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Research Article

Harmonically Convex Fuzzy-Interval-Valued Functions and Fuzzy-Interval Riemann–Liouville Fractional Integral Inequalities

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ABSTRACT

It is well known that the concept of convexity establishes strong relationship with integral inequality for single-valued and interval-valued function. The single-valued function and interval-valued function both are special cases of fuzzy interval-valued function. The aim of this paper is to introduce a new class of convex fuzzy interval-valued functions, which is called harmonically convex fuzzy interval-valued functions (harmonically convex fuzzy-IVFs) by means of fuzzy order relation and to investigate this new class via fuzzy-interval Riemann–Liouville fractional operator. With the help of fuzzy order relation and fuzzy-interval Riemann–Liouville fractional, we derive some integrals inequalities of Hermite–Hadamard (H-H) type and Hermite–Hadamard–Fejér (H-H Fejér) type as well as some product inequities for harmonically convex fuzzy-IVFs. Our results represent a significant improvement and refinement of the known results. We hope that these interesting outcomes may open a new direction for fuzzy optimization, modeling and interval-valued function.

1. INTRODUCTION

The Hermite–Hadamard (H-H) inequality was firstly introduced by Hadamard [1] and Hermite [2] for convex functions. This inequality is used as a most useful tool in mathematical analysis and optimization because convex functions establish strong relationship with H-H inequality. Therefore, many authors have discussed the relation of H-H inequality with different kinds of convex and nonconvex functions and many papers have provided refinements, generalizations and extensions, see [3–8]. Besides, fractional integrals have played a critical role in different branches of sciences. It is also a familiar fact that inequalities have become a very popular method using fractional integrals, and that this method has been the driving force behind many studies in recent years. Many forms of inequality have been studied, resulting in the introduction of new trend in inequality theory. Firstly, by using fractional integrals, Sarikaya *et al.* [9] discovered fractional H-H inequality for classical convex function. After that, many scholars devoted their efforts to present fractional H-H type inequalities for different classes of convex and nonconvex functions see [10–15].

It is well known that interval analysis provides tools to deal with data uncertainty. In general, interval analysis is typically used to deal with the models whose data are composed of inaccuracies that may occur from certain kinds of measurements. In 1966, the concept of interval analysis was firstly introduced by late American mathematician Ramon E. Moore in [16]. Since its inception, various authors in the mathematical community have paid close attention to this area of research. Interval analysis has been found to be useful in global optimization and constraint solution algorithms, according to experts. It has slowly risen in popularity over the last few decades. Scientists and engineers engaged in scientific computation have discovered that interval analysis is useful, especially in terms of accuracy, round-off error affects and automatic validation of results. After the invention of interval analysis, the researchers working in the area of inequalities wants to know whether the inequalities in abovementioned results can be found substituted with inclusions relation. In certain cases, the question is answered correctively. Recently, through interval Riemann integral, interval Riemann–Liouville fractional integrals and fuzzy Riemann integral, several authors presented new versions of various inequalities for interval and fuzzy-interval-valued functions like, as one can see Costa [17], Costa and Roman-Flores [18], Roman-Flores *et al.* [19,20], and Chalco-Cano *et al.* [21,22], An *et al.* [23], Zhao *et al.* [24], but also to more general set-valued maps by Nikodem *et al.* [25], Matkowski and Nikodem [26]. In particular, Zhang *et al.* [27] derived the new version of Jensen's inequalities for set-valued and fuzzy set-valued

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functions by means of a pseudo order relation and proved that these Jensen's inequalities generalized form of Costa Jensen's inequalities [17]. As a further extension, more and more, H-H type inequalities have been obtained through interval Riemann–Liouville fractional integrals, see for convex-IVFs [28,29], for harmonically convex-IVFs [30]. Moreover, recently, Khan *et al.* [31] introduced the new class of convex fuzzy mappings is known as (h_1, h_2)-convex fuzzy-IVFs by means fuzzy order relation and presented the following new version of H-H-type inequality for (h_1, h_2)-convex fuzzy-IVF involving fuzzy-interval Riemann integrals:

Let $\tilde{\mathcal{Q}} : [u, v] \to \mathbb{F}_0$ be a (h_1, h_2)-convex fuzzy-IVF with $h_1, h_2 : [0,1] \to \mathbb{R}^+$ and $h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \neq 0$, whose β-levels define the family of IVFs $\mathcal{Q}_\beta : [u, v] \subset \mathbb{R} \to \mathcal{K}_C^+$ are given by $\mathcal{Q}_\beta(x) = \left[\mathcal{Q}_*(x, \beta), \mathcal{Q}^*(x, \beta)\right]$ for all $x \in [u, v]$, $\beta \in [0,1]$. If $\tilde{\mathcal{Q}}$ is fuzzy-interval Riemann integrable (in sort, FR-integrable), then

$$\begin{aligned} &\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\tilde{\mathcal{Q}}\left(\frac{u+v}{2}\right) \\ &\preccurlyeq \frac{1}{v-u}(FR)\int_u^v \tilde{\mathcal{Q}}(x)\,dx \\ &\preccurlyeq \left[\tilde{\mathcal{Q}}(u) \tilde{+} \tilde{\mathcal{Q}}(v)\right]\int_0^1 h_1(\varrho)\,h_2(1-\varrho)\,d\varrho. \end{aligned} \tag{1}$$

If $h_1(\varrho) = \varrho$ and $h_2(\varrho) \equiv 1$, then inequality (1) reduces to the following inequality:

$$\tilde{\mathcal{Q}}\left(\frac{u+v}{2}\right) \preccurlyeq \frac{1}{v-u}(FR)\int_u^v \tilde{\mathcal{Q}}(x)\,dx \preccurlyeq \frac{\tilde{\mathcal{Q}}(u) \tilde{+} \tilde{\mathcal{Q}}(v)}{2}, \tag{2}$$

where $\tilde{\mathcal{Q}}$ is a convex fuzzy-IVF. We urge the readers to [32–42] and the citations therein for further review of related literature on the implementations and characterization of fuzzy-interval, inequalities and generalized convex fuzzy mappings.

This study is organized as follows: Section 2 presents preliminary notions and results in interval space, and fuzzy-interval space. Moreover, Section 2 also discusses new class of convex fuzzy-IVF, which is known as harmonically convex fuzzy-IVF. Section 3 obtains fuzzy-interval H-H inequalities via convex fuzzy-IVFs. In particular, some intriguing examples are provided to support our outcomes. Conclusions and future plans are discussed in Section 4.

2. PRELIMINARIES

Let $\mathcal{K}_C$ be the space of all closed and bounded intervals of $\mathbb{R}$ and $\eta \in \mathcal{K}_C$ be defined by

$$\eta = \left[\eta_*, \eta^*\right] = \left\{x \in \mathbb{R} | \eta_* \leq x \leq \eta^*\right\}, \left(\eta_*, \eta^* \in \mathbb{R}\right). \tag{3}$$

If $\eta_* = \eta^*$ then, η is said to be degenerate. In this article, all intervals will be nondegenerate intervals. If $\eta_* \geq 0$, then $\left[\eta_*, \eta^*\right]$ is called positive interval. The set of all positive interval is denoted by $\mathcal{K}_C^+$ and defined as $\mathcal{K}_C^+ = \left\{\left[\eta_*, \eta^*\right] : \left[\eta_*, \eta^*\right] \in \mathcal{K}_C \text{ and } \eta_* \geq 0\right\}$.

Let $\varrho \in \mathbb{R}$ and $\varrho\eta$ be defined by

$$\varrho.\eta = \begin{cases} \left[\varrho\eta_*, \varrho\eta^*\right] & \text{if } \varrho \geq 0, \\ \left[\varrho\eta^*, \varrho\eta_*\right] & \text{if } \varrho < 0. \end{cases} \tag{4}$$

Then the Minkowski difference $\zeta - \eta$, addition $\eta + \zeta$ and $\eta \times \zeta$ for $\eta, \zeta \in \mathcal{K}_C$ are defined by

$$\begin{aligned} \left[\zeta_*, \zeta^*\right] - \left[\eta_*, \eta^*\right] &= \left[\zeta_* - \eta_*, \zeta^* - \eta^*\right], \\ \left[\zeta_*, \zeta^*\right] + \left[\eta_*, \eta^*\right] &= \left[\zeta_* + \eta_*, \zeta^* + \eta^*\right], \end{aligned} \tag{5}$$

and

$$\begin{aligned} &\left[\zeta_*, \zeta^*\right] \times \left[\eta_*, \eta^*\right] \\ &= \left[min\left\{\begin{matrix} \zeta_*\eta_*, \zeta^*\eta_*, \\ \zeta_*\eta^*, \zeta^*\eta^* \end{matrix}\right\}, max\left\{\begin{matrix} \zeta_*\eta_*, \zeta^*\eta_*, \\ \zeta_*\eta^*, \zeta^*\eta^* \end{matrix}\right\}\right]. \end{aligned} \tag{6}$$

The inclusion "$\subseteq$" means that

$$\begin{aligned} &\zeta \subseteq \eta \text{ If and only if, } \left[\zeta_*, \zeta^*\right] \subseteq \left[\eta_*, \eta^*\right], \text{if and only if} \\ &\eta_* \leq \zeta_*, \zeta^* \leq \eta^*. \end{aligned} \tag{7}$$

Remark 2.1. [32] The relation "$\leq_I$" defined on $\mathcal{K}_C$ by

$$\left[\zeta_*, \zeta^*\right] \leq_I \left[\eta_*, \eta^*\right] \text{ if and only if } \zeta_* \leq \eta_*, \zeta^* \leq \eta^*, \tag{8}$$

for all $\left[\zeta_*, \zeta^*\right], \left[\eta_*, \eta^*\right] \in \mathcal{K}_C$, it is an order relation. For given $\left[\zeta_*, \zeta^*\right], \left[\eta_*, \eta^*\right] \in \mathcal{K}_C$, we say that $\left[\zeta_*, \zeta^*\right] \leq_I \left[\eta_*, \eta^*\right]$ if and only if $\zeta_* \leq \eta_*, \zeta^* \leq \eta^*$ or $\zeta_* \leq \eta_*, \zeta^* < \eta^*$.

For $\left[\zeta_*, \zeta^*\right], \left[\eta_*, \eta^*\right] \in \mathbb{R}_I$, the Hausdorff–Pompeiu distance between intervals $\left[\zeta_*, \zeta^*\right]$ and $\left[\eta_*, \eta^*\right]$ is defined by

$$d\left(\left[\zeta_*, \zeta^*\right], \left[\eta_*, \eta^*\right]\right) = max\left\{|\zeta_* - \eta_*|, |\zeta^* - \eta^*|\right\}. \tag{9}$$

It is familiar fact that $\left(\mathbb{R}_I, d\right)$ is a complete metric space.

Assume $\mathbb{R}$ is a set of real numbers. The membership function is a mapping $\tilde{\zeta} : \mathbb{R} \to [0,1]$ that characterizes a fuzzy subset A of $\mathbb{R}$, for each fuzzy set and $\beta \in (0,1]$, then β-level sets of $\tilde{\zeta}$ is denoted and defined as follows: $\zeta_\beta = \left\{u \in \mathbb{R} | \tilde{\zeta}(u) \geq \beta\right\}$. If $\beta = 0$, then$supp\left(\tilde{\zeta}\right) = \left\{x \in \mathbb{R} | \tilde{\zeta}(x) > 0\right\}$ is called support of $\tilde{\zeta}$. By $\left[\tilde{\zeta}\right]^0$ we define the closure of supp $\left(\tilde{\zeta}\right)$.

Let $\mathbb{F}(\mathbb{R})$ be the family of all fuzzy sets and $\tilde{\zeta} \in \mathbb{F}(\mathbb{R})$ denote the family of all nonempty sets. $\tilde{\zeta} \in \mathbb{F}(\mathbb{R})$ be a fuzzy set. Then we define the following:

(1) $\tilde{\zeta}$ is said to be normal if there exists $x \in \mathbb{R}$ and $\tilde{\zeta}(x) = 1$;

(2) $\tilde{\zeta}$ is said to be upper semi continuous on $\mathbb{R}$ if for given $x \in \mathbb{R}$, there exist $\varepsilon > 0$ there exist $\delta > 0$ such that $\tilde{\zeta}(x) - \tilde{\zeta}(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $|x - y| < \delta$;

(3) $\tilde{\zeta}$ is said to be fuzzy convex if ζ_β is convex for every $\beta \in [0,1]$;

(4) $\tilde{\zeta}$ is compactly supported if $supp\left(\tilde{\zeta}\right)$ is compact.

A fuzzy set is called a fuzzy number or fuzzy interval if it has properties (1), (2), (3) and (4). We denote by $\mathbb{F}_0$ the family of all interval.

Let $\tilde{\zeta} \in \mathbb{F}_0$ be a fuzzy-interval, if and only if, β levels $\left[\tilde{\zeta}\right]^\beta$ is a nonempty compact convex set of $\mathbb{R}$. From these definitions, we have

$$\left[\tilde{\zeta}\right]^\beta = \left[\zeta_*(\beta), \zeta^*(\beta)\right],$$

where

$$\zeta_*(\beta) = inf\left\{\varkappa \in \mathbb{R} | \tilde{\zeta}(\varkappa) \geq \beta\right\},$$
$$\zeta^*(\beta) = sup\left\{\varkappa \in \mathbb{R} | \tilde{\zeta}(\varkappa) \geq \beta\right\}.$$

Proposition 2.2. [18] If $\tilde{\zeta}, \tilde{\eta} \in \mathbb{F}_0$ then relation "$\preccurlyeq$" defined on $\mathbb{F}_0$ by

$$\tilde{\zeta} \preccurlyeq \tilde{\eta} \text{ if and only if, } \left[\tilde{\zeta}\right]^\beta \leq_I [\tilde{\eta}]^\beta, \text{for all } \beta \in [0,1], \tag{10}$$

this relation is known as partial order relation.

For $\tilde{\zeta}, \tilde{\eta} \in \mathbb{F}_0$ and $\varrho \in \mathbb{R}$, the sum $\tilde{\zeta} \tilde{+} \tilde{\eta}$, product $\tilde{\zeta} \tilde{\times} \tilde{\eta}$, scalar product $\varrho.\tilde{\zeta}$ and sum with scalar are defined by

Then, for all $\beta \in [0,1]$, we have

$$\left[\tilde{\zeta} \tilde{+} \tilde{\eta}\right]^\beta = \left[\tilde{\zeta}\right]^\beta + [\tilde{\eta}]^\beta, \tag{11}$$

$$\left[\tilde{\zeta} \tilde{\times} \tilde{\eta}\right]^\beta = \left[\tilde{\zeta}\right]^\beta \times [\tilde{\eta}]^\beta, \tag{12}$$

$$\left[\varrho.\tilde{\zeta}\right]^\beta = \varrho.\left[\tilde{\zeta}\right]^\beta. \tag{13}$$

$$\left[\varrho \tilde{+} \tilde{\zeta}\right]^\beta = \varrho + \left[\tilde{\zeta}\right]^\beta. \tag{14}$$

For $\tilde{\zeta} \in \mathbb{F}_0$ such that $\tilde{\xi} = \tilde{\eta} \tilde{+} \tilde{\zeta}$, then by this result we have existence of Hukuhara difference of $\tilde{\xi}$ and $\tilde{\eta}$, and we say that $\tilde{\zeta}$ is the H-difference of $\tilde{\xi}$ and $\tilde{\eta}$, and denoted by $\tilde{\xi} \tilde{-} \tilde{\eta}$. If H-difference exists, then

$$\begin{aligned}(\zeta)^*(\beta) &= (\xi - \eta)^*(\beta) = \xi^*(\beta) - \eta^*(\beta),\\ (\zeta)_*(\beta) &= (\xi - \eta)_*(\beta) = \zeta_*(\beta) - \eta_*(\beta).\end{aligned} \tag{15}$$

A partition of $[u, v]$ is any finite ordered subset P having the form

$$P = \left\{u = \varkappa_1 < \varkappa_2 < \varkappa_3 < \varkappa_4 < \varkappa_5 \ldots\ldots < \varkappa_k = v\right\}.$$

The mesh of a partition P is the maximum length of the subintervals containing P, that is,

$$\text{mash}(P) = max\left\{\varkappa_j - \varkappa_{j-1} : j = 1, 2, 3, \ldots\ldots k\right\}.$$

Let $\mathscr{P}(\delta, [u, v])$ be the set of all $P \in \mathscr{P}(\delta, [u, v])$ such that mesh $(P) < \delta$. For each interval $\left[\varkappa_{j-1}, \varkappa_j\right]$, where $1 \leq j \leq k$, choose an arbitrary point μ_j and taking the sum

$$S(\mathcal{Q}, P, \delta) = \sum_{j=1}^{k} \mathcal{Q}\left(\mu_j\right)\left(\varkappa_j - \varkappa_{j-1}\right),$$

where $\mathcal{Q} : [u, v] \to \mathbb{R}_I$. We call $S(\mathcal{Q}, P, \delta)$ a Riemann sum of $\mathcal{Q}$ corresponding to $P \in \mathscr{P}(\delta, [u, v])$.

Definition 2.3. [24] A function $\mathcal{Q} : [u, v] \to \mathbb{R}_I$ is called interval Riemann integrable (IR-integrable) on $[u, v]$ if there exists $B \in \mathbb{R}_I$ such that, foe each ϵ, there exists $\delta > 0$ such that

$$d(S(\mathcal{Q}, P, \delta), B) < \epsilon,$$

for every Riemann sum of $\mathcal{Q}$ corresponding to $P \in \mathscr{P}(\delta, [u, v])$ and for arbitrary choice of $\mu_j \in \left[\varkappa_{j-1}, \varkappa_j\right]$ for $1 \leq j \leq k$. Then we say that B is the IR-integral of $\mathcal{Q}$ on $[u, v]$ and is denoted by $B = (IR)\int_u^v \mathcal{Q}(\varkappa)\, d\varkappa$.

Moore [16] firstly proposed the concept of Riemann integral for IVF and it is defined as follows:

Theorem 2.4. *[16] If $\mathcal{Q} : [u, v] \subset \mathbb{R} \to \mathbb{R}_I$ is an IVF on such that $\mathcal{Q}(\varkappa) = \left[\mathcal{Q}_*, \mathcal{Q}^*\right]$. Then $\mathcal{Q}$ is Riemann integrable over $[u, v]$ if and only if, $\mathcal{Q}_*$ and $\mathcal{Q}^*$ both are Riemann integrable over $[u, v]$ such that*

$$(IR)\int_u^v \mathcal{Q}(\varkappa)\, d\varkappa = \left[(R)\int_u^v \mathcal{Q}_*(u)\, d\varkappa, (R)\int_u^v \mathcal{Q}^*(u)\, d\varkappa\right]. \tag{16}$$

Definition 2.5. [33] A fuzzy map $\tilde{\mathcal{Q}} : K \subset \mathbb{R} \to \mathbb{F}_0$ is also known as fuzzy-IVF. For each $\beta \in [0,1]$, whose β levels characterize the family of IVFs $\mathcal{Q}_\beta : K \subset \mathbb{R} \to \mathscr{K}_C$ are given by $\mathcal{Q}_\beta(\varkappa) = \left[\mathcal{Q}_*(\varkappa, \beta), \mathcal{Q}^*(\varkappa, \beta)\right]$ for all $\varkappa \in K$. Here, for each $\beta \in [0,1]$, the left and right real-valued functions $\mathcal{Q}_*(\varkappa, \beta), \mathcal{Q}^*(\varkappa, \beta) : K \to \mathbb{R}$ are also called lower and upper functions of $\tilde{\mathcal{Q}}$.

Remark 2.6. If $\tilde{\mathcal{Q}} : K \subset \mathbb{R} \to \mathbb{F}_0$ is a fuzzy-IVF then, $\tilde{\mathcal{Q}}(\varkappa)$ is called continuous function at $\varkappa \in K$, if for each $\beta \in [0,1]$, both left and right real-valued functions $\mathcal{Q}_*(\varkappa, \beta)$ and $\mathcal{Q}^*(\varkappa, \beta)$ are continuous at $\varkappa \in K$.

The following conclusion can be drawn from the above literature review, see [16,24,33].

Definition 2.7. Let $\tilde{\mathcal{Q}} : \left[c, d\right] \subset \mathbb{R} \to \mathbb{F}_0$ be a fuzzy-IVF. Then, fuzzy Riemann integral of $\tilde{\mathcal{Q}}$ over $\left[c, d\right]$, denoted by $(FR)\int_c^d \tilde{\mathcal{Q}}(\varkappa)\, d\varkappa$, it is defined level by level

$$\begin{aligned}&\left[(FR)\int_c^d \tilde{\mathcal{Q}}(\varkappa)\, d\varkappa\right]^\beta\\ &= (IR)\int_c^d \mathcal{Q}_\beta(\varkappa)\, d\varkappa\\ &= \left\{\int_c^d \mathcal{Q}(\varkappa, \beta)\, d\varkappa : \mathcal{Q}(\varkappa, \beta) \in \mathscr{R}_{[c,d]}\right\},\end{aligned} \tag{17}$$

for all $\beta \in [0,1]$, where $\mathscr{R}_{[c,d]}$ contains the family of left and right functions of IVFs. $\tilde{\mathcal{Q}}$ is (FR)-integrable over $\left[c, d\right]$ if $(FR)\int_c^d \tilde{\mathcal{Q}}(\varkappa)\, d\varkappa \in \mathbb{F}_0$. Note that, if left and right real-valued functions are Lebesgue-integrable, then $\tilde{\mathcal{Q}}$ is fuzzy Aumann-integrable over $\left[c, d\right]$, denoted by $(FA)\int_c^d \tilde{\mathcal{Q}}(\varkappa)\, d\varkappa$, see [33].

Theorem 2.8. *Let $\tilde{\mathcal{Q}} : \left[c, d\right] \subset \mathbb{R} \to \mathbb{F}_0$ be a fuzzy-IVF, whose β levels characterize the collection of IVFs $\mathcal{Q}_\beta : \left[c, d\right] \subset \mathbb{R} \to \mathscr{K}_C$ are defined by $\mathcal{Q}_\beta(\varkappa) = \left[\mathcal{Q}_*(\varkappa, \beta), \mathcal{Q}^*(\varkappa, \beta)\right]$ for all $\varkappa \in \left[c, d\right]$ and for all $\beta \in [0,1]$. Then $\tilde{\mathcal{Q}}$ is (FR)-integrable over $\left[c, d\right]$ if and only if, $\mathcal{Q}_*(\varkappa, \beta)$ and $\mathcal{Q}^*(\varkappa, \beta)$ both are R-integrable over $\left[c, d\right]$. Moreover,*

if $\tilde{\mathcal{Q}}$ is (FR)-integrable over $[c,d]$, then

$$\begin{aligned}&\left[(FR)\int_c^d \tilde{\mathcal{Q}}(x)\,dx\right]^{\beta}\\&=\left[(R)\int_c^d \mathcal{Q}_*(x,\beta)\,dx,(R)\int_c^d \mathcal{Q}^*(x,\beta)\,dx\right]\\&=(IR)\int_c^d \mathcal{Q}_\beta(x)\,dx,\end{aligned}\tag{18}$$

for all $\beta\in[0,1]$. For each $\beta\in[0,1]$, $\mathcal{Q}\mathcal{R}_{([c,d],\beta)}$ and $\mathcal{R}_{([c,d],\beta)}$ denote the collection of all (FR)-integrable fuzzy-IVFs and, R-integrable left and right functions over $[c,d]$.

Allahviranloo *et al.* [39] introduced the following fuzzy-interval Riemann–Liouville fractional integral operators:

Let $\alpha>0$ and $L([u,v],\mathbb{F}_0)$ be the collection of all Lebesgue measurable fuzzy-IVFs on $[u,v]$. Then the fuzzy-interval left and right Riemann–Liouville fractional integral of $\tilde{\mathcal{Q}}\in L([u,v],\mathbb{F}_0)$ with order $\alpha>0$ are defined by

$$\mathcal{J}^{\alpha}_{u^+}\tilde{\mathcal{Q}}(x)=\frac{1}{\Gamma(\alpha)}\int_u^x (x-\rho)^{\alpha-1}\tilde{\mathcal{Q}}(\rho)\,d(\rho),(x>u),\tag{19}$$

and

$$\mathcal{J}^{\alpha}_{v^-}\tilde{\mathcal{Q}}(x)=\frac{1}{\Gamma(\alpha)}\int_x^v (\rho-x)^{\alpha-1}\tilde{\mathcal{Q}}(\rho)\,d(\rho),(x<v),\tag{20}$$

respectively, where $\Gamma(x)=\int_0^\infty \rho^{x-1}u^{-\rho}d(\rho)$ is the Euler gamma function. The fuzzy-interval left and right Riemann–Liouville fractional integral x based on left and right end point functions can be defined, that is,

$$\begin{aligned}&\left[\mathcal{J}^{\alpha}_{u^+}\mathcal{Q}(x)\right]^{\beta}\\&=\frac{1}{\Gamma(\alpha)}\int_u^x (x-\rho)^{\alpha-1}\mathcal{Q}_\beta(\rho)\,d(\rho)\\&=\frac{1}{\Gamma(\alpha)}\int_u^x (x-\rho)^{\alpha-1}\left[\mathcal{Q}_*(\rho,\beta),\mathcal{Q}^*(\rho,\beta)\right]d(\rho),(x>u),\end{aligned}\tag{21}$$

where

$$\begin{aligned}&\mathcal{J}^{\alpha}_{u^+}\mathcal{Q}_*(x,\beta)\\&=\frac{1}{\Gamma(\alpha)}\int_u^x (x-\rho)^{\alpha-1}\mathcal{Q}_*(\rho,\beta)\,d(\rho),(x>u),\end{aligned}\tag{22}$$

and

$$\begin{aligned}&\mathcal{J}^{\alpha}_{u^+}\mathcal{Q}^*(x,\beta)\\&=\frac{1}{\Gamma(\alpha)}\int_u^x (x-\rho)^{\alpha-1}\mathcal{Q}^*(\rho,\beta)\,d(\rho),(x>u),\end{aligned}\tag{23}$$

Similarly, we can define right Riemann–Liouville fractional integral $\mathcal{Q}$ of x based on left and right end point functions.

Definition 2.9. A set $K=[u,v]\subset\mathbb{R}^+=(0,\infty)$ is said to be harmonically convex set, if, for all $x,y\in K,\rho\in[0,1]$, we have

$$\frac{xy}{\rho x+(1-\rho)y}\in K.\tag{24}$$

Definition 2.10. [3] The fuzzy-IVF $\mathcal{Q}:[u,v]\to\mathbb{F}_0$ is called harmonically convex **f**uzzy-IVF on $[u,v]$ if

$$\mathcal{Q}\left(\frac{xy}{\rho x+(1-\rho)y}\right)\le(1-\rho)\mathcal{Q}(x)+\rho\mathcal{Q}(y),\tag{25}$$

for all $x,y\in[u,v],\rho\in[0,1]$, where $\mathcal{Q}(x)\ge 0$ for all $x\in[u,v]$. If (25) is reversed then, $\mathcal{Q}$ is called harmonically concave fuzzy-IVF on $[u,v]$.

Definition 2.11. The fuzzy-IVF $\tilde{\mathcal{Q}}:[u,v]\to\mathbb{F}_0$ is called harmonically convex **f**uzzy-IVF on $[u,v]$ if

$$\tilde{\mathcal{Q}}\left(\frac{xy}{\rho x+(1-\rho)y}\right)\preccurlyeq(1-\rho)\tilde{\mathcal{Q}}(x)\tilde{+}\rho\tilde{\mathcal{Q}}(y),\tag{26}$$

for all $x,y\in[u,v],\rho\in[0,1]$, where $\tilde{\mathcal{Q}}(x)\succcurlyeq\tilde{0}$, for all $x\in[u,v]$. If (26) is reversed then, $\tilde{\mathcal{Q}}$ is called concave fuzzy-IVF on $[u,v]$.

Theorem 2.12. *Let K be harmonically convex set, and let $\tilde{\mathcal{Q}}:K\to\mathbb{F}_C(\mathbb{R})$ be a fuzzy-IVF whose β levels define the family of IVFs $\mathcal{Q}_\beta:K\subset\mathbb{R}\to\mathcal{K}_C^+\subset\mathcal{K}_C$ are given by*

$$\mathcal{Q}_\beta(x)=\left[\mathcal{Q}_*(x,\beta),\mathcal{Q}^*(x,\beta)\right],\forall x\in K,\tag{27}$$

for all $x\in K,\beta\in[0,1]$. Then $\tilde{\mathcal{Q}}$ is harmonically convex on K, if and only if, for all $\beta\in[0,1]$, $\mathcal{Q}_*(x,\beta)$ and $\mathcal{Q}^*(x,\beta)$ are harmonically convex.

Proof. Assume that for each $\beta\in[0,1]$, $\mathcal{Q}_*(x,\beta)$ and $\mathcal{Q}^*(x,\beta)$ are harmonically convex on K. Then from (25), we have

$$\mathcal{Q}_*\left(\frac{xy}{\rho x+(1-\rho)y},\beta\right)\le(1-\rho)\mathcal{Q}_*(x,\beta)+\rho\mathcal{Q}_*(y,\beta),$$

and

$$\mathcal{Q}^*\left(\frac{xy}{\rho x+(1-\rho)y},\beta\right)\le(1-\rho)\mathcal{Q}^*(x,\beta)+\rho\mathcal{Q}^*(y,\beta).$$

Then by (26), (19) and (21), we obtain

$$\begin{aligned}&\mathcal{Q}_\beta\left(\frac{xy}{\rho x+(1-\rho)y}\right)\\&=\left[\mathcal{Q}_*(\rho x+(1-\rho)y,\beta),\mathcal{Q}^*(\rho x+(1-\rho)y,\beta)\right],\\&\le_I(1-\rho)\left[\mathcal{Q}_*(x,\beta),\mathcal{Q}^*(x,\beta)\right]\\&\quad+\rho\left[\mathcal{Q}_*(y,\beta),\mathcal{Q}^*(y,\beta)\right],\end{aligned}$$

that is,

$$\tilde{\mathcal{Q}}\left(\frac{xy}{\rho x+(1-\rho)y}\right)\preccurlyeq(1-\rho)\tilde{\mathcal{Q}}(x)\tilde{+}\rho\tilde{\mathcal{Q}}(y),$$

$\forall x,y\in K,\rho\in[0,1]$. Hence, $\tilde{\mathcal{Q}}$ is harmonically convex fuzzy-IVF on K.

Conversely, let $\tilde{\mathcal{Q}}$ be harmonically convex fuzzy-IVF on K. Then for all $x, y \in K$, $\varrho \in [0,1]$, we have

$$\tilde{\mathcal{Q}}\left(\frac{xy}{\varrho x+(1-\varrho)y}\right) \preccurlyeq (1-\varrho)\tilde{\mathcal{Q}}(x) \tilde{+} \varrho\tilde{\mathcal{Q}}(y).$$

Therefore, from (26), for each $\beta \in [0,1]$, left side of above inequality, we have

$$\mathcal{Q}_\beta\left(\frac{xy}{\varrho x+(1-\varrho)y}\right) = \left[\mathcal{Q}_*\left(\frac{xy}{\varrho x+(1-\varrho)y},\beta\right), \mathcal{Q}^*\left(\frac{xy}{\varrho x+(1-\varrho)y},\beta\right)\right].$$

Again, from (26), we obtain

$$(1-\varrho)\mathcal{Q}_\beta(x) + \varrho\mathcal{Q}_\beta(x) = (1-\varrho)\left[\mathcal{Q}_*(x,\beta), \mathcal{Q}^*(x,\beta)\right] + \varrho\left[\mathcal{Q}_*(y,\beta), \mathcal{Q}^*(y,\beta)\right],$$

for all $x, y \in K$, $\varrho \in [0,1]$. Then by harmonically convexity of $\tilde{\mathcal{Q}}$, we have for all $x, y \in K$, $\varrho \in [0,1]$ such that

$$\mathcal{Q}_*\left(\frac{xy}{\varrho x+(1-\varrho)y},\beta\right) \le (1-\varrho)\mathcal{Q}_*(x,\beta) + \varrho\mathcal{Q}_*(y,\beta),$$

and

$$\mathcal{Q}^*\left(\frac{xy}{\varrho x+(1-\varrho)y},\beta\right) \le (1-\varrho)\mathcal{Q}^*(x,\beta) + \varrho\mathcal{Q}^*(y,\beta),$$

for each $\beta \in [0,1]$. Hence, the result follows.

Remark 2.12. If $\mathcal{Q}_*(x,\beta) = \mathcal{Q}^*(x,\beta)$ and $\beta = 1$ then from Definition 210, we obtain Definition 2.10.

Example 2.13. We consider the fuzzy-IVFs $\tilde{\mathcal{Q}} : [0,2] \to \mathbb{F}_C(\mathbb{R})$ defined by

$$\tilde{\mathcal{Q}}(x)(\sigma) = \begin{cases} \dfrac{\sigma}{\sqrt{x}} & \sigma \in \left[0, \sqrt{x}\right] \\ \dfrac{2-\sigma}{2x^2} & \sigma \in (\sqrt{x}, 2\sqrt{x}] \\ 0 & \text{otherwise.} \end{cases}$$

Then, for each $\beta \in [0,1]$, we have $\mathcal{Q}_\beta(x) = \left[\beta\sqrt{x}, (2-\beta)\sqrt{x}\right]$. Since end point functions $\mathcal{Q}_*(x,\beta)$, $\mathcal{Q}^*(x,\beta)$ are harmonically convex functions for each $\beta \in [0,1]$. Hence $\tilde{\mathcal{Q}}(x)$ is harmonically convex fuzzy-IVF.

In next result, we will establish a relation between convex fuzzy-IVF and harmonically convex fuzzy-IVF.

Theorem 2.14. *Let $\tilde{\mathcal{Q}} : K \to \mathbb{F}_C(\mathbb{R})$ be a fuzzy-IVF, where for all $\beta \in [0,1]$, whose β levels define the family of IVFs $\mathcal{Q}_\beta : K \subset \mathbb{R} \to \mathcal{K}_C^+ \subset \mathcal{K}_C$ are given by $\mathcal{Q}_\beta(x) = \left[\mathcal{Q}_*(x,\beta), \mathcal{Q}^*(x,\beta)\right]$, for all $x \in K$. Then $\tilde{\mathcal{Q}}(x)$ is harmonically convex fuzzy-IVF on K, if and only if, $\tilde{\mathcal{Q}}\left(\frac{1}{x}\right)$ is convex fuzzy-IVF on K.*

Proof. Since $\tilde{\mathcal{Q}}(x)$ is a harmonically convex fuzzy-IVF then, for $x, y \in [u,v]$, $\varrho \in [0,1]$, we have

$$\tilde{\mathcal{Q}}\left(\frac{xy}{\varrho x+(1-\varrho)y}\right) \preccurlyeq (1-\varrho)\tilde{\mathcal{Q}}(x) \tilde{+} \varrho\tilde{\mathcal{Q}}(y).$$

Therefore, for each $\beta \in [0,1]$, we have

$$\begin{aligned} \mathcal{Q}_*\left(\frac{xy}{\varrho x+(1-\varrho)y},\beta\right) &\le (1-\varrho)\mathcal{Q}_*(x,\beta) + \varrho\mathcal{Q}_*(y,\beta), \\ \mathcal{Q}^*\left(\frac{xy}{\varrho x+(1-\varrho)y},\beta\right) &\le (1-\varrho)\mathcal{Q}^*(x,\beta) + \varrho\mathcal{Q}^*(y,\beta). \end{aligned} \tag{28}$$

Consider $\tilde{\varphi}(x) = \tilde{\mathcal{Q}}\left(\frac{1}{x}\right)$. Taking $m = \frac{1}{x}$ and $n = \frac{1}{y}$ to replace x and y, respectively. Then for each $\beta \in [0,1]$, applying (28)

$$\begin{aligned} &\mathcal{Q}_*\left(\frac{\frac{1}{xy}}{\varrho\frac{1}{x}+(1-\varrho)\frac{1}{y}},\beta\right) \\ &= \mathcal{Q}_*\left(\frac{1}{(1-\varrho)x+\varrho y},\beta\right) \\ &= \varphi_*\left((1-\varrho)x+\varrho y,\beta\right) \\ &\le \varrho\mathcal{Q}_*\left(\tfrac{1}{y},\beta\right) + (1-\varrho)\mathcal{Q}_*\left(\tfrac{1}{x},\beta\right) \\ &= \varrho\varphi_*(y,\beta) + (1-\varrho)\varphi_*(x,\beta), \\ &\mathcal{Q}^*\left(\frac{\frac{1}{xy}}{\varrho\frac{1}{x}+(1-\varrho)\frac{1}{y}},\beta\right) \\ &= \mathcal{Q}^*\left(\frac{1}{(1-\varrho)x+\varrho y},\beta\right) \\ &= \varphi^*\left((1-\varrho)x+\varrho y,\beta\right) \\ &\le \varrho\mathcal{Q}^*\left(\tfrac{1}{y},\beta\right) + (1-\varrho)\mathcal{Q}^*\left(\tfrac{1}{x},\beta\right) \\ &= \varrho\varphi^*(y,\beta) + (1-\varrho)\varphi^*(x,\beta). \end{aligned}$$

It follows that

$$\begin{aligned} &\left[\mathcal{Q}_*\left(\frac{\frac{1}{xy}}{\varrho\frac{1}{x}+(1-\varrho)\frac{1}{y}},\beta\right), \mathcal{Q}^*\left(\frac{\frac{1}{xy}}{\varrho\frac{1}{x}+(1-\varrho)\frac{1}{y}},\beta\right)\right] \\ &= \left[\varphi_*\left((1-\varrho)x+\varrho y,\beta\right), \varphi^*\left((1-\varrho)x+\varrho y,\beta\right)\right] \\ &\le \varrho\left[\varphi_*(y,\beta), \varphi^*(y,\beta)\right] + (1-\varrho)\left[\varphi_*(x,\beta), \varphi^*(x,\beta)\right]. \end{aligned}$$

which implies that

$$\varphi_\beta\left((1-\varrho)x+\varrho y\right) \le_I \varrho\varphi_\beta(y) + (1-\varrho)\varphi_\beta(x),$$

that is,

$$\tilde{\varphi}\left((1-\varrho)x+\varrho y\right) \preccurlyeq \varrho\tilde{\varphi}(y) \tilde{+} (1-\varrho)\tilde{\varphi}(x).$$

This concludes that $\tilde{\varphi}(x)$ is a convex fuzzy-IVF.

Conversely, let $\tilde{\varphi}$ is convex fuzzy-IVF on K. Then, for all $x, y \in K$, $\varrho \in [0,1]$, we have

$$\tilde{\varphi}\left(\varrho x+(1-\varrho)y\right) \preccurlyeq \varrho\tilde{\varphi}(x) \tilde{+} (1-\varrho)\tilde{\varphi}(y).$$

By using same steps as above, for each $\beta \in [0,1]$, we have

$$\begin{aligned}
&\varphi_*\left(\varrho\tfrac{1}{x}+(1-\varrho)\tfrac{1}{y},\beta\right)\\
&=\mathcal{Q}_*\left(\frac{1}{\varrho\frac{1}{x}+(1-\varrho)\frac{1}{y}},\beta\right)\\
&=\mathcal{Q}_*\left(\frac{xy}{(1-\varrho)x+\varrho y},\beta\right)\\
&\leq \varrho\varphi_*\left(\tfrac{1}{x},\beta\right)+(1-\varrho)\varphi_*\left(\tfrac{1}{y},\beta\right)\\
&=\varrho\mathcal{Q}_*(x,\beta)+(1-\varrho)\mathcal{Q}_*(y,\beta)
\end{aligned}$$

$$\begin{aligned}
&\varphi^*\left(\varrho\tfrac{1}{x}+(1-\varrho)\tfrac{1}{y},\beta\right)\\
&=\mathcal{Q}_*\left(\frac{1}{\varrho\frac{1}{x}+(1-\varrho)\frac{1}{y}},\beta\right)\\
&=\mathcal{Q}_*\left(\frac{xy}{(1-\varrho)x+\varrho y},\beta\right)\\
&\leq \varrho\varphi^*\left(\tfrac{1}{x},\beta\right)+(1-\varrho)\varphi^*\left(\tfrac{1}{y},\beta\right)\\
&=\varrho\mathcal{Q}_*(x,\beta)+(1-\varrho)\mathcal{Q}_*(y,\beta).
\end{aligned}$$

It follows that

$$\mathcal{Q}_\beta\left(\frac{zy}{\varrho z+(1-\varrho)y}\right)\leq_I(1-\varrho)\mathcal{Q}_\beta(z)+\varrho\mathcal{Q}_\beta(y),$$

that is,

$$\tilde{\mathcal{Q}}\left(\frac{zy}{\varrho z+(1-\varrho)y}\right)\preccurlyeq(1-\varrho)\tilde{\mathcal{Q}}(z)\tilde{+}\varrho\tilde{\mathcal{Q}}(y),$$

the proof the theorem has been completed.

Remark 2.15. If $\mathcal{Q}_*(z,\beta)=\mathcal{Q}^*(z,\beta)$ and $\beta=1$ then from Theorem 2.14, we obtain Lemma 2.1 of [13].

3. FUZZY-INTERVAL FRACTIONAL HERMITE–HADAMARD INEQUALITIES

In this section, we shall continue with the following the fractional $H-H$ inequality for harmonically convex fuzzy-IVFs and we also give fractional $H-H$ Fejér inequality for harmonically convex fuzzy-IVF through fuzzy order relation. In what follows, we denote by $L\left([u,v],\mathbb{F}_0\right)$ the family of Lebesgue measureable fuzzy-IVFs.

Theorem 3.1. *Let $\tilde{\mathcal{Q}}:[u,v]\to\mathbb{F}_0$ be a harmonically convex fuzzy-IVF on $[u,v]$, whose β levels define the family of IVFs $\mathcal{Q}_\beta:[u,v]\subset\mathbb{R}\to\mathcal{K}_C^+$ are given by $\mathcal{Q}_\beta(x)=\left[\mathcal{Q}_*(x,\beta),\mathcal{Q}^*(x,\beta)\right]$ for all $x\in[u,v]$, $\beta\in[0,1]$. If $\tilde{\mathcal{Q}}\in L\left([u,v],\mathbb{F}_0\right)$, then*

$$\begin{aligned}
\tilde{\mathcal{Q}}\left(\frac{2uv}{u+v}\right)&\preccurlyeq\frac{\Gamma(\alpha+1)}{2(v-u)^\alpha}\left[\mathcal{J}^\alpha_{\frac{1}{u}^-}\left(\tilde{\mathcal{Q}}\circ\psi\right)\left(\tfrac{1}{v}\right)\tilde{+}\mathcal{J}^\alpha_{\frac{1}{v}^+}\left(\tilde{\mathcal{Q}}\circ\psi\right)\left(\tfrac{1}{u}\right)\right]\\
&\preccurlyeq\frac{\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(v)}{2}.
\end{aligned}\tag{29}$$

If $\tilde{\mathcal{Q}}(x)$ is concave fuzzy-IVF then

$$\begin{aligned}
\tilde{\mathcal{Q}}\left(\frac{2uv}{u+v}\right)&\succcurlyeq\frac{\Gamma(\alpha+1)}{2(v-u)^\alpha}\left[\mathcal{J}^\alpha_{\frac{1}{u}^-}\left(\tilde{\mathcal{Q}}\circ\psi\right)\left(\tfrac{1}{v}\right)\tilde{+}\mathcal{J}^\alpha_{\frac{1}{v}^+}\left(\tilde{\mathcal{Q}}\circ\psi\right)\left(\tfrac{1}{u}\right)\right]\\
&\succcurlyeq\frac{\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(v)}{2}.
\end{aligned}\tag{30}$$

where $\psi(x)=\frac{1}{x}$.

Proof. Let $\tilde{\mathcal{Q}}:[u,v]\to\mathbb{F}_0$ be harmonically convex fuzzy-IVF. Then, by hypothesis, we have

$$2\tilde{\mathcal{Q}}\left(\frac{2uv}{u+v}\right)\preccurlyeq\tilde{\mathcal{Q}}\left(\frac{uv}{\varrho u+(1-\varrho)v}\right)\tilde{+}\tilde{\mathcal{Q}}\left(\frac{uv}{(1-\varrho)u+\varrho v}\right).$$

Therefore, for each $\beta\in[0,1]$, we have

$$2\mathcal{Q}_*\left(\frac{2uv}{u+v},\beta\right)\leq\mathcal{Q}_*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)+\mathcal{Q}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right),$$

$$2\mathcal{Q}^*\left(\frac{2uv}{u+v},\beta\right)\leq\mathcal{Q}^*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)+\mathcal{Q}^*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right).$$

Consider $\tilde{\varphi}(x)=\tilde{\mathcal{Q}}\left(\frac{1}{x}\right)$. By Theorem 2.14 we have $\tilde{\varphi}(x)$ is convex fuzzy-IVF then for each $\beta\in[0,1]$, above inequality, we have

$$2\varphi_*\left(\frac{u+v}{2uv},\beta\right)\leq\varphi_*\left(\tfrac{\varrho u+(1-\varrho)v}{uv},\beta\right)+\varphi_*\left(\tfrac{(1-\varrho)u+\varrho v}{uv},\beta\right).$$

Multiplying both sides by $\varrho^{\alpha-1}$ and integrating the obtained result with respect to ϱ over $(0,1)$, we have

$$\begin{aligned}
&2\int_0^1\varrho^{\alpha-1}\varphi_*\left(\frac{u+v}{2uv},\beta\right)d\varrho\\
&\leq\int_0^1\varrho^{\alpha-1}\varphi_*\left(\frac{\varrho u+(1-\varrho)v}{uv},\beta\right)d\varrho\\
&+\int_0^1\varrho^{\alpha-1}\varphi_*\left(\frac{(1-\varrho)u+\varrho v}{uv},\beta\right)d\varrho.
\end{aligned}$$

Let $x=\dfrac{(1-\varrho)u+\varrho v}{uv}$ and $y=\dfrac{\varrho u+(1-\varrho)v}{uv}$. Then we have

$$\begin{aligned}
\tfrac{2}{\alpha}\varphi_*\left(\tfrac{u+v}{2uv},\beta\right)&\leq\left(\tfrac{uv}{v-u}\right)^\alpha\int_{\frac{1}{v}}^{\frac{1}{u}}\left(\frac{1}{u}-y\right)^{\alpha-1}\varphi_*(y,\beta)\,dy\\
&+\left(\tfrac{uv}{v-u}\right)^\alpha\int_{\frac{1}{v}}^{\frac{1}{u}}\left(x-\frac{1}{v}\right)^{\alpha-1}\varphi_*(x,\beta)\,dx\\
&=\Gamma(\alpha)\left(\tfrac{uv}{v-u}\right)^\alpha\left[\mathcal{J}^\alpha_{\left(\frac{1}{u}\right)^-}\varphi_*\left(\tfrac{1}{v},\beta\right)+\mathcal{J}^\alpha_{\left(\frac{1}{v}\right)^+}\varphi_*\left(\tfrac{1}{u},\beta\right)\right].
\end{aligned}$$

Similarly, for $\mathcal{Q}^*(x,\gamma)$, we have

$$\begin{aligned}
&\tfrac{2}{\alpha}\varphi_*\left(\tfrac{u+v}{2uv},\beta\right)\\
&\leq\Gamma(\alpha)\left(\tfrac{uv}{v-u}\right)^\alpha\left[\mathcal{J}^\alpha_{\left(\frac{1}{u}\right)^-}\varphi^*\left(\tfrac{1}{v},\beta\right)+\mathcal{J}^\alpha_{\left(\frac{1}{v}\right)^+}\varphi^*\left(\tfrac{1}{u},\beta\right)\right].
\end{aligned}$$

It follows that

$$\begin{aligned}&2\left[\varphi_*\left(\tfrac{u+v}{2uv},\beta\right),\varphi^*\left(\tfrac{u+v}{2uv},\beta\right)\right]\\&\leq_I \Gamma(\alpha+1)\left(\tfrac{uv}{v-u}\right)^{\alpha}\left[\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\varphi_*\left(\tfrac{1}{v},\beta\right)+\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\right.\\&\left.\varphi_*\left(\tfrac{1}{u},\beta\right),\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\varphi^*\left(\tfrac{1}{v},\beta\right)+\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\varphi^*\left(\tfrac{1}{u},\beta\right)\right].\end{aligned}$$

That is,

$$\begin{aligned}&2\tilde{\varphi}\left(\tfrac{u+v}{2uv}\right)\\&\preccurlyeq \Gamma(\alpha+1)\left(\tfrac{uv}{v-u}\right)^{\alpha}\left[\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\tilde{\varphi}\left(\tfrac{1}{v}\right)\tilde{+}\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\tilde{\varphi}\left(\tfrac{1}{u}\right)\right].\end{aligned}\tag{31}$$

In a similar way as above, we have

$$\begin{aligned}&\Gamma(\alpha)\left(\tfrac{uv}{v-u}\right)^{\alpha}\left[\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\tilde{\varphi}\left(\tfrac{1}{v}\right)\tilde{+}\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\tilde{\varphi}\left(\tfrac{1}{u}\right)\right]\\&\preccurlyeq\frac{\tilde{\varphi}\left(\frac{1}{u}\right)\tilde{+}\tilde{\varphi}\left(\frac{1}{v}\right)}{\alpha}.\end{aligned}\tag{32}$$

Combining (31) and (32), we have

$$\begin{aligned}\tilde{\varphi}\left(\tfrac{u+v}{2uv}\right)&\preccurlyeq\frac{\Gamma(\alpha+1)\left(\frac{uv}{v-u}\right)^{\alpha}}{2}\left[\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\tilde{\varphi}\left(\tfrac{1}{v}\right)\tilde{+}\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\tilde{\varphi}\left(\tfrac{1}{u}\right)\right]\\&\preccurlyeq\frac{\tilde{\varphi}\left(\frac{1}{u}\right)\tilde{+}\tilde{\varphi}\left(\frac{1}{v}\right)}{2}.\end{aligned}$$

That is,

$$\begin{aligned}\tilde{\mathcal{Q}}\left(\tfrac{2uv}{u+v}\right)&\preccurlyeq\frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\frac{1}{u}^-}\left(\tilde{\mathcal{Q}}\circ\psi\right)\left(\tfrac{1}{v}\right)\tilde{+}\mathcal{J}^{\alpha}_{\frac{1}{v}^+}\left(\tilde{\mathcal{Q}}\circ\psi\right)\left(\tfrac{1}{u}\right)\right]\\&\preccurlyeq\frac{\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(v)}{2}.\end{aligned}$$

Hence, the required result.

Remark 3.2. If $\alpha=1$, then inequality (29) reduces to the following inequality which is also new one:

$$\tilde{\mathcal{Q}}\left(\frac{2uv}{u+v}\right)\preccurlyeq\frac{uv}{v-u}\int_u^v\frac{\tilde{\mathcal{Q}}(x)}{x^2}dx\preccurlyeq\frac{\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(v)}{2}.\tag{33}$$

If $\mathcal{Q}_*(x,\beta)=\mathcal{Q}^*(x,\beta)$ with $\beta=1$ then, we obtain classical fractional H-H inequality for harmonically convex function which is given in [13]:

$$\begin{aligned}\mathcal{Q}\left(\tfrac{2uv}{u+v}\right)&\leq\frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}}\left[\mathcal{J}^{\alpha}_{\frac{1}{u}^-}\left(\mathcal{Q}\circ\psi\right)\left(\tfrac{1}{v}\right)+\mathcal{J}^{\alpha}_{\frac{1}{v}^+}\left(\mathcal{Q}\circ\psi\right)\left(\tfrac{1}{u}\right)\right]\\&\leq\frac{\mathcal{Q}(u)+\mathcal{Q}(v)}{2}.\end{aligned}\tag{34}$$

If $\mathcal{Q}_*(x,\beta)=\mathcal{Q}^*(x,\beta)$ with $\beta=1$ and $\alpha=1$ then, we obtain classical H-H inequality for harmonically convex function which is given in [3].

$$\mathcal{Q}\left(\frac{2uv}{u+v}\right)\leq\frac{uv}{v-u}\int_u^v\frac{\mathcal{Q}(x)}{x^2}dx\leq\frac{\mathcal{Q}(u)+\mathcal{Q}(v)}{2}.\tag{35}$$

Theorem 3.3. *(Second fuzzy fractional $H-H$ Fejér inequality) Let $\tilde{\mathcal{Q}}:[u,v]\to\mathbb{F}_0$ be a harmonically convex fuzzy-IVF with $u<v$, whose β levels define the family of IVFs $\mathcal{Q}_\beta:[u,v]\subset\mathbb{R}\to\mathcal{K}_C^+$ are given by $\mathcal{Q}_\beta(x)=\left[\mathcal{Q}_*(x,\beta),\mathcal{Q}^*(x,\beta)\right]$ for all $x\in[u,v]$, $\beta\in[0,1]$. If $\tilde{\mathcal{Q}}\in L\left([u,v],\mathbb{F}_0\right)$ and $\Omega:[u,v]\to\mathbb{R}$, $\Omega\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}\right)=\Omega(x)\geq0$, then*

$$\begin{aligned}&\left[\mathcal{J}^{\alpha}_{u^+}\tilde{\mathcal{Q}}\Omega\circ\psi(v)\tilde{+}\mathcal{J}^{\alpha}_{v^-}\tilde{\mathcal{Q}}\Omega\circ\psi(u)\right]\\&\preccurlyeq\frac{\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(v)}{2}\left[\mathcal{J}^{\alpha}_{\frac{1}{v}^+}(\Omega\circ\psi)\left(\tfrac{1}{u}\right)+\mathcal{J}^{\alpha}_{\frac{1}{u}^-}(\Omega\circ\psi)\left(\tfrac{1}{v}\right)\right].\end{aligned}\tag{36}$$

If $\tilde{\mathcal{Q}}$ is concave fuzzy-IVF then, inequality (36) is reversed.

Proof. Let $\tilde{\mathcal{Q}}$ be a harmonically convex fuzzy-IVF and $\rho^{\alpha-1}\Omega\left(\frac{uv}{\rho u+(1-\rho)v}\right)\geq0$. Then, for each $\beta\in[0,1]$, we have

$$\begin{aligned}&\rho^{\alpha-1}\mathcal{Q}_*\left(\tfrac{uv}{\rho u+(1-\rho)v},\beta\right)\Omega\left(\tfrac{uv}{\rho u+(1-\rho)v}\right)\\&\leq\rho^{\alpha-1}\left((1-\rho)\mathcal{Q}_*(u,\beta)+\rho\mathcal{Q}_*(v,\beta)\right)\Omega\left(\tfrac{uv}{\rho u+(1-\rho)v}\right),\end{aligned}\tag{37}$$

and

$$\begin{aligned}&\rho^{\alpha-1}\mathcal{Q}_*\left(\tfrac{uv}{(1-\rho)u+\rho v},\beta\right)\Omega\left(\tfrac{uv}{\rho u+(1-\rho)v}\right)\\&\leq\rho^{\alpha-1}\begin{pmatrix}\rho\mathcal{Q}_*(u,\beta)+\\(1-\rho)\mathcal{Q}_*(v,\beta)\end{pmatrix}\Omega\left(\tfrac{uv}{\rho u+(1-\rho)v}\right).\end{aligned}\tag{38}$$

After adding (37) and (38), and integrating over $[0,1]$, we get

$$\begin{aligned}&\int_0^1\rho^{\alpha-1}\mathcal{Q}_*\left(\frac{uv}{\rho u+(1-\rho)v},\beta\right)\Omega\left(\frac{uv}{\rho u+(1-\rho)v}\right)d\rho\\&+\int_0^1\rho^{\alpha-1}\mathcal{Q}_*\left(\frac{uv}{(1-\rho)u+\rho v},\beta\right)\Omega\left(\frac{uv}{\rho u+(1-\rho)v}\right)d\rho\\&\leq\int_0^1\begin{bmatrix}\rho^{\alpha-1}\mathcal{Q}_*(u,\beta)\{\rho+(1-\rho)\}\Omega\left(\frac{uv}{\rho u+(1-\rho)v}\right)\\+\rho^{\alpha-1}\mathcal{Q}_*(v,\beta)\{(1-\rho)+\rho\}\Omega\left(\frac{uv}{\rho u+(1-\rho)v}\right)\end{bmatrix}d\rho,\\&=\mathcal{Q}_*(u,\beta)\int_0^1\rho^{\alpha-1}\Omega\left(\frac{uv}{\rho u+(1-\rho)v}\right)d\rho\\&+\mathcal{Q}_*(v,\beta)\int_0^1\rho^{\alpha-1}\Omega\left(\frac{uv}{\rho u+(1-\rho)v}\right)d\rho,\end{aligned}$$

Similarly, for $\mathcal{Q}^*(x,\gamma)$, we have

$$\begin{aligned}&\int_0^1\rho^{\alpha-1}\mathcal{Q}^*\left(\frac{uv}{\rho u+(1-\rho)v},\beta\right)\Omega\left(\frac{uv}{\rho u+(1-\rho)v}\right)d\rho\\&+\int_0^1\rho^{\alpha-1}\mathcal{Q}^*\left(\frac{uv}{(1-\rho)u+\rho v},\beta\right)\Omega\left(\frac{uv}{\rho u+(1-\rho)v}\right)d\rho\\&=\mathcal{Q}^*(u,\beta)\int_0^1\rho^{\alpha-1}\Omega\left(\frac{uv}{\rho u+(1-\rho)v}\right)d\rho\\&+\mathcal{Q}^*(v,\beta)\int_0^1\rho^{\alpha-1}\Omega\left(\frac{uv}{\rho u+(1-\rho)v}\right)d\rho.\end{aligned}$$

From which, we have

$$\Gamma(\alpha)\left(\frac{uv}{v-u}\right)^{\alpha}\left[\mathcal{J}^{\alpha}_{u^+}\mathcal{Q}_{\beta}\Omega\circ\psi(v)+\mathcal{J}^{\alpha}_{v^-}\mathcal{Q}_{\beta}\Omega\circ\psi(u)\right]$$
$$\leq_I \Gamma(\alpha)\left(\frac{uv}{v-u}\right)^{\alpha}\frac{\mathcal{Q}_{\beta}(u)+\mathcal{Q}_{\beta}(v)}{2}\left[\begin{array}{l}\mathcal{J}^{\alpha}_{\frac{1}{v}^+}(\Omega\circ\psi)\left(\frac{1}{u}\right)\\ +\mathcal{J}^{\alpha}_{\frac{1}{u}^-}(\Omega\circ\psi)\left(\frac{1}{v}\right)\end{array}\right],$$

that is,

$$\left[\mathcal{J}^{\alpha}_{u^+}\tilde{\mathcal{Q}}\Omega\circ\psi(v)\tilde{+}\mathcal{J}^{\alpha}_{v^-}\tilde{\mathcal{Q}}\Omega\circ\psi(u)\right]$$
$$\preccurlyeq \frac{\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(v)}{2}\left[\mathcal{J}^{\alpha}_{\frac{1}{v}^+}(\Omega\circ\psi)\left(\frac{1}{u}\right)+\mathcal{J}^{\alpha}_{\frac{1}{u}^-}(\Omega\circ\psi)\left(\frac{1}{v}\right)\right]. \tag{39}$$

Theorem 3.4. *(First fuzzy fractional $H-H$ Fejér inequality) Let $\tilde{\mathcal{Q}}:[u,v]\to\mathbb{F}_0$ be a harmonically convex fuzzy-IVF with $u<v$, whose β levels define the family of IVFs $\mathcal{Q}_{\beta}:[u,v]\subset\mathbb{R}\to\mathcal{K}_C^+$ are given by $\mathcal{Q}_{\beta}(x)=\left[\mathcal{Q}_*(x,\beta),\mathcal{Q}^*(x,\beta)\right]$ for all $x\in[u,v]$, $\beta\in[0,1]$. If $\tilde{\mathcal{Q}}\in L\left([u,v],\mathbb{F}_0\right)$ and $\Omega:[u,v]\to\mathbb{R},\Omega\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}\right)=\Omega(x)\geq 0$, then*

$$\tilde{\mathcal{Q}}\left(\frac{2uv}{u+v}\right)\left[\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}(\Omega\circ\psi)\left(\frac{1}{u}\right)+\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}(\Omega\circ\psi)\left(\frac{1}{v}\right)\right]$$
$$\preccurlyeq\left[\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\left(\tilde{\mathcal{Q}}\Omega\circ\psi\right)\left(\frac{1}{u}\right)\tilde{+}\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\left(\tilde{\mathcal{Q}}\Omega\circ\psi\right)\left(\frac{1}{v}\right)\right] \tag{40}$$
$$\preccurlyeq\frac{\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(v)}{2}\left[\mathcal{J}^{\alpha}_{\frac{1}{v}^+}(\Omega\circ\psi)\left(\frac{1}{u}\right)+\mathcal{J}^{\alpha}_{\frac{1}{u}^-}(\Omega\circ\psi)\left(\frac{1}{v}\right)\right].$$

If $\tilde{\mathcal{Q}}$ is concave fuzzy-IVF then, inequality (40) is reversed.

Proof. Since $\tilde{\mathcal{Q}}$ is a harmonically convex fuzzy-IVF, then for $\beta\in[0,1]$, we have

$$\mathcal{Q}_*\left(\frac{2uv}{u+v},\beta\right)$$
$$\leq\frac{1}{2}\left(\mathcal{Q}_*\left(\frac{uv}{\rho u+(1-\rho)v},\beta\right)+\mathcal{Q}_*\left(\frac{uv}{(1-\rho)u+\rho v},\beta\right)\right). \tag{41}$$

Multiplying both sides by (41) by $\rho^{\alpha-1}\Omega\left(\frac{uv}{(1-\rho)u+\rho v}\right)$ and then integrating the resultant with respect to ρ over $[0,1]$, we obtain

$$\mathcal{Q}_*\left(\frac{2uv}{u+v},\beta\right)\int_0^1\rho^{\alpha-1}\Omega\left(\frac{uv}{(1-\rho)u+\rho v}\right)d\rho$$
$$\leq\frac{1}{2}\left(\begin{array}{l}\int_0^1\rho^{\alpha-1}\mathcal{Q}_*\left(\frac{uv}{\rho u+(1-\rho)v},\beta\right)\Omega\left(\frac{uv}{(1-\rho)u+\rho v}\right)d\rho\\ +\int_0^1\rho^{\alpha-1}\mathcal{Q}_*\left(\frac{uv}{(1-\rho)u+\rho v},\beta\right)\Omega\left(\frac{uv}{(1-\rho)u+\rho v}\right)d\rho\end{array}\right). \tag{42}$$

Let $x=\frac{uv}{\rho u+(1-\rho)v}$. Then, we have

$$2\left(\frac{uv}{v-u}\right)^{\alpha}\mathcal{Q}_*\left(\frac{2uv}{u+v},\beta\right)\int_{\frac{1}{v}}^{\frac{1}{u}}\left(x-\frac{1}{v}\right)^{\alpha-1}\Omega\left(\frac{1}{x},\beta\right)dx$$
$$\leq\left(\frac{uv}{v-u}\right)^{\alpha}\int_{\frac{1}{v}}^{\frac{1}{u}}\left(x-\frac{1}{v}\right)^{\alpha-1}\mathcal{Q}_*\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}},\beta\right)\Omega\left(\frac{1}{x}\right)dx$$
$$+\left(\frac{uv}{v-u}\right)^{\alpha}\int_{u}^{\frac{1}{u}}\left(x-\frac{1}{v}\right)^{\alpha-1}\mathcal{Q}_*\left(\frac{1}{x},\beta\right)\Omega\left(\frac{1}{x}\right)dx$$
$$=\left(\frac{uv}{v-u}\right)^{\alpha}\int_{\frac{1}{v}}^{\frac{1}{u}}\left(\frac{1}{u}-x\right)^{\alpha-1}\mathcal{Q}_*(x,\beta)\Omega\left(\frac{1}{\frac{1}{u}+\frac{1}{v}-\frac{1}{x}}\right)dx$$
$$+\left(\frac{uv}{v-u}\right)^{\alpha}\int_{\frac{1}{v}}^{\frac{1}{u}}\left(x-\frac{1}{v}\right)^{\alpha-1}\mathcal{Q}_*\left(\frac{1}{x},\beta\right)\Omega\left(\frac{1}{x}\right)dx$$
$$=\Gamma(\alpha)\left(\frac{uv}{v-u}\right)^{\alpha}\left[\mathcal{I}^{\alpha}_{\left(\frac{1}{v}\right)^+}\mathcal{Q}_*\Omega\left(\frac{1}{u}\right)+\mathcal{I}^{\alpha}_{\left(\frac{1}{u}\right)^-}\mathcal{Q}_*\Omega\left(\frac{1}{v}\right)\right], \tag{43}$$

Similarly, for $\mathcal{Q}^*(x,\gamma)$, we have

$$2\left(\frac{uv}{v-u}\right)^{\alpha}\mathcal{Q}^*\left(\frac{2uv}{u+v},\beta\right)\int_{\frac{1}{v}}^{\frac{1}{u}}\left(x-\frac{1}{v}\right)^{\alpha-1}\Omega\left(\frac{1}{x},\beta\right)dx$$
$$\leq\Gamma(\alpha)\left(\frac{uv}{v-u}\right)^{\alpha}\left[\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\mathcal{Q}^*\Omega\left(\frac{1}{u}\right)+\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\mathcal{Q}^*\Omega\left(\frac{1}{v}\right)\right]. \tag{44}$$

From (43) and (44), we have

$$\Gamma(\alpha)\left(\frac{uv}{v-u}\right)^{\alpha}\left[\mathcal{Q}_*\left(\frac{2uv}{u+v},\beta\right),\mathcal{Q}^*\left(\frac{2uv}{u+v},\beta\right)\right]$$
$$.\left[\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\Omega\left(\frac{1}{u}\right)+\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\Omega\left(\frac{1}{v}\right)\right]$$
$$\leq_I\Gamma(\alpha)\left(\frac{uv}{v-u}\right)^{\alpha}\left[\begin{array}{l}\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\mathcal{Q}_*\Omega\left(\frac{1}{u}\right)+\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\mathcal{Q}_*\Omega\left(\frac{1}{v}\right),\\ \mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\mathcal{Q}^*\Omega\left(\frac{1}{u}\right)+\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\mathcal{Q}^*\Omega\left(\frac{1}{v}\right)\end{array}\right],$$

that is,

$$\tilde{\mathcal{Q}}\left(\frac{2uv}{u+v}\right)\left[\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}(\Omega\circ\psi)\left(\frac{1}{u}\right)+\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}(\Omega\circ\psi)\left(\frac{1}{v}\right)\right]$$
$$\preccurlyeq\left[\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\left(\tilde{\mathcal{Q}}\Omega\circ\psi\right)\left(\frac{1}{u}\right)\tilde{+}\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\left(\tilde{\mathcal{Q}}\Omega\circ\psi\right)\left(\frac{1}{v}\right)\right]. \tag{45}$$

Similarly, if $\tilde{\mathcal{Q}}$ be a harmonically convex fuzzy-IVF and $\rho^{\alpha-1}\Omega\left(\frac{uv}{\rho u+(1-\rho)v}\right)\geq 0$, then, for each $\beta\in[0,1]$, we have

$$\rho^{\alpha-1}\mathcal{Q}_*\left(\frac{uv}{\rho u+(1-\rho)v},\beta\right)\Omega\left(\frac{uv}{\rho u+(1-\rho)v}\right)$$
$$\leq\rho^{\alpha-1}\left((1-\rho)\mathcal{Q}_*(u,\beta)+\rho\mathcal{Q}_*(v,\beta)\right)\Omega\left(\frac{uv}{\rho u+(1-\rho)v}\right) \tag{46}$$

and

$$\begin{aligned}&\varrho^{\alpha-1}\mathcal{Q}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right)\Omega\left(\tfrac{uv}{\varrho u+(1-\varrho)v}\right)\\&\leq\varrho^{\alpha-1}\left(\varrho\mathcal{Q}_*(u,\beta)+(1-\varrho)\mathcal{Q}_*(v,\beta)\right)\Omega\left(\tfrac{uv}{\varrho u+(1-\varrho)v}\right).\end{aligned}\tag{47}$$

After adding (46) and (47), and integrating the resultant over $[0,1]$, we get

$$\begin{aligned}&\int_0^1\varrho^{\alpha-1}\mathcal{Q}_*\left(\frac{uv}{\varrho u+(1-\varrho)v},\beta\right)\Omega\left(\frac{uv}{\varrho u+(1-\varrho)v}\right)d\varrho\\&+\int_0^1\varrho^{\alpha-1}\mathcal{Q}_*\left(\frac{uv}{(1-\varrho)u+\varrho v},\beta\right)\Omega\left(\frac{uv}{\varrho u+(1-\varrho)v}\right)d\varrho\\&\leq\int_0^1\left[\begin{array}{l}\varrho^{\alpha-1}\mathcal{Q}_*(u,\beta)\{\varrho+(1-\varrho)\}\left(\frac{uv}{\varrho u+(1-\varrho)v}\right)\\+\varrho^{\alpha-1}\mathcal{Q}_*(v,\beta)\{(1-\varrho)+\varrho\}\left(\frac{uv}{\varrho u+(1-\varrho)v}\right)\end{array}\right]d\varrho,\\&=\mathcal{Q}_*(u,\beta)\int_0^1\varrho^{\alpha-1}\Omega\left(\frac{uv}{\varrho u+(1-\varrho)v}\right)d\varrho\\&+\mathcal{Q}_*(v,\beta)\int_0^1\varrho^{\alpha-1}\Omega\left(\frac{uv}{\varrho u+(1-\varrho)v}\right)d\varrho.\end{aligned}$$

Similarly, for $\mathcal{Q}^*(x,\gamma)$, we have

$$\begin{aligned}&\int_0^1\varrho^{\alpha-1}\mathcal{Q}^*\left(\frac{uv}{\varrho u+(1-\varrho)v},\beta\right)\Omega\left(\frac{uv}{\varrho u+(1-\varrho)v}\right)d\varrho\\&+\int_0^1\varrho^{\alpha-1}\mathcal{Q}^*\left(\frac{uv}{(1-\varrho)u+\varrho v},\beta\right)\Omega\left(\frac{uv}{\varrho u+(1-\varrho)v}\right)d\varrho\\&=\mathcal{Q}^*(u,\beta)\int_0^1\varrho^{\alpha-1}\Omega\left(\frac{uv}{\varrho u+(1-\varrho)v}\right)d\varrho\\&+\mathcal{Q}^*(v,\beta)\int_0^1\varrho^{\alpha-1}\Omega\left(\frac{uv}{\varrho u+(1-\varrho)v}\right)d\varrho.\end{aligned}$$

From which, we have

$$\begin{aligned}&\Gamma(\alpha)\left(\tfrac{uv}{v-u}\right)^{\alpha}\left[\mathcal{J}^{\alpha}_{\frac{1}{v}^+}\mathcal{Q}_\beta\Omega\circ\psi(v)+\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\mathcal{Q}_\beta\Omega\circ\psi\left(\tfrac{1}{v}\right)\right]\\&\leq_I\Gamma(\alpha)\left(\tfrac{uv}{v-u}\right)^{\alpha}\tfrac{\mathcal{Q}_\beta(u)+\mathcal{Q}_\beta(v)}{2}\\&\quad\left[\mathcal{J}^{\alpha}_{\frac{1}{v}^+}(\Omega\circ\psi)\left(\tfrac{1}{u}\right)+\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}(\Omega\circ\psi)\left(\tfrac{1}{v}\right)\right],\end{aligned}$$

that is,

$$\begin{aligned}&\left[\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\tilde{\mathcal{Q}}\Omega\circ\psi\left(\tfrac{1}{u}\right)\tilde{+}\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\tilde{\mathcal{Q}}\Omega\circ\psi\left(\tfrac{1}{v}\right)\right]\\&\preccurlyeq\tfrac{\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(v)}{2}\left[\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}(\Omega\circ\psi)\left(\tfrac{1}{u}\right)+\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}(\Omega\circ\psi)\left(\tfrac{1}{v}\right)\right].\end{aligned}\tag{48}$$

By combining (45) and (48), we obtain the required inequality (40).

Remark 3.5. Let $\alpha=1$. Then from Theorems 3.3 and 3.4, we get following H-H inequality for harmonically convex fuzzy-IVF which is also new one:

$$\begin{aligned}\tilde{\mathcal{Q}}\left(\tfrac{2uv}{u+v}\right)\int_u^v\frac{\Omega(x)}{x^2}dx&\preccurlyeq\int_u^v\frac{\tilde{\mathcal{Q}}(x)}{x^2}\Omega(x)\,dx\\&\preccurlyeq\tfrac{\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(v)}{2}\int_u^v\frac{\Omega(x)}{x^2}dx.\end{aligned}$$

Let $\Omega(x)=1$. Then from Theorems 3.3 and 3.4, we obtain inequality (29).

Let $\Omega(x)=1$ and $\alpha=1$, then from Theorems 3.3 and 3.4, we get H-H inequality for harmonically convex fuzzy-IVF:

$$\tilde{\mathcal{Q}}\left(\frac{2uv}{u+v}\right)\preccurlyeq\frac{uv}{v-u}\int_u^v\frac{\tilde{\mathcal{Q}}(x)}{x^2}dx\preccurlyeq\frac{\tilde{\mathcal{Q}}(u)\tilde{+}\tilde{\mathcal{Q}}(v)}{2}.$$

If $\mathcal{Q}_*(x,\beta)=\mathcal{Q}^*(x,\beta)$ with $\beta=1$ then from Theorems 3.3 and 3.4, we obtain classical fractional H-H Fejér inequality for harmonically convex function, given in [10].

Let $\mathcal{Q}_*(x,\beta)=\mathcal{Q}^*(x,\beta)$ with $\beta=1$ and $\alpha=1$. Then, from Theorems 3.3 and 3.4, we obtain classical H-H -Fejér inequality for harmonically convex function, given in [4].

If $\mathcal{Q}_*(x,\beta)=\mathcal{Q}^*(x,\beta)$ with $\Omega(x)=\beta=1$ then from Theorems 3.3 and 3.4, we obtain classical fractional H-H inequality for harmonically convex function.

If $\mathcal{Q}_*(x,\beta)=\mathcal{Q}^*(x,\beta)$ and $\Omega(x)=\beta=\alpha=1$ then from Theorems 3.3 and 3.4, we obtain classical H-H inequality for harmonically convex function.

Now in next results, we will establish some H-H type inequalities for the products of two harmonically convex fuzzy-IVFs involving fuzzy-interval Riemann–Liouville fractional integral. These inequalities about harmonically convex fuzzy-IVFs are analogous generalization for some classical results provided by Noor [7], and Chen [6,13] for convex and generalized harmonically convex functions.

Theorem 3.6. *Let $\tilde{\mathcal{Q}},\tilde{\mathcal{P}}:[u,v]\to\mathbb{F}_0$ be two harmonically convex fuzzy-IVFs on $[u,v]$, whose β levels $\mathcal{Q}_\beta,\mathcal{P}_\beta:[u,v]\subset\mathbb{R}\to\mathcal{K}_C^+$ are defined by $\mathcal{Q}_\beta(x)=\left[\mathcal{Q}_*(x,\beta),\mathcal{Q}^*(x,\beta)\right]$ and $\mathcal{P}_\beta(x)=\left[\mathcal{P}_*(x,\beta),\mathcal{P}^*(x,\beta)\right]$ for all $x\in[u,v]$, $\beta\in[0,1]$. If $\tilde{\mathcal{Q}}\tilde{\times}\tilde{\mathcal{P}}\in L\left([u,v],\mathbb{F}_0\right)$, then*

$$\begin{aligned}&\tfrac{\Gamma(\alpha+1)}{2}\left(\tfrac{uv}{v-u}\right)^{\alpha}\left[\begin{array}{l}\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\tilde{\mathcal{Q}}\circ\psi\left(\tfrac{1}{u}\right)\tilde{\times}\tilde{\mathcal{P}}\circ\psi\left(\tfrac{1}{u}\right)\\+\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\tilde{\mathcal{Q}}\circ\psi\left(\tfrac{1}{v}\right)\tilde{\times}\tilde{\mathcal{P}}\circ\psi\left(\tfrac{1}{v}\right)\end{array}\right]\\&\preccurlyeq\left(\tfrac{1}{2}-\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)\tilde{M}(u,v)+\left(\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)\tilde{N}(u,v).\end{aligned}$$

where $\tilde{M}(u,v)=\tilde{\mathcal{Q}}(u)\tilde{\times}\tilde{\mathcal{P}}(u)\tilde{+}\tilde{\mathcal{Q}}(v)\tilde{\times}\tilde{\mathcal{P}}(v)$, $\tilde{N}(u,v)=\tilde{\mathcal{Q}}(u)\tilde{\times}\tilde{\mathcal{P}}(v)\tilde{+}\tilde{\mathcal{Q}}(v)\tilde{\times}\tilde{\mathcal{P}}(u)$, and $M_\beta(u,v)=\left[M_*((u,v),\beta),M^*((u,v),\beta)\right]$ and $N_\beta(u,v)=\left[N_*((u,v),\beta),N^*((u,v),\beta)\right]$.

Proof. Since $\tilde{\mathcal{Q}}, \tilde{\mathcal{P}}$ both are harmonically convex fuzzy-IVFs then, for each $\beta \in [0,1]$ we have

$$\mathcal{Q}_*\left(\frac{uv}{\varrho u+(1-\varrho)v},\beta\right) \leq (1-\varrho)\,\mathcal{Q}_*(u,\beta)+\varrho\mathcal{Q}_*(v,\beta)$$

and

$$\mathcal{P}_*\left(\frac{uv}{\varrho u+(1-\varrho)v},\beta\right) \leq (1-\varrho)\,\mathcal{P}_*(u,\beta)+\varrho\mathcal{P}_*(v,\beta).$$

From the definition of harmonically convex fuzzy-IVFs it follows that $\tilde{0} \preccurlyeq \tilde{\mathcal{Q}}(x)$ and $\tilde{0} \preccurlyeq \tilde{\mathcal{P}}(x)$, so

$$\begin{aligned}
&\mathcal{Q}_*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)\times\mathcal{P}_*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)\\
&\leq \left((1-\varrho)\,\mathcal{Q}_*(u,\beta)+\varrho\mathcal{Q}_*(v,\beta)\right)\begin{pmatrix}(1-\varrho)\,\mathcal{P}_*(u,\beta)\\+\varrho\mathcal{P}_*(v,\beta)\end{pmatrix}\\
&=(1-\varrho)^2\mathcal{Q}_*(u,\beta)\times\mathcal{P}_*(u,\beta)+\varrho^2\mathcal{Q}_*(v,\beta)\times\mathcal{P}_*(v,\beta)\\
&\quad+\varrho(1-\varrho)\,\mathcal{Q}_*(u,\beta)\times\mathcal{P}_*(v,\beta)\\
&\quad+\varrho(1-\varrho)\,\mathcal{Q}_*(v,\beta)\times\mathcal{P}_*(u,\beta)
\end{aligned}\tag{49}$$

Analogously, we have

$$\begin{aligned}
&\mathcal{Q}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right)\mathcal{P}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right)\\
&\leq \varrho^2\mathcal{Q}_*(u,\beta)\times\mathcal{P}_*(u,\beta)\\
&\quad+(1-\varrho)^2\mathcal{Q}_*(v,\beta)\times\mathcal{P}_*(v,\beta)\\
&\quad+\varrho(1-\varrho)\,\mathcal{Q}_*(u,\beta)\times\mathcal{P}_*(v,\beta)\\
&\quad+\varrho(1-\varrho)\,\mathcal{Q}_*(v,\beta)\times\mathcal{P}_*(u,\beta)
\end{aligned}\tag{50}$$

Adding (49) and (50), we have

$$\begin{aligned}
&\mathcal{Q}_*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)\times\mathcal{P}_*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)\\
&\quad+\mathcal{Q}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right)\times\mathcal{P}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right)\\
&\leq\left[\varrho^2+(1-\varrho)^2\right]\left[\mathcal{Q}_*(u,\beta)\times\mathcal{P}_*(u,\beta)+\mathcal{Q}_*(v,\beta)\times\mathcal{P}_*(v,\beta)\right]\\
&\quad+2\varrho(1-\varrho)\left[\mathcal{Q}_*(v,\beta)\times\mathcal{P}_*(u,\beta)+\mathcal{Q}_*(u,\beta)\times\mathcal{P}_*(v,\beta)\right]
\end{aligned}\tag{51}$$

Taking multiplication of (51) by $\varrho^{\alpha-1}$ and integrating the obtained result with respect to ϱ over $(0,1)$, we have

$$\begin{aligned}
&\int_0^1\varrho^{\alpha-1}\mathcal{Q}_*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)\times\mathcal{P}_*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)d\varrho\\
&\quad+\int_0^1\varrho^{\alpha-1}\mathcal{Q}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right)\times\mathcal{P}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right)d\varrho\\
&\leq M_*((u,v),\beta)\int_0^1\varrho^{\alpha-1}\left[\varrho^2+(1-\varrho)^2\right]d\varrho\\
&\quad+2N_*((u,v),\beta)\int_0^1\varrho^{\alpha-1}\varrho(1-\varrho)\,d\varrho.
\end{aligned}$$

It follows that,

$$\begin{aligned}
&\Gamma(\alpha)\left(\tfrac{uv}{v-u}\right)^{\alpha}\left[\begin{array}{l}\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\mathcal{Q}_*\left(\frac{1}{u},\beta\right)\times\mathcal{P}_*\left(\frac{1}{u},\beta\right)\\+\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\mathcal{Q}_*\left(\frac{1}{v},\beta\right)\times\mathcal{P}_*\left(\frac{1}{v},\beta\right)\end{array}\right]\\
&\leq\tfrac{2}{\alpha}\left(\tfrac{1}{2}-\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)M_*((u,v),\beta)\\
&\quad+\tfrac{2}{\alpha}\left(\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)N_*((u,v),\beta)
\end{aligned}$$

Similarly, for $\mathcal{Q}^*(x,\gamma)$, we have

$$\begin{aligned}
&\Gamma(\alpha)\left(\tfrac{uv}{v-u}\right)^{\alpha}\left[\begin{array}{l}\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\mathcal{Q}^*\left(\frac{1}{u},\beta\right)\times\mathcal{P}^*\left(\frac{1}{u},\beta\right)\\+\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\mathcal{Q}^*\left(\frac{1}{v},\beta\right)\times\mathcal{P}^*\left(\frac{1}{v},\beta\right)\end{array}\right]\\
&\leq\tfrac{2}{\alpha}\left(\tfrac{1}{2}-\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)M^*((u,v),\beta)\\
&\quad+\tfrac{2}{\alpha}\left(\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)N^*((u,v),\beta),
\end{aligned}$$

that is,

$$\begin{aligned}
&\Gamma(\alpha)\left(\tfrac{uv}{v-u}\right)^{\alpha}\left[\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\mathcal{Q}_*\left(\tfrac{1}{u},\beta\right)\times\mathcal{P}_*\left(\tfrac{1}{u},\beta\right)\right.\\
&+\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\mathcal{Q}_*\left(\tfrac{1}{v},\beta\right)\times\mathcal{P}_*\left(\tfrac{1}{v},\beta\right),\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\mathcal{Q}^*\left(\tfrac{1}{u},\beta\right)\\
&\left.\times\mathcal{P}^*\left(\tfrac{1}{u},\beta\right)+\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\mathcal{Q}^*\left(\tfrac{1}{v},\beta\right)\times\mathcal{P}^*\left(\tfrac{1}{v},\beta\right)\right]\\
&\leq_I\tfrac{2}{\alpha}\left(\tfrac{1}{2}-\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left[M_*((u,v),\beta),M^*((u,v),\beta)\right]\\
&\quad+\tfrac{2}{\alpha}\left(\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left[N_*((u,v),\beta),N^*((u,v),\beta)\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
&\tfrac{\Gamma(\alpha+1)}{2}\left(\tfrac{uv}{v-u}\right)^{\alpha}\left[\begin{array}{l}\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\tilde{\mathcal{Q}}\circ\psi\left(\frac{1}{u}\right)\tilde{\times}\tilde{\mathcal{P}}\circ\psi\left(\frac{1}{u}\right)\\+\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\tilde{\mathcal{Q}}\circ\psi\left(\frac{1}{v}\right)\tilde{\times}\tilde{\mathcal{P}}\circ\psi\left(\frac{1}{v}\right)\end{array}\right]\\
&\preccurlyeq\left(\tfrac{1}{2}-\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)\tilde{M}(u,v)+\left(\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)\tilde{N}(u,v).
\end{aligned}$$

and the theorem has been established.

Theorem 3.7. *Let $\tilde{\mathcal{Q}},\tilde{\mathcal{P}}:[u,v]\to\mathbb{F}_0$ be two harmonically convex fuzzy-IVFs, whose β levels define the family of IVFs $\mathcal{Q}_\beta,\mathcal{P}_\beta:[u,v]\subset\mathbb{R}\to\mathcal{K}_C^+$ are given by $\mathcal{Q}_\beta(x)=\left[\mathcal{Q}_*(x,\beta),\mathcal{Q}^*(x,\beta)\right]$ and $\mathcal{P}_\beta(x)=\left[\mathcal{P}_*(x,\beta),\mathcal{P}^*(x,\beta)\right]$ for all $x\in[u,v]$, $\beta\in[0,1]$. If $\tilde{\mathcal{Q}}\tilde{\times}\tilde{\mathcal{P}}\in L\left([u,v],\mathbb{F}_0\right)$, then*

$$\begin{aligned}
&\tilde{\mathcal{Q}}\left(\tfrac{2uv}{u+v}\right)\tilde{\times}\tilde{\mathcal{P}}\left(\tfrac{2uv}{u+v}\right)\\
&\preccurlyeq\tfrac{\Gamma(\alpha+1)}{4}\left(\tfrac{uv}{v-u}\right)^{\alpha}\left[\begin{array}{l}\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\tilde{\mathcal{Q}}\left(\frac{1}{u}\right)\tilde{\times}\tilde{\mathcal{P}}\left(\frac{1}{u}\right)\\+\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\tilde{\mathcal{Q}}\left(\frac{1}{v}\right)\tilde{\times}\tilde{\mathcal{P}}\left(\frac{1}{v}\right)\end{array}\right]\\
&\quad+\tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)\tilde{N}(u,v)\\
&\quad+\tfrac{1}{2}\left(\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)\tilde{M}(u,v).
\end{aligned}$$

where $\tilde{M}(u,v) = \tilde{\mathcal{Q}}(u)\tilde{\times}\tilde{\mathcal{P}}(u)\tilde{+}\tilde{\mathcal{Q}}(v)\tilde{\times}\tilde{\mathcal{P}}(v)$, $\tilde{N}(u,v) = \tilde{\mathcal{Q}}(u)\tilde{\times}\tilde{\mathcal{P}}(v)\tilde{+}\tilde{\mathcal{Q}}(v)\tilde{\times}\tilde{\mathcal{P}}(u)$, *and* $M_\beta(u,v) = \left[M_*((u,v),\beta), M^*((u,v),\beta)\right]$ *and* $N_\beta(u,v) = \left[N_*((u,v),\beta), N^*((u,v),\beta)\right]$.

Proof. Consider $\tilde{\mathcal{Q}},\tilde{\mathcal{P}} : [u,v] \to \mathbb{F}_0$ are harmonically convex fuzzy-IVFs. Then by hypothesis, for each $\beta \in [0,1]$, we have

$$\begin{aligned}
&\mathcal{Q}_*\left(\tfrac{2uv}{u+v},\beta\right)\times\mathcal{P}_*\left(\tfrac{2uv}{u+v},\beta\right)\\
&\le \frac{1}{4}\left[\begin{array}{l}\mathcal{Q}_*\left(\frac{uv}{\rho u+(1-\rho)v},\beta\right)\times\mathcal{P}_*\left(\frac{uv}{\rho u+(1-\rho)v},\beta\right)\\ +\mathcal{Q}_*\left(\frac{uv}{\rho u+(1-\rho)v},\beta\right)\times\mathcal{P}_*\left(\frac{uv}{(1-\rho)u+\rho v},\beta\right)\end{array}\right]\\
&\quad+\frac{1}{4}\left[\begin{array}{l}\mathcal{Q}_*\left(\frac{uv}{(1-\rho)u+\rho v},\beta\right)\times\mathcal{P}_*\left(\frac{uv}{\rho u+(1-\rho)v},\beta\right)\\ +\mathcal{Q}_*\left(\frac{uv}{(1-\rho)u+\rho v},\beta\right)\times\mathcal{P}_*\left(\frac{uv}{(1-\rho)u+\rho v},\beta\right)\end{array}\right],\\
&\le \frac{1}{4}\left[\begin{array}{l}\mathcal{Q}_*\left(\frac{uv}{\rho u+(1-\rho)v},\beta\right)\times\mathcal{P}_*\left(\frac{uv}{\rho u+(1-\rho)v},\beta\right)\\ +\mathcal{Q}_*\left(\frac{uv}{(1-\rho)u+\rho v},\beta\right)\times\mathcal{P}_*\left(\frac{uv}{(1-\rho)u+\rho v},\beta\right)\end{array}\right]\\
&\quad+\frac{1}{4}\left[\begin{array}{l}\left(\rho\mathcal{Q}_*(u,\beta)+(1-\rho)\mathcal{Q}_*(v,\beta)\right)\\ \times\left((1-\rho)\mathcal{P}_*(u,\beta)+\rho\mathcal{P}_*(v,\beta)\right)\\ +\left((1-\rho)\mathcal{Q}_*(u,\beta)+\rho\mathcal{Q}_*(v,\beta)\right)\\ \times\left(\rho\mathcal{P}_*(u,\beta)+(1-\rho)\mathcal{P}_*(v,\beta)\right)\end{array}\right],\\
&= \frac{1}{4}\left[\begin{array}{l}\mathcal{Q}_*\left(\frac{uv}{\rho u+(1-\rho)v},\beta\right)\times\mathcal{P}_*\left(\frac{uv}{\rho u+(1-\rho)v},\beta\right)\\ +\mathcal{Q}_*\left(\frac{uv}{(1-\rho)u+\rho v},\beta\right)\times\mathcal{P}_*\left(\frac{uv}{(1-\rho)u+\rho v},\beta\right)\end{array}\right]\\
&\quad+\frac{1}{4}\left[\begin{array}{l}\left\{\rho^2+(1-\rho)^2\right\}N_*((u,v),\beta)\\ +\left\{\rho(1-\rho)+(1-\rho)\rho\right\}M_*((u,v),\beta)\end{array}\right]
\end{aligned}\tag{52}$$

Multiplying inequality (52) by $\rho^{\alpha-1}$ and integrating over $(0,1)$,

$$\begin{aligned}
&\mathcal{Q}_*\left(\tfrac{2uv}{u+v},\beta\right)\times\mathcal{P}_*\left(\tfrac{2uv}{u+v},\beta\right)\\
&\le \frac{1}{4}\left[\begin{array}{c}\int_0^1 \rho^{\alpha-1}\mathcal{Q}_*\left(\frac{uv}{\rho u+(1-\rho)v},\beta\right)\\ \times\mathcal{P}_*\left(\frac{uv}{\rho u+(1-\rho)v},\beta\right)d\rho\\ +\int_0^1 \rho^{\alpha-1}\mathcal{Q}_*\left(\frac{uv}{(1-\rho)u+\rho v},\beta\right)\\ \times\mathcal{P}_*\left(\frac{uv}{(1-\rho)u+\rho v},\beta\right)\end{array}\right]d\rho\\
&\quad+\left[\begin{array}{c}\frac{1}{4}N_*((u,v),\beta)\int_0^1 \rho^{\alpha-1}\left[\rho^2+(1-\rho)^2\right]d\rho\\ +2M_*((u,v),\beta)\int_0^1 \rho^{\alpha-1}\rho(1-\rho)\,d\rho\end{array}\right]
\end{aligned}$$

Taking $z = \frac{uv}{\rho u+(1-\rho)v}$ and $y = \frac{uv}{(1-\rho)u+\rho v}$, then we get

$$\begin{aligned}
&\frac{1}{\alpha}\mathcal{Q}_*\left(\tfrac{2uv}{u+v},\beta\right)\times\mathcal{P}_*\left(\tfrac{2uv}{u+v},\beta\right)\\
&\le \frac{\Gamma(\alpha)}{4}\left(\frac{uv}{v-u}\right)^\alpha\left[\begin{array}{l}\mathcal{J}^\alpha_{\left(\frac{1}{v}\right)^+}\mathcal{Q}_*\circ\psi\left(\frac{1}{u}\right)\times\mathcal{P}_*\circ\psi\left(\frac{1}{u}\right)\\ +\mathcal{J}^\alpha_{\left(\frac{1}{u}\right)^-}\mathcal{Q}_*\circ\psi\left(\frac{1}{v}\right)\times\mathcal{P}_*\circ\psi\left(\frac{1}{v}\right)\end{array}\right]\\
&\quad+\frac{1}{2\alpha}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)N_*((u,v),\beta)\\
&\quad+\frac{1}{2\alpha}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)M_*((u,v),\beta),
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\alpha}\mathcal{Q}^*\left(\tfrac{2uv}{u+v},\beta\right)\times\mathcal{P}^*\left(\tfrac{2uv}{u+v},\beta\right)\\
&\le \frac{\Gamma(\alpha)}{4}\left(\frac{uv}{v-u}\right)^\alpha\left[\begin{array}{l}\mathcal{J}^\alpha_{\left(\frac{1}{v}\right)^+}\mathcal{Q}^*\circ\psi\left(\frac{1}{u}\right)\times\mathcal{P}^*\circ\psi\left(\frac{1}{u}\right)+\\ +\mathcal{J}^\alpha_{\left(\frac{1}{u}\right)^-}\mathcal{Q}^*\circ\psi\left(\frac{1}{v},\beta\right)\times\mathcal{P}^*\circ\psi\left(\frac{1}{v},\beta\right)\end{array}\right]\\
&\quad+\frac{1}{2\alpha}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)N^*((u,v),\beta)\\
&\quad+\frac{1}{2\alpha}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)M^*((u,v),\beta),
\end{aligned}$$

Similarly, for $\mathcal{Q}^*(z,\gamma)$, we have

$$\begin{aligned}
&\frac{1}{\alpha}\mathcal{Q}^*\left(\tfrac{2uv}{u+v},\beta\right)\times\mathcal{P}^*\left(\tfrac{2uv}{u+v},\beta\right)\\
&\le \frac{\Gamma(\alpha)}{4}\left(\frac{uv}{v-u}\right)^\alpha\left[\begin{array}{l}\mathcal{J}^\alpha_{\left(\frac{1}{v}\right)^+}\mathcal{Q}^*\circ\psi\left(\frac{1}{u}\right)\times\mathcal{P}^*\circ\psi\left(\frac{1}{u}\right)\\ +\mathcal{J}^\alpha_{\left(\frac{1}{u}\right)^-}\mathcal{Q}^*\circ\psi\left(\frac{1}{v},\beta\right)\times\mathcal{P}^*\circ\psi\left(\frac{1}{v},\beta\right)\end{array}\right]\\
&\quad+\frac{1}{2\alpha}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)N^*((u,v),\beta)\\
&\quad+\frac{1}{2\alpha}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)M^*((u,v),\beta),
\end{aligned}$$

that is,

$$\begin{aligned}
&\tilde{\mathcal{Q}}\left(\tfrac{2uv}{u+v}\right)\tilde{\times}\tilde{\mathcal{P}}\left(\tfrac{2uv}{u+v}\right)\\
&\preccurlyeq \frac{\Gamma(\alpha+1)}{4}\left(\frac{uv}{v-u}\right)^\alpha\left[\begin{array}{l}\mathcal{J}^\alpha_{\left(\frac{1}{v}\right)^+}\tilde{\mathcal{Q}}\left(\frac{1}{u}\right)\tilde{\times}\tilde{\mathcal{P}}\left(\frac{1}{u}\right)\\ +\mathcal{J}^\alpha_{\left(\frac{1}{u}\right)^-}\tilde{\mathcal{Q}}\left(\frac{1}{v}\right)\tilde{\times}\tilde{\mathcal{P}}\left(\frac{1}{v}\right)\end{array}\right]\\
&\quad+\frac{1}{2}\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\tilde{N}(u,v)\\
&\quad+\frac{1}{2}\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\tilde{M}(u,v).
\end{aligned}$$

Hence, the required result.

Theorem 3.8. *Let* $\tilde{\mathcal{Q}},\tilde{\mathcal{P}} : [u,v] \to \mathbb{F}_0$ *be two harmonically convex fuzzy-IVFs, whose* β *levels define the family of IVFs* $\mathcal{Q}_\beta,\mathcal{P}_\beta : [u,v] \subset \mathbb{R} \to \mathcal{K}_C^+$ *are given by* $\mathcal{Q}_\beta(z) = \left[\mathcal{Q}_*(z,\beta),\mathcal{Q}^*(z,\beta)\right]$ *and* $\mathcal{P}_\beta(z) = \left[\mathcal{P}_*(z,\beta),\mathcal{P}^*(z,\beta)\right]$ *for all* $z \in [u,v]$, $\beta \in [0,1]$. *If* $\tilde{\mathcal{Q}}\tilde{\times}\tilde{\mathcal{P}} \in L\left([u,v],\mathbb{F}_0\right)$, *then*

$$\begin{aligned}
&2\tilde{\mathcal{Q}}\left(\tfrac{2uv}{u+v}\right)\tilde{\times}\tilde{\mathcal{P}}\left(\tfrac{2uv}{u+v}\right)\\
&\preccurlyeq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{uv}{v-u}\right)^\alpha\left[\begin{array}{l}\mathcal{J}^\alpha_{\left(\frac{u+v}{2uv}\right)^+}\tilde{\mathcal{Q}}\circ\psi\left(\frac{1}{u}\right)\tilde{\times}\tilde{\mathcal{P}}\circ\psi\left(\frac{1}{u}\right)\\ +\mathcal{J}^\alpha_{\left(\frac{u+v}{2uv}\right)^-}\tilde{\mathcal{Q}}\circ\psi\left(\frac{1}{v}\right)\tilde{\times}\tilde{\mathcal{P}}\circ\psi\left(\frac{1}{v}\right)\end{array}\right]\\
&\quad+\left(\frac{1}{2}-\frac{\alpha^2+3\alpha}{4(\alpha+1)(\alpha+2)}\right)\tilde{N}(u,v)\\
&\quad+\frac{\alpha^2+3\alpha}{4(\alpha+1)(\alpha+2)}\tilde{M}(u,v).
\end{aligned}$$

where $\tilde{M}(u,v) = \tilde{\mathcal{Q}}(u)\tilde{\times}\tilde{\mathcal{P}}(u)\tilde{+}\tilde{\mathcal{Q}}(v)\tilde{\times}\tilde{\mathcal{P}}(v)$, $\tilde{N}(u,v) = \tilde{\mathcal{Q}}(u)\tilde{\times}\tilde{\mathcal{P}}(v)\tilde{+}\tilde{\mathcal{Q}}(v)\tilde{\times}\tilde{\mathcal{P}}(u)$, *and* $M_\beta(u,v) = \left[M_*((u,v),\beta), M^*((u,v),\beta)\right]$ *and* $N_\beta(u,v) = \left[N_*((u,v),\beta), N^*((u,v),\beta)\right]$.

Proof. Consider $\tilde{\mathcal{Q}}, \tilde{\mathcal{P}} : [u,v] \to \mathbb{F}_0$ are harmonically convex fuzzy-IVFs. Then by hypothesis, for each $\beta \in [0,1]$, we have

$$\begin{aligned}
&\mathcal{Q}_*\left(\tfrac{2uv}{u+v},\beta\right)\times\mathcal{P}_*\left(\tfrac{2uv}{u+v},\beta\right)\\
&\le \tfrac{1}{4}\begin{bmatrix}\mathcal{Q}_*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)\times\mathcal{P}_*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)\\ +\mathcal{Q}_*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)\times\mathcal{P}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right)\end{bmatrix}\\
&\quad+\tfrac{1}{4}\begin{bmatrix}\mathcal{Q}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right)\times\mathcal{P}_*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)\\ +\mathcal{Q}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right)\times\mathcal{P}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right)\end{bmatrix},\\
&\le \tfrac{1}{4}\begin{bmatrix}\mathcal{Q}_*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)\times\mathcal{P}_*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)\\ +\mathcal{Q}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right)\times\mathcal{P}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right)\end{bmatrix}\\
&\quad+\tfrac{1}{4}\begin{bmatrix}\left(\varrho\mathcal{Q}_*(u,\beta)+(1-\varrho)\mathcal{Q}_*(v,\beta)\right)\\ \times\left((1-\varrho)\mathcal{P}_*(u,\beta)+\varrho\mathcal{P}_*(v,\beta)\right)\\ +\left((1-\varrho)\mathcal{Q}_*(u,\beta)+\varrho\mathcal{Q}_*(v,\beta)\right)\\ \times\left(\varrho\mathcal{P}_*(u,\beta)+(1-\varrho)\mathcal{P}_*(v,\beta)\right)\end{bmatrix},\\
&= \tfrac{1}{4}\begin{bmatrix}\mathcal{Q}_*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)\times\mathcal{P}_*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)\\ +\mathcal{Q}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right)\times\mathcal{P}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right)\end{bmatrix}\\
&\quad+\tfrac{1}{4}\begin{bmatrix}\left\{\varrho^2+(1-\varrho)^2\right\}N_*((u,v),\beta)\\ +2\varrho(1-\varrho)M_*((u,v),\beta)\end{bmatrix}
\end{aligned}\tag{53}$$

Multiplying inequality (53) by $2^{1+\alpha}\alpha\varrho^{\alpha-1}$ and then integrating the obtain outcome over $\left[0,\frac{1}{2}\right]$,

$$\begin{aligned}
&\mathcal{Q}_*\left(\tfrac{2uv}{u+v},\beta\right)\times\mathcal{P}_*\left(\tfrac{2uv}{u+v},\beta\right)\\
&\le \tfrac{1}{4}\int_0^{\frac{1}{2}}2^{1+\alpha}\alpha\varrho^{\alpha-1}\begin{bmatrix}\mathcal{Q}_*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)\\ \times\mathcal{P}_*\left(\tfrac{uv}{\varrho u+(1-\varrho)v},\beta\right)\\ +\mathcal{Q}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right)\\ \times\mathcal{P}_*\left(\tfrac{uv}{(1-\varrho)u+\varrho v},\beta\right)\end{bmatrix}d\varrho.\\
&\quad+\tfrac{1}{4}\begin{bmatrix}N_*((u,v),\beta)\displaystyle\int_0^{\frac{1}{2}}2^{1+\alpha}\alpha\varrho^{\alpha-1}\left[\varrho^2+(1-\varrho)^2\right]d\varrho\\ +2M_*((u,v),\beta)\displaystyle\int_0^{\frac{1}{2}}2^{1+\alpha}\alpha\varrho^{\alpha-1}\varrho(1-\varrho)\,d\varrho\end{bmatrix}
\end{aligned}$$

Taking $z = \frac{uv}{\varrho u+(1-\varrho)v}$ and $y = \frac{uv}{(1-\varrho)u+\varrho v}$, then we get

$$\begin{aligned}
&2\mathcal{Q}_*\left(\tfrac{2uv}{u+v},\beta\right)\times\mathcal{P}_*\left(\tfrac{2uv}{u+v},\beta\right)\\
&\le \tfrac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\tfrac{uv}{v-u}\right)^{\alpha}\begin{bmatrix}\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\mathcal{Q}_*\circ\psi\left(\tfrac{1}{u}\right)\times\mathcal{P}_*\circ\psi\left(\tfrac{1}{u}\right)\\ +\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\mathcal{Q}_*\circ\psi\left(\tfrac{1}{v}\right)\times\mathcal{P}_*\circ\psi\left(\tfrac{1}{v}\right)\end{bmatrix}\\
&\quad+\left(\tfrac{1}{2}-\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)N_*((u,v),\beta)\\
&\quad+\left(\tfrac{\alpha}{(\alpha+1)(\alpha+2)}\right)M_*((u,v),\beta)
\end{aligned}$$

Similarly, for $\mathcal{Q}^*(z,\gamma)$, we have

$$\begin{aligned}
&2\mathcal{Q}^*\left(\tfrac{2uv}{u+v},\beta\right)\times\mathcal{P}^*\left(\tfrac{2uv}{u+v},\beta\right)\\
&\le \tfrac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\tfrac{uv}{v-u}\right)^{\alpha}\begin{bmatrix}\mathcal{J}^{\alpha}_{\left(\frac{1}{v}\right)^+}\mathcal{Q}^*\circ\psi\left(\tfrac{1}{u}\right)\times\mathcal{P}^*\circ\psi\left(\tfrac{1}{u}\right)\\ +\mathcal{J}^{\alpha}_{\left(\frac{1}{u}\right)^-}\mathcal{Q}^*\circ\psi\left(\tfrac{1}{v}\right)\times\mathcal{P}^*\circ\psi\left(\tfrac{1}{v},\beta\right)\end{bmatrix}\\
&\quad+\left(\tfrac{1}{2}-\tfrac{\alpha^2+3\alpha}{4(\alpha+1)(\alpha+2)}\right)N^*((u,v),\beta)\\
&\quad+\tfrac{\alpha^2+3\alpha}{4(\alpha+1)(\alpha+2)}M^*((u,v),\beta),
\end{aligned}$$

that is,

$$\begin{aligned}
&2\tilde{\mathcal{Q}}\left(\tfrac{2uv}{u+v}\right)\tilde{\times}\tilde{\mathcal{P}}\left(\tfrac{2uv}{u+v}\right)\\
&\preccurlyeq \tfrac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\tfrac{uv}{v-u}\right)^{\alpha}\begin{bmatrix}\mathcal{J}^{\alpha}_{\left(\frac{u+v}{2uv}\right)^+}\tilde{\mathcal{Q}}\circ\psi\left(\tfrac{1}{u}\right)\tilde{\times}\tilde{\mathcal{P}}\circ\psi\left(\tfrac{1}{u}\right)\\ \tilde{+}\mathcal{J}^{\alpha}_{\left(\frac{u+v}{2uv}\right)^-}\tilde{\mathcal{Q}}\circ\psi\left(\tfrac{1}{v}\right)\tilde{\times}\tilde{\mathcal{P}}\circ\psi\left(\tfrac{1}{v}\right)\end{bmatrix}\\
&\quad+\left(\tfrac{1}{2}-\tfrac{\alpha^2+3\alpha}{4(\alpha+1)(\alpha+2)}\right)\tilde{N}(u,v)+\tfrac{\alpha^2+3\alpha}{4(\alpha+1)(\alpha+2)}\tilde{M}(u,v).
\end{aligned}$$

4. CONCLUSION AND FUTURE STUDY

In this study, firstly we introduced the class of harmonically convex fuzzy-IVFs by means of fuzzy-order relation. Then we established *H-H* and *H-H* Fejér type inequalities for convex fuzzy-IVFs involving fuzzy Riemann–Liouville fractional integrals, and *H-H* inequalities are true for this concept of harmonically convex fuzzy-IVFs. As a future research, we try to explore this concept for generalized harmonically convex fuzzy-IVFs and some applications in fuzzy-interval nonlinear programing. By using this concept, the new direction of study can be found in optimization theory and convex analysis. We hope that this concept will be helpful for other authors to pay their roles in different fields of sciences.

AVAILABILITY OF DATA AND MATERIALS

Not applicable.

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

AUTHORS' CONTRIBUTIONS

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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