \ln L(\boldsymbol{\Omega}|y)}{\partial \theta_1^2} = \frac{2s_1}{\theta_1^2} - \frac{3 \sum_{h=1}^{s_1} y_{1h}^2}{\theta_1^4} + \frac{3(n-s) w_1 t^4 \exp\left(-\frac{t^2}{2\theta_1^2}\right) \left\{ w_2 \exp\left(-\frac{t^2}{2\theta_2^2}\right) + (1-w_1-w_2) \exp\left(-\frac{t^2}{2\theta_3^2}\right) \right\}}{\theta_1^7 \left\{ w_1 \exp\left(-\frac{t^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{t^2}{2\theta_2^2}\right) + (1-w_1-w_2) \exp\left(-\frac{t^2}{2\theta_3^2}\right) \right\}^2}.$$

$$\frac{\partial^2 \ln L(\boldsymbol{\Omega}|y)}{\partial \theta_2^2} = \frac{2s_2}{\theta_2^2} - \frac{3 \sum_{h=1}^{s_2} y_{2h}^2}{\theta_2^4} + \frac{3(n-s) w_2 t^4 \exp\left(-\frac{t^2}{2\theta_2^2}\right) \left\{ w_1 \exp\left(-\frac{t^2}{2\theta_1^2}\right) + (1-w_1-w_2) \exp\left(-\frac{t^2}{2\theta_3^2}\right) \right\}}{\theta_2^7 \left\{ w_1 \exp\left(-\frac{t^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{t^2}{2\theta_2^2}\right) + (1-w_1-w_2) \exp\left(-\frac{t^2}{2\theta_3^2}\right) \right\}^2}.$$

$$\frac{\partial^2 \ln L(\boldsymbol{\Omega}|y)}{\partial \theta_3^2} = \frac{2s_3}{\theta_3^2} - \frac{3 \sum_{h=1}^{s_3} y_{3h}^2}{\theta_3^4} + \frac{3(n-s) (1-w_1-w_2) t^4 \exp\left(-\frac{t^2}{2\theta_3^2}\right) \left\{ w_1 \exp\left(-\frac{t^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{t^2}{2\theta_2^2}\right) \right\}}{\theta_3^7 \left\{ w_1 \exp\left(-\frac{t^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{t^2}{2\theta_2^2}\right) + (1-w_1-w_2) \exp\left(-\frac{t^2}{2\theta_3^2}\right) \right\}^2}.$$

$$\frac{\partial^2 \ln L(\boldsymbol{\Omega}|y)}{\partial w_1^2} = -\frac{s_1}{w_1^2} - \frac{s_3}{(1-w_1-w_2)^2} + \frac{(n-s) \left\{ \exp\left(-\frac{t^2}{2\theta_1^2}\right) - \exp\left(-\frac{t^2}{2\theta_3^2}\right) \right\}}{\left\{ w_1 \exp\left(-\frac{t^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{t^2}{2\theta_2^2}\right) + (1-w_1-w_2) \exp\left(-\frac{t^2}{2\theta_3^2}\right) \right\}^2}.$$

$$\frac{\partial^2 \ln L(\boldsymbol{\Omega}|y)}{\partial w_2^2} = -\frac{s_2}{w_2^2} - \frac{s_3}{(1-w_1-w_2)^2} + \frac{(n-s) \left\{ \exp\left(-\frac{t^2}{2\theta_2^2}\right) - \exp\left(-\frac{t^2}{2\theta_3^2}\right) \right\}}{\left\{ w_1 \exp\left(-\frac{t^2}{2\theta_1^2}\right) + w_2 \exp\left(-\frac{t^2}{2\theta_2^2}\right) + (1-w_1-w_2) \exp\left(-\frac{t^2}{2\theta_3^2}\right) \right\}^2}.$$

### 8.3. ML Estimators and their Variances for Complete Data

All the censored observations become uncensored if  $t$  approaches to  $\infty$  and  $s_l$  approaches to  $n_l$  ( $l = 1, 2, 3$ ). The proficiency of the ML estimators is increased due to the fact that in complete data all the observations are merged in our data. The mathematical expressions of the ML estimators and as well as their variances expression based on complete data are shown below:

$$\begin{aligned} \hat{\theta}_1 &= \sqrt{\frac{\sum_{h=1}^{n_1} y_{1h}^2}{2n_1}}, \hat{\theta}_2 = \sqrt{\frac{\sum_{h=1}^{n_2} y_{2h}^2}{2n_2}}, \hat{\theta}_3 = \sqrt{\frac{\sum_{h=1}^{n_3} y_{3h}^2}{2n_3}}, \hat{w}_1 = \frac{n_1}{n}, \hat{w}_2 = \frac{n_2}{n}, \\ \text{Var} \left( \hat{\theta}_1 \right) &= \frac{\sum_{h=1}^{n_1} y_{1h}^2}{8n_1^2}, \text{Var} \left( \hat{\theta}_2 \right) = \frac{\sum_{h=1}^{n_2} y_{2h}^2}{8n_2^2}, \text{Var} \left( \hat{\theta}_3 \right) = \frac{\sum_{h=1}^{n_3} y_{3h}^2}{8n_3^2}, \\ \text{Var} \left( \hat{w}_1 \right) &= \frac{n_1 n_3}{n^2 (n_1 + n_3)}, \text{Var} \left( \hat{w}_2 \right) = \frac{n_2 n_3}{n^2 (n_2 + n_3)}. \end{aligned}$$



## 9. MONTE CARLO SIMULATION

In this section, we have used Monte Carlo simulation to find the MLEs and MLVs for the proposed mixture distribution. Through the following steps, we obtained the MLEs and MLVs, as follows:

1. First we generate  $w_1 n$ ,  $w_2 n$  and  $w_3 n$  observation from  $f_1(y; \theta_1)$ ,  $f_2(y; \theta_2)$  and  $f_3(y; \theta_3)$ , respectively.
2. A censored sample is selected at a fixed  $t$  and observations which are greater than  $t$  will be considered as censored observations. When generated an uncensored (complete) sample then this step is neglected.
3. The Steps 1 and 2 are repeated 1000 times for all selected choices of parameters.
4. To find the MLEs and MLVs of  $\theta_1, \theta_2, \theta_3, w_1$  and  $w_2$  based on the samples obtained in Step 3.

The abovementioned Steps 1–4 used for different sample size  $n = 50, 100, 200, 500$ , parameters values  $(\theta_1, \theta_2, \theta_3, w_1, w_2) = \{(13, 14, 15, 0.3, 0.5), (16, 15, 14, 0.5, 0.3)\}$  and test termination time  $t = 24, 32$ . In such condition, the choice of  $t = 24, 32$  was done to have 10% to 20% censored rate in the resulting sample.

From Tables 2 and 3, if  $\theta_1 > \theta_2 > \theta_3$  and  $w_1 > w_2$ , it is observed that parameters  $\theta_2, \theta_3$  and  $w_1$  are over-estimated but  $\theta_1$  and  $w_2$  are under-estimated at different values of  $n$  and  $t$ . On the other hand, if  $\theta_1 < \theta_2 < \theta_3$  and  $w_1 < w_2$ , it is pointed that parameters  $\theta_2, \theta_3$  and  $w_2$  are over-estimated whereas  $\theta_1$  and  $w_1$  are under-estimated at various values of  $n$  and  $t$ . Also, the amount of over-estimation of  $\theta_1, \theta_2, \theta_3, w_1$  and  $w_2$  is smaller for a large  $n$  at different values of  $t$ , and an opposite behavior was noticed for a small  $t$  at a given value of  $n$ . Moreover, the parameters  $\theta_1, \theta_2, \theta_3, w_1$  and  $w_2$  were pointed under-estimated to a smaller degree when the true values of  $\theta_1, \theta_2, \theta_3$  were larger at various choices of  $t$  for a fixed  $n$ . The difference of the MLEs of parameters  $\theta_1, \theta_2, \theta_3, w_1$  and  $w_2$  from the nominal values becomes the least with the increase of  $n$  at a fixed  $t$ .

From Tables 4 and 5, it is revealed that the  $\theta_1, \theta_2$  and  $\theta_3$  are under-estimated as the value of  $t \rightarrow \infty$ . The degree of under-estimation of  $\theta_1, \theta_2$  and  $\theta_3$  is lesser for larger values of  $n$ . The degree of under-estimation or over-estimation of  $\theta_1, \theta_2$  and  $\theta_3$  is greater for censored data than the complete data at various choices of  $n$ . It is also revealed that for  $t \rightarrow \infty$  the difference in the values of MLEs and assumed values of parameters will be reduced with the increase in  $n$ .

## 10. APPLICATION BASED ON REAL DATA SET

In this section, we present the real-life application of the proposed mixture distribution based on three components, i.e., Transmitter Tube (V805), Transmitter Tube and Indicator Tube (V600) related to the aircraft (cf. Davis [11]). Davis [11] showed that the distribution of these components is exponential. For an exponential random data ( $x$ ), the suitable transformation  $y = \sqrt{2}x$  gives the Rayleigh random data

**Table 2** The MLEs values of the proposed mixture distribution for  $\theta_1 = 13, \theta_2 = 14, \theta_3 = 15$ ,  $w_1 = 0.3, w_2 = 0.5$  and  $t = 24, 32$  under censored data.

$t$	$n$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{w}_1$	$\hat{w}_2$
24	50	11.80089	14.61541	15.01973	0.311001	0.450384
24	100	12.14090	14.47142	15.79121	0.309475	0.459040
24	200	12.32481	14.41900	15.71900	0.306542	0.471284
24	500	12.64047	14.30591	15.49235	0.305908	0.483244
32	50	12.30870	14.50639	15.28540	0.307408	0.473077
32	100	12.572412	14.32247	15.64205	0.305157	0.482970
32	200	12.70826	14.24910	15.39890	0.303980	0.489530
32	500	12.81997	14.11887	15.19401	0.302209	0.496001

**Table 3** The MLEs values of the proposed mixture distribution for  $\theta_1 = 16, \theta_2 = 15, \theta_3 = 14$ ,  $w_1 = 0.5, w_2 = 0.3$  and  $t = 24, 32$  under censored data.

$t$	$n$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{w}_1$	$\hat{w}_2$
24	50	14.95881	15.60520	14.95971	0.442387	0.314019
24	100	15.20851	15.47148	14.84121	0.459401	0.311701
24	200	15.32170	15.40986	14.55412	0.470451	0.310951
24	500	15.71638	15.29501	14.42015	0.482201	0.308541
32	50	15.41810	15.55680	14.21510	0.461050	0.309451
32	100	15.54207	15.39238	14.74209	0.485094	0.306105
32	200	15.70844	15.25910	14.43770	0.489451	0.304404
32	500	15.85159	15.13786	14.21241	0.495001	0.302199

**Table 4** The MLEs values of the proposed mixture distribution for  $\theta_1 = 13, \theta_2 = 14, \theta_3 = 15$ ,  $w_1 = 0.3, w_2 = 0.5$  under complete data.

$n$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{w}_1$	$\hat{w}_2$
50	12.88630	13.89810	14.68280	0.30000	0.50000
100	12.90350	13.93310	14.90500	0.30000	0.50000
200	12.96220	13.97070	14.94290	0.30000	0.50000
500	12.96660	13.99730	14.99860	0.30000	0.50000

**Table 5** The MLEs values of the proposed mixture distribution for  $\theta_1 = 16, \theta_2 = 15, \theta_3 = 14$ ,  $w_1 = 0.5, w_2 = 0.3$  under complete data.

$n$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{w}_1$	$\hat{w}_2$
50	15.91150	14.93810	13.87220	0.50000	0.30000
100	15.94880	14.95030	13.92910	0.50000	0.30000
200	15.95560	14.97220	13.98150	0.50000	0.30000
500	15.98630	14.98730	13.99820	0.50000	0.30000

**Table 6** MLEs and MLVs using lifetime mixture under censored and complete data.

	$t$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{w}_1$	$\hat{w}_2$
MLEs	600	12.87401	11.73240	16.62847	0.642813	0.240017
MLVs	600	0.054882	0.116176	0.809115	0.000055	0.000048
ML estimates	$\infty$	13.36638	12.22623	17.24819	0.673880	0.251492
Variances	$\infty$	0.049463	0.110891	0.743750	0.000050	0.000043

(y). Thus, the suggested mixture distribution can be a fair choice to model the abovementioned mixture lifetime data. In addition, it is not known that which component of an aircraft radar set fails until the condition of disappointment of a set of radar arises before or at  $t = 600$  hours. The summary of the data for  $t = 600$  is as follows:

$$n = 1340, s_1 = 866, s_2 = 337, s_3 = 83, s = s_1 + s_2 + s_3 = 1286, n - s = 54,$$

$$\sum_{k=1}^{s_1} y_{1k}^2 = 2 \sum_{k=1}^{s_1} x_{1k} = 268160, \sum_{k=1}^{s_2} y_{2k}^2 = 2 \sum_{k=1}^{s_2} x_{2k} = 100750, \sum_{k=1}^{s_3} y_{3k}^2 = 2 \sum_{k=1}^{s_3} x_{3k} = 32500.$$

when  $t \rightarrow \infty$ , then the summary of a complete data set is as follows:

$$n = 1340, n_1 = 903, n_2 = 337, n_3 = 100, n = n_1 + n_2 + n_3 = 1340,$$

$$\sum_{k=1}^{n_1} y_{1k}^2 = 2 \sum_{k=1}^{n_1} x_{1k} = 322660, \sum_{k=1}^{n_2} y_{2k}^2 = 2 \sum_{k=1}^{n_2} x_{2k} = 100750, \sum_{k=1}^{n_3} y_{3k}^2 = 2 \sum_{k=1}^{n_3} x_{3k} = 59500.$$

The MLEs and MLVs are shown in Table 6.

From Table 6, it is observed that the ML estimators based on complete data are more efficient than the ML estimator using censored data due to the lesser values of MLVs.

## 11. CONCLUSION

In this study, we have discussed some basic statistical properties, various statistical functions, some important entropies and different order statistics for the proposed 3-component mixture of Rayleigh distributions. A Monte Carlo simulation is used to evaluate the performance of the unknown parameter based on the ML estimator under censored and uncensored sampling schemes. To explain a practical application of the proposed mixture model, a real-life example has also been analyzed.

From simulated results, it has been observed that an increase in  $t$  under a fixed  $n$  yield very efficient ML estimators and vice versa. It is also noticed that the parameters are over-estimated to a small degree with relatively larger  $n$ . However, the degree of over-estimation (under-estimation) of parameters is smaller for a relatively large parameter value and vice versa. Finally, it is concluded that the results are more efficient under complete data as compared to censored data due to associated least MLVs.

## CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

## AUTHORS' CONTRIBUTIONS

Conceptualization: Muhammad Tahir, Muhammad Abid; Formal analysis: Muhammad Abid, Sidra Mohsin; Methodology: Muhammad Tahir; Original draft: Muhammad Tahir; Review & Editing: Muhammad Abid, Mohammad Ahsanullah.

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