

## Research Article

# Higher-Order Strongly Preinvex Fuzzy Mappings and Fuzzy Mixed Variational-Like Inequalities

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## ABSTRACT

A family of fuzzy mappings is called higher-order strongly preinvex fuzzy mappings (HOS-preinvex fuzzy mappings), which take the place of generalization of the notion of nonconvexity is introduced through the “fuzzy-max” order among fuzzy numbers. This family properly includes the family of preinvex fuzzy mappings and is included in the family of quasi preinvex fuzzy mappings. With the support of examples, we have discussed some special cases. Some properties are derived and relations among the HOS-preinvex fuzzy mappings, HOS-invex fuzzy mappings, and fuzzy HOS-monotonicity are obtained under some mild conditions. Then, we have shown that optimality conditions of generalized differentiable HOS-preinvex fuzzy mappings and for the sum of generalized differentiable (briefly, G-differentiable) preinvex fuzzy mappings and nongeneralized differentiable HOS-preinvex fuzzy mappings can be distinguished by HOS-fuzzy variational-like inequalities and HOS-fuzzy mixed variational-like inequalities, respectively which can be viewed as novel applications. These inequalities are very interesting outcome of our main results and appear to be new ones. Several exceptional cases are debated. Presented results in this paper can be considered and development of previously established results.

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## 1. INTRODUCTION

In past few decades, the ideas of convexity and nonconvexity are well acknowledged in optimization concepts and gifted a vital role in operation research, economics, decision-making, and management. Hanson [1] initiated to introduce a generalized class of convexity which is known as an invex function. The invex function played a significant role in mathematical programming. A step forward, the invex set and preinvex function were introduced and studied by Israel and Mond [2]. Also, Noor [3] examined the optimality conditions of differentiable preinvex functions and proved that variational-like inequalities would characterize the minimum. Many classical convexity generalizations and extensions have been investigated by several authors. The concept of strongly convex functions on the convex set was studied and implemented by Polyak [4], which plays a crucial role in the theory of optimization. The unique existence of a solution to nonlinear complementary problems was explored by Karmardian [5] by the use of strongly convex functions. Qu and Li [6] and Rufián-Lizana *et al.* [7] studied a convergence analysis to solve equilibrium problems and variational inequalities with the help of strongly convex functions. Noor and Noor [8–11] derived the useful properties of strongly preinvex function and investigated its applications. For further study, we

refer to the readers about applications and properties of the convex functions and generalized convex functions, see [12–16] and the references therein. The notions of higher-order strongly (HOS)-preinvex functions were initiated by Lin and Fokushim [17], and utilized these notions in the analysis of mathematical programming with equilibrium constraints. HOS-preinvex functions and their generalization have many applications in different areas such as multilevel games, engineering design, and economical equilibrium. Some of the authors presented different types of HOS-preinvex functions and discussed their characterizations like Alabdali *et al.* [18], Noor and Noor [9,19,20], and Mohsen *et al.* [21] considered different classes and prove that the optimality condition of HOS-preinvex functions can be distinguished by different kinds of variational inequalities like strongly variational inequality, strongly variational-like inequality, HOS-variational inequality, and HOS-variational-like inequality.

Similarly, the notions of convexity and nonconvexity play a vital role in optimization under fuzzy domain because during characterization of the optimality condition of convexity, we obtain fuzzy variational inequalities so variational inequality theory and fuzzy complementary problem theory established powerful mechanism of the mathematical problems and they have a friendly relationship. Many authors contributed to this fascinating and interesting field. In 1989, Nanda and Kar [22] were initiated to introduce convex fuzzy mappings and characterized the notion of convex fuzzy

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mapping through the idea of epigraph. A step forward, Furukawa [23] and Syau [24] proposed and examined fuzzy mapping from space  $\mathbb{R}^n$  to the set of fuzzy numbers, fuzzy valued Lipschitz continuity, logarithmic convex fuzzy mappings, and quasi-convex fuzzy mappings. Besides, Chang [25] discussed the idea of convex fuzzy mapping and find its optimality condition with the support of fuzzy variational inequality. Generalization and extension of fuzzy convexity play a vital and significant implementation in diverse directions. So let's note that, one of the most considered classes of nonconvex fuzzy mapping is preinvex fuzzy mapping. Noor [26] introduced this idea and proved some results that distinguish the fuzzy optimality condition of differentiable fuzzy preinvex mappings by fuzzy variational-like inequality. Fuzzy variational inequality theory and complementary problem theory established a strong relationship with mathematical problems. Recently, Li and Noor [27] established an equivalence condition of preinvex fuzzy mapping and characterizations about preinvex fuzzy mappings under some mild conditions. With the support of examples, Wu and Xu [28] updated the definition of convex fuzzy mappings and established a new approach regarding to the existence of a fuzzy preinvex mapping under the condition of lower or upper semicontinuity. In 2012, Rufian-Lizana *et al.* [7] reviewed the existing literature and made appropriate modifications to the results obtained by Wu and Xu [28] regarding invex fuzzy mappings. For differentiable and twice differentiable preinvex fuzzy mappings, Rufian-Lizana *et al.* [29] given the required and sufficient conditions. They demonstrated the validity of characterizations with help of examples, and improved the previous results provided by Li and Noor [27]. We refer to the readers for further analysis of literature on the applications and properties of variational-like inequalities and generalized convex fuzzy mappings, see [2,9,16,30–47] and the references therein.

Motivated and inspired by the ongoing research work, we note that convex and generalized convex fuzzy mappings play an important role in fuzzy optimization. The paper is organized as follows. Section 2 recalls some basic definitions, preliminary notations, and results which will be helpful for further study. Section 3 introduces and considers a family of classes of nonconvex fuzzy mappings is called HOS-preinvex fuzzy mappings and investigates some properties. Section 4 derives some relations among the HOS-preinvex fuzzy mappings, HOS-invex fuzzy mappings, and HOS-monotonicity under some mild conditions. Section 5 introduces the new classes of fuzzy variational-like inequality which is known as HOS-fuzzy variational-like inequality and HOS-fuzzy mixed variational-like inequality. Several special cases are also discussed. These inequalities are an interesting outcome of our main results.

## 2. PRELIMINARIES

Let  $\mathbb{R}$  be the set of real numbers whose standard element is symbolized by  $u$ . A fuzzy set  $\varphi$  in  $\mathbb{R}$  is a mapping  $\varphi : \mathbb{R} \rightarrow [0, 1]$  and support of  $\varphi$  is denoted by  $\text{supp}(\varphi)$  and is defined as

$$\text{supp}(\varphi) = \{u \in \mathbb{R} | \varphi(u) > 0\}.$$

**Definition 2.1.** If  $\varphi$  be a fuzzy set in  $\mathbb{R}$  and  $\gamma \in [0, 1]$ , then  $\gamma$ -level sets of  $\varphi$  is denoted and is defined as follows:

$$\varphi_\gamma = \{u \in \mathbb{R} | \varphi(u) \geq \gamma\}. \quad (1)$$

**Definition 2.2.** A fuzzy number  $\varphi$  is a fuzzy set in  $\mathbb{R}$  with the following properties:

- (1)  $\varphi$  is normal, i.e., there exists  $u \in \mathbb{R}$  such that  $\varphi(u) = 1$ ;
- (2)  $\varphi$  is upper semi continuous;
- (3)  $\varphi((1 - \tau)u + \tau\vartheta) \geq \min(\varphi(u), \varphi(\vartheta))$ ,  $\forall u, \vartheta \in \mathbb{R}$ ,  $\tau \in [0, 1]$ ;
- (4)  $\text{supp}(\varphi)$  is compact.

The  $LR$ -fuzzy numbers first introduced by Dubois and Prade [34] are defined as follows:

**Definition 2.3.** Let  $L, R : [0, 1] \rightarrow [0, 1]$  be two decreasing and upper semicontinuous functions with  $L(0) = R(0) = 1$  and  $L(1) = R(1) = 0$ . Then the fuzzy number is defined as

$$\varphi(u) = \begin{cases} L\left(\frac{c-u}{\rho}\right), & c - \rho \leq u < d, \\ 1, & c \leq u < d, \\ L\left(\frac{u-d}{r}\right), & c \leq u < d + r, \\ 0, & \text{otherwise,} \end{cases}$$

where  $r, \rho > 0$ , and  $d \geq c$ .

Let  $\mathfrak{F}_0$  denotes the set of all fuzzy numbers and let  $\varphi \in \mathfrak{F}_0$  be fuzzy number, if and only if,  $\gamma$ -levels  $\varphi_\gamma$  is a nonempty compact convex set of  $\mathbb{R}$ . This is represented by

$$\varphi_\gamma = [\varphi_*(\gamma), \varphi^*(\gamma)]$$

where

$$\varphi_*(\gamma) = \inf\{u \in \mathbb{R} | \varphi(u) \geq \gamma\},$$

$$\varphi^*(\gamma) = \sup\{u \in \mathbb{R} | \varphi(u) \geq \gamma\}.$$

Since each  $\rho \in \mathbb{R}$  is also a fuzzy number, defined as

$$\bar{\rho}(u) = \begin{cases} 1 & \text{if } u = \rho \\ 0 & \text{if } u \neq \rho \end{cases}.$$

Thus a fuzzy number  $\varphi$  can be identified by a parametrized triples

$$\{(\varphi_*(\gamma), \varphi^*(\gamma), \gamma) : \gamma \in [0, 1]\}.$$

This leads the following characterization of a fuzzy number in terms of the two end point functions  $\varphi_*(\gamma)$  and  $\varphi^*(\gamma)$ .

**Theorem 2.1.** [48] Suppose that  $\varphi_*(\gamma) : [0, 1] \rightarrow \mathbb{R}$  and  $\varphi^*(\gamma) : [0, 1] \rightarrow \mathbb{R}$  satisfy the following conditions:

- (1)  $\varphi_*(\gamma)$  is a nondecreasing function.
- (2)  $\varphi^*(\gamma)$  is a nonincreasing function.
- (3)  $\varphi_*(1) \leq \varphi^*(1)$ .
- (4)  $\varphi_*(\gamma)$  and  $\varphi^*(\gamma)$  are bounded and left continuous on  $(0, 1]$  and right continuous at  $\gamma = 0$ .

Then  $\varphi : \mathbb{R} \rightarrow [0, 1]$ , defined by

$$\varphi(u) = \sup \{ \gamma : \varphi_*(\gamma) \leq u \leq \varphi^*(\gamma) \},$$

is a fuzzy number parameterization is given by  $\{(\varphi_*(\gamma), \varphi^*(\gamma), \gamma) : \gamma \in [0, 1]\}$ . Moreover, If  $\varphi : \mathbb{R} \rightarrow [0, 1]$  is a fuzzy number with parametrization given by  $\{(\varphi_*(\gamma), \varphi^*(\gamma), \gamma) : \gamma \in [0, 1]\}$ , then function  $\varphi_*(\gamma)$  and  $\varphi^*(\gamma)$  find the conditions (1)–(4).

Let  $\varphi, \phi \in \mathfrak{F}_0$  represented parametrically  $\{(\varphi_*(\gamma), \varphi^*(\gamma), \gamma) : \gamma \in [0, 1]\}$  and  $\{(\phi_*(\gamma), \phi^*(\gamma), \gamma) : \gamma \in [0, 1]\}$ , respectively. We say that  $\varphi \leq \phi$  if for all  $\gamma \in (0, 1]$ ,  $\varphi^*(\gamma) \leq \phi^*(\gamma)$ , and  $\varphi_*(\gamma) \leq \phi_*(\gamma)$ . If  $\varphi \leq \phi$ , then there exist  $\gamma \in (0, 1]$  such that  $\varphi^*(\gamma) < \phi^*(\gamma)$  or  $\varphi_*(\gamma) < \phi_*(\gamma)$ . We say comparable if for any  $\varphi, \phi \in \mathfrak{F}_0$ , we have  $\varphi \leq \phi$  or  $\varphi \geq \phi$  otherwise they are non-comparable. Some time we may write  $\varphi \leq \phi$  instead of  $\phi \geq \varphi$  and note that, we may say that  $\mathfrak{F}_0$  is a partial-ordered set under the relation  $\leq$ .

If  $\varphi, \phi \in \mathfrak{F}_0$ , there exist  $\psi \in \mathfrak{F}_0$  such that  $\varphi = \phi \dot{+} \psi$ , then by this result we have existence of Hukuhara difference of  $\varphi$  and  $\phi$ , and we say that  $\psi$  is the H-difference of  $\varphi$  and  $\phi$ , and denoted by  $\varphi \dot{-} \phi$ , see [49]. If H-difference exists, then

$$\begin{aligned} (\psi)^*(\gamma) &= (\varphi \dot{-} \phi)^*(\gamma) = \varphi^*(\gamma) - \phi^*(\gamma), \\ (\psi)_*(\gamma) &= (\varphi \dot{-} \phi)_*(\gamma) = \varphi_*(\gamma) - \phi_*(\gamma). \end{aligned}$$

Now we discuss some properties of fuzzy numbers under addition and scalar multiplication, if  $\varphi, \phi \in \mathfrak{F}_0$  and  $\rho \in \mathbb{R}$  then  $\varphi \dot{+} \phi$  and  $\rho\varphi$  define as

$$\varphi \dot{+} \phi = \{(\varphi_*(\gamma) + \phi_*(\gamma), \varphi^*(\gamma) + \phi^*(\gamma), \gamma) : \gamma \in [0, 1]\}, \quad (2)$$

$$\rho\varphi = \{(\rho\varphi_*(\gamma), \rho\varphi^*(\gamma), \gamma) : \gamma \in [0, 1]\}. \quad (3)$$

**Remark 2.1.** Obviously,  $\mathfrak{F}_0$  is closed under addition and nonnegative scalar multiplication and above defined properties on  $\mathfrak{F}_0$  are equivalent to those derived from the usual extension principle. Furthermore, for each scalar number  $\rho \in \mathbb{R}$ ,

$$\varphi \dot{+} \rho = \{(\varphi_*(\gamma) + \rho, \varphi^*(\gamma) + \rho, \gamma) : \gamma \in [0, 1]\}. \quad (4)$$

**Definition 2.4.** A mapping  $\mathcal{F} : K \subset \mathbb{R} \rightarrow \mathfrak{F}_0$  is called fuzzy mapping. For each  $\gamma \in [0, 1]$ , denote  $[\mathcal{F}(u)]^\gamma = [\mathcal{F}_*(u, \gamma), \mathcal{F}^*(u, \gamma)]$ . Thus a fuzzy mapping  $\mathcal{F}$  can be identified by a parametrized triples

$$\mathcal{F}(u) = \{(\mathcal{F}_*(u, \gamma), \mathcal{F}^*(u, \gamma), \gamma) : \gamma \in [0, 1]\}.$$

**Definition 2.5.** Let  $\mathcal{F} : K \subset \mathbb{R} \rightarrow \mathfrak{F}_0$  be a fuzzy mapping. Then  $\mathcal{F}(u)$  is said to be continuous at  $u \in K$ , if for each  $\gamma \in [0, 1]$ , both end point functions  $\mathcal{F}_*(u, \gamma)$  and  $\mathcal{F}^*(u, \gamma)$  are continuous at  $u \in K$ .

**Definition 2.6.** [30] Let  $L = (m, n)$  and  $u \in L$ . Then fuzzy mapping  $\mathcal{F} : (m, n) \rightarrow \mathfrak{F}_0$  is said to be a generalized differentiable (in short, G-differentiable) at  $u$  if there exists an element  $\mathcal{F}'(u) \in \mathfrak{F}_0$  such that for all  $0 < \tau$ , sufficiently small, there exist  $\mathcal{F}(u + \tau) \dot{-} \mathcal{F}(u)$ ,  $\mathcal{F}(u) \dot{-} \mathcal{F}(u - \tau)$  and the limits (in the metric  $D$ )

$$\begin{aligned} \lim_{\tau \rightarrow 0^+} \frac{\mathcal{F}(u + \tau) \dot{-} \mathcal{F}(u)}{\tau} &= \lim_{\tau \rightarrow 0^+} \frac{\mathcal{F}(u) \dot{-} \mathcal{F}(u - \tau)}{\tau} = \mathcal{F}'(u) \text{ or} \\ \lim_{\tau \rightarrow 0^+} \frac{\mathcal{F}(u) \dot{-} \mathcal{F}(u + \tau)}{-\tau} &= \lim_{\tau \rightarrow 0^+} \frac{\mathcal{F}(u - \tau) \dot{-} \mathcal{F}(u)}{-\tau} = \mathcal{F}'(u) \\ \text{or } \lim_{\tau \rightarrow 0^+} \frac{\mathcal{F}(u + \tau) \dot{-} \mathcal{F}(u)}{\tau} &= \lim_{\tau \rightarrow 0^+} \frac{\mathcal{F}(u - \tau) \dot{-} \mathcal{F}(u)}{-\tau} = \mathcal{F}'(u) \text{ or} \\ \lim_{\tau \rightarrow 0^+} \frac{\mathcal{F}(u) \dot{-} \mathcal{F}(u + \tau)}{-\tau} &= \lim_{\tau \rightarrow 0^+} \frac{\mathcal{F}(u) \dot{-} \mathcal{F}(u - \tau)}{\tau} = \mathcal{F}'(u) \end{aligned}$$

where the limits are taken in the metric space  $(\mathfrak{F}_0, D)$ , for  $\varphi, \phi \in \mathfrak{F}_0$

$$D(\varphi, \phi) = \sup_{0 \leq \gamma \leq 1} H(\varphi_\gamma, \phi_\gamma),$$

and  $H$  denote the well-known Hausdorff metric on space of intervals.

**Definition 2.7.** [26] A fuzzy mapping  $\mathcal{F} : K \rightarrow \mathfrak{F}_0$  is called convex on the convex set  $K$  if

$$\begin{aligned} &\mathcal{F}((1 - \tau)u + \tau\vartheta) \\ &\leq (1 - \tau)\mathcal{F}(u) \dot{+} \tau\mathcal{F}(\vartheta) \forall u, \vartheta \in K, \tau \in [0, 1]. \end{aligned}$$

**Definition 2.8.** [26] A fuzzy mapping  $\mathcal{F} : K \rightarrow \mathfrak{F}_0$  is called quasi-convex on the convex set  $K$  if

$$\begin{aligned} &\mathcal{F}((1 - \tau)u + \tau\vartheta) \\ &\leq \max(\mathcal{F}(u), \mathcal{F}(\vartheta)) \forall u, \vartheta \in K, \tau \in [0, 1]. \end{aligned}$$

**Definition 2.9.** [26] A fuzzy mapping  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  is called preinvex on the invex set  $K_\partial$  w.r.t. bi-function  $\partial$  if

$$\begin{aligned} &\mathcal{F}(u + \tau\partial(\vartheta, u)) \\ &\leq (1 - \tau)\mathcal{F}(u) \dot{+} \tau\mathcal{F}(\vartheta) \forall u, \vartheta \in K_\partial, \tau \in [0, 1], \end{aligned}$$

where  $\partial : K_\partial \times K_\partial \rightarrow \mathbb{R}$ .

**Lemma 2.1.** [27] Let  $K_\partial$  be an invex set w.r.t.  $\partial$  and let  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  be a fuzzy mapping parametrized by

$$\mathcal{F}(u) = \{(\mathcal{F}_*(u, \gamma), \mathcal{F}^*(u, \gamma), \gamma) : \gamma \in [0, 1]\}, \forall u \in K_\partial.$$

Then  $\mathcal{F}(u)$  is preinvex fuzzy mapping on  $K_\partial$  if and only if, for all  $\gamma \in [0, 1]$ ,  $\mathcal{F}_*(u, \gamma)$  and  $\mathcal{F}^*(u, \gamma)$  are preinvex functions w.r.t.  $\partial$  on  $K_\partial$ .

**Definition 2.10.** [26] A fuzzy mapping  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  is called quasi-preinvex on the invex set  $K_\partial$  w.r.t.  $\partial$  if

$$\begin{aligned} &\mathcal{F}(u + \tau\partial(\vartheta, u)) \\ &\leq \max(\mathcal{F}(u), \mathcal{F}(\vartheta)) \forall u, \vartheta \in K_\partial, \tau \in [0, 1]. \end{aligned}$$

For further study, let  $K_\partial$  be a nonempty invex set in  $\mathbb{R}$ . Let  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  be a fuzzy mapping and  $\partial : K_\partial \times K_\partial \rightarrow \mathbb{R}$  be an arbitrary bifunction. We denote  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$  be the norm and inner product, respectively. Furthermore, throughout in this article fuzzy mappings are discussed through the so-called “fuzzy-max” order among fuzzy numbers. As it is well-known, the fuzzy-max order is a partial order relation “ $\leq$ ” on the set of fuzzy numbers.

### 3. HIGHER-ORDER STRONGLY PREINVEX FUZZY MAPPINGS

In this section, we propose a family of nonconvex fuzzy mappings which is known as HOS-preinvex fuzzy mappings. We define some different classes of HOS-preinvex fuzzy mappings and investigate some basic properties.

**Definition 3.1.** Let  $K_\partial$  be an invex set and let  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  be fuzzy mapping. Then  $\mathcal{F}(u)$  is said to be HOS-preinvex fuzzy mapping w.r.t. bi-function  $\partial(\cdot, \cdot)$ , if there exist a constant  $\Omega > 0$  such that

$$\begin{aligned} &\mathcal{F}(u + \tau\partial(\vartheta, u)) \leq (1 - \tau)\mathcal{F}(u) \tilde{+} \tau\mathcal{F}(\vartheta) \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned} \quad (5)$$

$\forall u, \vartheta \in K_\partial, \tau \in [0, 1]$ , where  $\partial : K_\partial \times K_\partial \rightarrow \mathbb{R}$  and  $p \geq 1$ .

Similarly,  $\mathcal{F}(u)$  is said to be HOS-preconcave fuzzy mapping on  $K_\partial$  if inequality (5) is reversed.

Now we discuss some special cases of HOS-preinvex fuzzy mappings:

- If  $p = 2$ , then (5) becomes

$$\begin{aligned} &\mathcal{F}(u + \tau\partial(\vartheta, u)) \leq (1 - \tau)\mathcal{F}(u) \tilde{+} \tau\mathcal{F}(\vartheta) \\ &\quad - \Omega \tau(1 - \tau) \|\partial(\vartheta, u)\|^2, \forall u, \vartheta \in K_\partial, \tau \in [0, 1]. \end{aligned}$$

Which is called the strongly preinvex fuzzy mapping. This is itself a very interesting problem to study its applications in pure and applied science like fuzzy optimization.

- If  $\Omega = 0$ , then HOS-preinvex fuzzy mapping becomes preinvex fuzzy mapping w.r.t. bi-function  $\partial(\cdot, \cdot)$ , i.e.,

$$\begin{aligned} &\mathcal{F}(u + \tau\partial(\vartheta, u)) \\ &\leq (1 - \tau)\mathcal{F}(u) \tilde{+} \tau\mathcal{F}(\vartheta), \forall u, \vartheta \in K_\partial, \tau \in [0, 1]. \end{aligned}$$

- If  $\partial(\vartheta, u) = \vartheta - u$  and  $p \geq 1$ , then (5) becomes

$$\begin{aligned} &\mathcal{F}(u + \tau\partial(\vartheta, u)) \leq (1 - \tau)\mathcal{F}(u) \tilde{+} \tau\mathcal{F}(\vartheta) \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\vartheta - u\|^p, \forall u, \vartheta \in K_\partial, \tau \in [0, 1]. \end{aligned}$$

Which is called the HOS-convex fuzzy mapping. When  $p = 2$ , then  $\mathcal{F}(u)$  is called strongly convex fuzzy mapping.

- If  $\partial(\vartheta, u) = \vartheta - u$  and  $\Omega = 0$ , then HOS-preinvex fuzzy mapping becomes convex fuzzy mapping, i.e.,

$$\begin{aligned} &\mathcal{F}(u + \tau(\vartheta - u)) \leq (1 - \tau)\mathcal{F}(u) \tilde{+} \tau\mathcal{F}(\vartheta), \\ &\quad \forall u, \vartheta \in K_\partial, \tau \in [0, 1]. \end{aligned}$$

- If  $\tau = \frac{1}{2}$ , then (5) becomes
- $\mathcal{F}\left(\frac{2u + \partial(\vartheta, u)}{2}\right) \leq \frac{\mathcal{F}(u) \tilde{+} \mathcal{F}(\vartheta)}{2} - \frac{1}{2^p} \Omega \|\partial(\vartheta, u)\|^p,$   
 $\forall u, \vartheta \in K_\partial.$

The mapping  $\mathcal{F}(u)$  is called the HOS-Jensen preinvex (in short,  $J$ -preinvex) fuzzy mapping. We also define the HOS-affine  $J$ -preinvex fuzzy mapping.

**Definition 3.2.** A fuzzy mapping  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  is said to be HOS-affine preinvex fuzzy mapping on  $K_\partial$  w.r.t. bi-function  $\partial$ , if there exist  $\Omega > 0$  such that

$$\begin{aligned} &\mathcal{F}(u + \tau\partial(\vartheta, u)) = (1 - \tau)\mathcal{F}(u) \tilde{+} \tau\mathcal{F}(\vartheta) \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned} \quad (6)$$

for all  $u, \vartheta \in K_\partial, \tau \in [0, 1]$ , where  $\partial : K_\partial \times K_\partial \rightarrow \mathbb{R}$  and  $p \geq 1$ .

In other words, a fuzzy mapping  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  is said to be HOS-affine preinvex fuzzy mapping on  $K_\partial$  w.r.t. bi-function  $\partial$ , if  $\mathcal{F}(u)$  is both HOS-preinvex fuzzy mapping and HOS-preconcave fuzzy mapping w.r.t. same bi-function  $\partial$ .

If  $\tau = \frac{1}{2}$ , then we also say that  $\mathcal{F}(u)$  is HOS-affine  $J$ -preinvex fuzzy mapping such that

$$\begin{aligned} &\mathcal{F}\left(\frac{2u + \partial(\vartheta, u)}{2}\right) = \frac{\mathcal{F}(u) \tilde{+} \mathcal{F}(\vartheta)}{2} - \frac{1}{2^p} \Omega \|\partial(\vartheta, u)\|^p, \\ &\quad \forall u, \vartheta \in K_\partial. \end{aligned}$$

**Remark 3.1.** The HOS-preinvex fuzzy mappings have some very nice properties similar to convex fuzzy mappings:

- If  $\mathcal{F}(u)$  is HOS-preinvex fuzzy mapping, then  $\sigma\mathcal{F}(u)$  is also HOS-preinvex for  $\sigma \geq 0$ .
- If  $\mathcal{F}(u)$  and  $\mathcal{G}(u)$  both are HOS-preinvex fuzzy mappings w.r.t. bi-function  $\partial(\cdot, \cdot)$ , then  $\max(\mathcal{F}(u), \mathcal{G}(u))$  is also HOS-preinvex fuzzy mapping w.r.t.  $\partial$ .

We now prove a special result for HOS-preinvex fuzzy mapping which establish a equivalence relation between HOS-preinvex fuzzy mapping  $\mathcal{F}(u)$ , and end point functions  $\mathcal{F}_*(u, \gamma)$  and  $\mathcal{F}^*(u, \gamma)$ .

**Theorem 3.1.** Let  $K_\partial$  be an invex set w.r.t.  $\partial$  and let  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  be a fuzzy mapping parametrized by

$$\mathcal{F}(u) = \{(\mathcal{F}_*(u, \gamma), \mathcal{F}^*(u, \gamma), \gamma) : \gamma \in [0, 1]\}, \forall u \in K_\partial. \quad (7)$$

Then  $\mathcal{F}(u)$  is HOS-preinvex fuzzy mapping on  $K_\partial$  with modulus  $\Omega$ , if and only if, for all  $\gamma \in [0, 1]$ ,  $\mathcal{F}_*(u, \gamma)$  and  $\mathcal{F}^*(u, \gamma)$  are HOS-preinvex functions w.r.t.  $\partial$  with modulus  $\Omega$ . (8)

**Proof.** Assume that for each  $\gamma \in [0, 1]$ ,  $\mathcal{F}_*(u, \gamma)$  and  $\mathcal{F}^*(u, \gamma)$  are HOS-preinvex functions w.r.t.  $\partial$  and modulus  $\Omega$  on  $K_\partial$ . Then from (5), we have

$$\begin{aligned} & \mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) \\ & \leq (1 - \tau) \mathcal{F}_*(u, \gamma) + \tau \mathcal{F}_*(\vartheta, \gamma) \\ & \quad - \Omega \{ \tau^p (1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \forall u, \vartheta \in K_\partial, \tau \in [0, 1]. \end{aligned}$$

And

$$\begin{aligned} & \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) \\ & \leq (1 - \tau) \mathcal{F}^*(u, \gamma) + \tau \mathcal{F}^*(\vartheta, \gamma) \\ & \quad - \Omega \{ \tau^p (1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \forall u, \vartheta \in K_\partial, \tau \in [0, 1]. \end{aligned}$$

Then by (8), (2), (3) and (4), we obtain

$$\begin{aligned} & \mathcal{F}(u + \tau\partial(\vartheta, u)) \\ & = \{ (\mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma), \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma)) : \gamma \in [0, 1] \}, \\ & \leq \{ ((1 - \tau) \mathcal{F}_*(u, \gamma), (1 - \tau) \mathcal{F}^*(u, \gamma), \gamma) : \gamma \in [0, 1] \} \\ & \quad \tilde{+} \{ (\tau \mathcal{F}_*(\vartheta, \gamma), \tau \mathcal{F}^*(\vartheta, \gamma), \gamma) : \gamma \in [0, 1] \} \\ & \quad \tilde{-} \Omega \{ \tau^p (1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \\ & = (1 - \tau) \mathcal{F}(u) \tilde{+} \tau \mathcal{F}(\vartheta) \tilde{-} \Omega \{ \tau^p (1 - \tau) + \tau(1 - \tau)^p \} \\ & \quad \|\partial(\vartheta, u)\|^p, \forall u, \vartheta \in K_\partial, \tau \in [0, 1]. \end{aligned}$$

Hence  $\mathcal{F}(u)$  is HOS-preinvex fuzzy mapping on  $K_\partial$  with modulus  $\Omega$ .

Conversely, let  $\mathcal{F}(u)$  be a strongly preinvex fuzzy mapping on  $K_\partial$  with modulus  $\Omega$ . Then for all  $u, \vartheta \in K_\partial$  and  $\tau \in [0, 1]$ , we have

$$\begin{aligned} & \mathcal{F}(u + \tau\partial(\vartheta, u)) \leq (1 - \tau) \mathcal{F}(u) \tilde{+} \tau \mathcal{F}(\vartheta) \\ & \quad \tilde{-} \Omega \{ \tau^p (1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p. \end{aligned}$$

From (8), we have

$$\begin{aligned} & \mathcal{F}(u + \tau\partial(\vartheta, u)) \\ & = \{ (\mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma), \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma)) : \gamma \in [0, 1] \}, \\ & \forall u, \vartheta \in K_\partial, \tau \in [0, 1]. \end{aligned}$$

From (8), (2), (3) and (4), we obtain

$$\begin{aligned} & (1 - \tau) \mathcal{F}(u) \tilde{+} \tau \mathcal{F}(\vartheta) \tilde{-} \Omega \{ \tau^p (1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p \\ & = \{ ((1 - \tau) \mathcal{F}_*(u, \gamma), (1 - \tau) \mathcal{F}^*(u, \gamma), \gamma) : \gamma \in [0, 1] \} \\ & \quad \tilde{+} \{ (\tau \mathcal{F}_*(\vartheta, \gamma), \tau \mathcal{F}^*(\vartheta, \gamma), \gamma) : \gamma \in [0, 1] \} \\ & \quad \tilde{-} \Omega \{ \tau^p (1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned} \quad (9)$$

for all  $u, \vartheta \in K_\partial$  and  $\tau \in [0, 1]$ . Then by HOS-preinvexity of  $\mathcal{F}(u)$ , we have for all  $u, \vartheta \in K_\partial$  and  $\tau \in [0, 1]$  such that

$$\begin{aligned} & \mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) \\ & \leq (1 - \tau) \mathcal{F}_*(u, \gamma) + \tau \mathcal{F}_*(\vartheta, \gamma) \\ & \quad - \Omega \{ \tau^p (1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned}$$

and

$$\begin{aligned} & \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) \\ & \leq (1 - \tau) \mathcal{F}^*(u, \gamma) + \tau \mathcal{F}^*(\vartheta, \gamma) \\ & \quad - \Omega \{ \tau^p (1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned}$$

for each  $\gamma \in [0, 1]$ . Hence, the result follows.

Note that, If  $\xi(\vartheta, u) = \vartheta - u$ , then Theorem 3.1, reduce to following statement such as

“Let  $K_\partial$  be a convex set and let  $F : K \rightarrow \mathbb{F}_0$  be a fuzzy mapping parametrized by

$$F(u) = \{ (F_*(u, \gamma), F^*(u, \gamma), \gamma) : \gamma \in [0, 1] \}, \forall u \in K_\partial$$

Then  $F$  is HOS-convex fuzzy mapping on  $K_\partial$  if and only if, for all  $\gamma \in [0, 1]$ ,

$F_*(u, \gamma)$  and  $F^*(u, \gamma)$  are HOS-convex function on  $K_\partial$ .”

### Example 3.1.

We consider the fuzzy mappings  $\mathcal{F} : (0, 1) \rightarrow \mathfrak{F}_0$  defined by

$$\mathcal{F}(u)(\sigma) = \begin{cases} \frac{\sigma}{u^2}, & \sigma \in [0, u^2] \\ \frac{2u^2 - \sigma}{u^2}, & \sigma \in (u^2, 2u^2] \\ 0, & \text{otherwise.} \end{cases}$$

Then, for each  $\gamma \in [0, 1]$ , we have  $\mathcal{F}_\gamma(u) = [\gamma u^2, (2 - \gamma) u^2]$ . Since end point functions  $\mathcal{F}_*(u, \gamma)$ ,  $\mathcal{F}^*(u, \gamma)$  are HOS-preinvex functions for each  $\gamma \in [0, 1]$ . Hence  $\mathcal{F}(u)$  is HOS-preinvex fuzzy mapping w.r.t.

$$\partial(\vartheta, u) = \vartheta - u,$$

with  $0 < \Omega \leq 1$  and  $p \geq 2$ . It can be easily seen that for each  $\Omega \in (0, 1)$  and  $p \geq 2$ , there exist a HOS-preinvex fuzzy mapping and  $\mathcal{F}(u)$  is neither convex fuzzy mapping and nor preinvex fuzzy mapping w.r.t. bifunction  $\partial(\vartheta, u) = \vartheta - u$  with  $0 < \Omega \leq 1$  and  $p \geq 2$ .

We now established a result for HOS-preinvex fuzzy mapping which shows that the difference of HOS-preinvex fuzzy mapping and HOS-affine preinvex fuzzy mapping is a preinvex fuzzy mapping.

**Theorem 3.2.** Let fuzzy mapping  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  be a HOS-affine preinvex w.r.t.  $\partial$ . Then  $\mathcal{F}(u)$  is HOS-preinvex fuzzy w.r.t same  $\partial$ , if and only if,  $\mathcal{G} = \mathcal{F} - \mathcal{J}$  is preinvex fuzzy mapping.



**Proof.** The “If” part is obvious. To prove the “only if” assume that,  $\mathcal{F} : K_{\partial} \rightarrow \mathfrak{F}_0$  be a HOS-affine preinvex w.r.t. bi-function  $\partial$ , then there exist  $\Omega > 0$  such that

$$\begin{aligned} \mathcal{F}(u + \tau\partial(\vartheta, u)) &= (1 - \tau)\mathcal{F}(u) \dot{+} \tau\mathcal{F}(\vartheta) \\ &\leq \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p. \end{aligned}$$

Therefore, for each  $\gamma \in [0, 1]$ , we have

$$\begin{aligned} \mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) &= (1 - \tau)\mathcal{F}_*(u, \gamma) + \tau\mathcal{F}_*(\vartheta, \gamma) \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \\ \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) &= (1 - \tau)\mathcal{F}^*(u, \gamma) + \tau\mathcal{F}^*(\vartheta, \gamma) \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p. \end{aligned} \quad (10)$$

Since  $\mathcal{F}(u)$  is HOS-preinvex fuzzy mapping, then, for each  $\gamma \in [0, 1]$ , we have

$$\begin{aligned} \mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) &\leq (1 - \tau)\mathcal{F}_*(u, \gamma) + \tau\mathcal{F}_*(\vartheta, \gamma) \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \\ \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) &\leq (1 - \tau)\mathcal{F}^*(u, \gamma) + \tau\mathcal{F}^*(\vartheta, \gamma) \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned} \quad (11)$$

from (10) and (11), we have

$$\begin{aligned} \mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) - \mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) &\leq (1 - \tau)\mathcal{F}_*(u, \gamma) + \tau\mathcal{F}_*(\vartheta, \gamma) - (1 - \tau)\mathcal{F}_*(u, \gamma) - \tau\mathcal{F}_*(\vartheta, \gamma), \\ \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) - \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) &\leq (1 - \tau)\mathcal{F}^*(u, \gamma) + \tau\mathcal{F}^*(\vartheta, \gamma) - (1 - \tau)\mathcal{F}^*(u, \gamma) - \tau\mathcal{F}^*(\vartheta, \gamma), \end{aligned}$$

$$\begin{aligned} \mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) - \mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) &\leq (1 - \tau) \{ \mathcal{F}_*(u, \gamma) - \mathcal{F}_*(u, \gamma) \} + \tau \{ \mathcal{F}_*(\vartheta, \gamma) - \mathcal{F}_*(\vartheta, \gamma) \}, \\ \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) - \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) &\leq (1 - \tau) \{ \mathcal{F}^*(u, \gamma) - \mathcal{F}^*(u, \gamma) \} + \tau \{ \mathcal{F}^*(\vartheta, \gamma) - \mathcal{F}^*(\vartheta, \gamma) \}, \end{aligned}$$

from which it follows that

$$\begin{aligned} \mathcal{G}_*(u + \tau\partial(\vartheta, u), \gamma) &= \mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) - \mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma), \\ \mathcal{G}^*(u + \tau\partial(\vartheta, u), \gamma) &= \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) - \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma), \end{aligned}$$

$$\begin{aligned} \mathcal{G}_*(u + \tau\partial(\vartheta, u), \gamma) &\leq (1 - \tau) \{ \mathcal{F}_*(u, \gamma) - \mathcal{F}_*(u, \gamma) \} + \tau \{ \mathcal{F}_*(\vartheta, \gamma) - \mathcal{F}_*(\vartheta, \gamma) \}, \\ \mathcal{G}^*(u + \tau\partial(\vartheta, u), \gamma) &\leq (1 - \tau) \{ \mathcal{F}^*(u, \gamma) - \mathcal{F}^*(u, \gamma) \} + \tau \{ \mathcal{F}^*(\vartheta, \gamma) - \mathcal{F}^*(\vartheta, \gamma) \}, \end{aligned}$$

which implies that

$$\begin{aligned} \mathcal{G}_*(u + \tau\partial(\vartheta, u), \gamma) &\leq (1 - \tau)\mathcal{G}_*(u, \gamma) + \tau\mathcal{G}_*(\vartheta, \gamma), \\ \mathcal{G}^*(u + \tau\partial(\vartheta, u), \gamma) &\leq (1 - \tau)\mathcal{G}^*(u, \gamma) + \tau\mathcal{G}^*(\vartheta, \gamma), \end{aligned}$$

i.e.,

$$\mathcal{G}(u + \tau\partial(\vartheta, u)) \leq (1 - \tau)\mathcal{G}(u) \dot{+} \tau\mathcal{G}(\vartheta).$$

Showing that  $\mathcal{G} = \mathcal{F} - \mathcal{F}$  is preinvex fuzzy mapping.

**Definition 3.3.** A fuzzy mapping  $\mathcal{F} : K_{\partial} \rightarrow \mathfrak{F}_0$  is said to be HOS-quasi-preinvex on  $K_{\partial}$  w.r.t. bi-function  $\partial$ , if there exist a  $\Omega > 0$  such that

$$\begin{aligned} \mathcal{F}(u + \tau\partial(\vartheta, u)) &\leq \max(\mathcal{F}(u), \mathcal{F}(\vartheta)) \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned} \quad (12)$$

$\forall u, \vartheta \in K_{\partial}, \tau \in [0, 1]$ , where  $\partial : K_{\partial} \times K_{\partial} \rightarrow \mathbb{R}$  and  $p \geq 1$ .

Similarly,  $\mathcal{F}(u)$  is said to be HOS-quasi-preconcave fuzzy mapping on  $K_{\partial}$  if inequality (12) is reversed.

**Remark 3.2.** From Definitions 3.1 and 3.3, for each  $\gamma \in [0, 1]$ , we have

$$\begin{aligned} \mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) &\leq (1 - \tau)\mathcal{F}_*(u, \gamma) + \tau\mathcal{F}_*(\vartheta, \gamma) \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \\ \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) &\leq (1 - \tau)\mathcal{F}^*(u, \gamma) + \tau\mathcal{F}^*(\vartheta, \gamma) \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned}$$

$$\begin{aligned} \mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) &\leq \max(\mathcal{F}_*(u, \gamma), \mathcal{F}_*(\vartheta, \gamma)) \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \\ \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) &\leq \max(\mathcal{F}^*(u, \gamma), \mathcal{F}^*(\vartheta, \gamma)) \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned}$$

i.e.,

$$\begin{aligned} \mathcal{F}(u + \tau\partial(\vartheta, u)) &\leq \max(\mathcal{F}(u), \mathcal{F}(\vartheta)) \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p. \end{aligned}$$

It can be easily note that each HOS-preinvex fuzzy mapping on  $K_{\partial}$  is HOS-quasi-preinvex fuzzy mapping, when  $\mathcal{F}$  is a fuzzy mapping.

**Definition 3.4.** A fuzzy mapping  $\mathcal{F} : K_{\partial} \rightarrow \mathfrak{F}_0$  is called pseudo preinvex on  $K_{\partial}$  if there exist a  $b : K_{\partial} \times K_{\partial} \rightarrow \mathfrak{F}_0$  such that

$$\begin{aligned} \mathcal{F}(\vartheta) &< \mathcal{F}(u) \Rightarrow \\ \mathcal{F}(u + \tau\partial(\vartheta, u)) &< \mathcal{F}(u) \dot{+} \tau(\tau - 1)b(u, \vartheta), \\ \forall u, \vartheta \in K_{\partial}, \tau \in [0, 1], \text{ where } b(., .) &> \tilde{0}. \end{aligned} \quad (13)$$

**Theorem 3.3.** Let  $\mathcal{F}(u)$  be a HOS-preinvex fuzzy mapping on  $K_{\partial}$  such that  $\mathcal{F}(\vartheta) < \mathcal{F}(u)$ . Then fuzzy mapping  $\mathcal{F}(u)$  is HOS-pseudo preinvex w.r.t. same  $\partial$ .

**Proof.** Let  $\mathcal{F}(\vartheta) < \mathcal{F}(u)$  and  $\mathcal{F}(u)$  be a HOS-preinvex fuzzy mapping. Then there exist modulus  $\Omega$  and for all  $u, \vartheta \in K_\partial$ ,  $\tau \in [0, 1]$ , such that

$$\begin{aligned} \mathcal{F}(u + \tau\partial(\vartheta, u)) &\leq (1 - \tau)\mathcal{F}(u) \tilde{+} \tau\mathcal{F}(\vartheta) \\ &\tilde{\leq} \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p. \end{aligned}$$

Therefore, for every  $\gamma \in [0, 1]$ , we have

$$\begin{aligned} &\mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) \\ &\leq (1 - \tau)\mathcal{F}_*(u, \gamma) + \tau\mathcal{F}_*(\vartheta, \gamma) \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \\ &\mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) \\ &\leq (1 - \tau)\mathcal{F}^*(u, \gamma) + \tau\mathcal{F}^*(\vartheta, \gamma) \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned}$$

$$\begin{aligned} &\mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) \\ &\leq \mathcal{F}_*(u, \gamma) + \tau \{ \mathcal{F}_*(\vartheta, \gamma) - \mathcal{F}_*(u, \gamma) \} \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \\ &\mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) \\ &\leq \mathcal{F}^*(u, \gamma) + \tau \{ \mathcal{F}^*(\vartheta, \gamma) - \mathcal{F}^*(u, \gamma) \} \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p. \end{aligned} \quad (14)$$

From (14), we have

$$\begin{aligned} &\mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) \\ &< \mathcal{F}_*(u, \gamma) + \tau(\tau - 1) \{ \mathcal{F}_*(u, \gamma) - \mathcal{F}_*(\vartheta, \gamma) \} \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \\ &\mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) \\ &< \mathcal{F}^*(u, \gamma) + \tau(\tau - 1) \{ \mathcal{F}^*(u, \gamma) - \mathcal{F}^*(\vartheta, \gamma) \} \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned}$$

$$\begin{aligned} &\mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) \\ &< \mathcal{F}_*(u, \gamma) + \tau(\tau - 1) b_*(u, \vartheta, \gamma) \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \\ &\mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) \\ &< \mathcal{F}^*(u, \gamma) + \tau(\tau - 1) b^*(u, \vartheta, \gamma) \\ &\quad - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned}$$

i.e.,

$$\begin{aligned} \mathcal{F}(u + \tau\partial(\vartheta, u)) &< \mathcal{F}(u) \tilde{+} \tau(\tau - 1) b(u, \vartheta) \\ &\tilde{\leq} \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned}$$

where  $b(u, \vartheta) = \mathcal{F}(u) \tilde{-} \mathcal{F}(\vartheta)$ . This prove that  $\mathcal{F}(u)$  is HOS-pseudo preinvex fuzzy mapping w.r.t. same  $\partial$ .

## 4. HIGHER-ORDER STRONGLY INVEX FUZZY MAPPINGS AND FUZZY MONOTONICITY

We need the following assumption regarding the function  $\partial$ , which plays an important role in G-differentiation of the main results.

**Condition C.** [22]

$$\partial(\vartheta, u + \tau\partial(\vartheta, u)) = (1 - \tau)\partial(\vartheta, u),$$

$$\partial(u, u + \tau\partial(\vartheta, u)) = -\tau\partial(\vartheta, u).$$

Clearly for  $\tau = 0$ , we have  $\xi(\vartheta, u) = 0$  if and only if  $\vartheta = u$ , for all  $u, \vartheta \in K$ . For the application of Condition C, see [7,8,22,26,28,29,50].

**Definition 4.1.** A G-differentiable fuzzy mapping  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  is called HOS-invex w.r.t.  $\partial$ , if there exist a constant  $\Omega > 0$  such that

$$\begin{aligned} \mathcal{F}(\vartheta) \tilde{-} \mathcal{F}(u) &\geq \langle \mathcal{F}^*(u), \partial(\vartheta, u) \rangle \\ &\quad \tilde{+} \Omega \|\partial(\vartheta, u)\|^p, \forall u, \vartheta \in K_\partial. \end{aligned} \quad (15)$$

If  $\Omega = 0$ , then HOS" invex fuzzy mapping is known as invex fuzzy mapping.

### Example 4.1.

We consider the fuzzy mappings  $\mathcal{F} : (0, 1) \rightarrow \mathfrak{F}_0$  defined by,  $\mathcal{F}_\gamma(u) = [\gamma u^2, (2 - \gamma)u^2]$ , as in Example 3.1, then  $\mathcal{F}(u)$  is HOS-invex fuzzy mapping w.r.t. bifunction  $\partial(\vartheta, u) = \vartheta - u$ , with  $0 < \Omega \leq 1$  and  $p \geq 2$ , where  $u \leq \vartheta$ . We have  $\mathcal{F}_*(u, \gamma) = \gamma u^2$  and  $\mathcal{F}^*(u, \gamma) = (2 - \gamma)u^2$ . Now we computing the following

$$\mathcal{F}_*(\vartheta, \gamma) - \mathcal{F}_*(u, \gamma) = \gamma\vartheta^2 - \gamma u^2,$$

while

$$\langle \mathcal{F}^*(u, \gamma), \partial(\vartheta, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p = 2\gamma(\vartheta - u) + \Omega \|\vartheta - u\|^p.$$

And  $\gamma\vartheta^2 - \gamma u^2 \geq 2\gamma(\vartheta - u) + \Omega \|\vartheta - u\|^p$ , with  $0 < \Omega \leq 1$  and  $p \geq 2$ , where  $u \leq \vartheta$ .

Similarly, it can be easily show that

$$\mathcal{F}^*(\vartheta, \gamma) - \mathcal{F}^*(u, \gamma) \geq \langle \mathcal{F}^*(u, \gamma), \partial(\vartheta, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p.$$

Hence,  $\mathcal{F}(u)$  is HOS" invex fuzzy mapping w.r.t. bifunction  $\partial(\vartheta, u) = \vartheta - u$ , with  $0 < \Omega \leq 1$  and  $p \geq 2$ . It can be easily seen that  $\mathcal{F}(u)$  is not invex fuzzy mapping w.r.t. bifunction  $\partial(\vartheta, u) = \vartheta - u$ .

**Definition 4.2.** A G-differentiable fuzzy mapping  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  is called HOS-pseudo invex w.r.t.  $\partial$ , if there exist a constant  $\Omega > 0$  such that

$$\begin{aligned} \langle \mathcal{F}^*(u), \partial(\vartheta, u) \rangle \tilde{+} \Omega \|\partial(\vartheta, u)\|^p &\geq \tilde{0} \Rightarrow \\ \mathcal{F}(\vartheta) \tilde{-} \mathcal{F}(u) &\geq \tilde{0}, \forall u, \vartheta \in K_\partial. \end{aligned} \quad (16)$$

If  $\Omega = 0$ , then HOS” pseudo invex fuzzy mapping is known as pseudo invex fuzzy mapping w.r.t.  $\partial$ .

**Definition 4.3.** A G-differentiable fuzzy mapping  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  is called HOS-quasi invex w.r.t.  $\partial$ , if there exist a constant  $\Omega > 0$  such that

$$\begin{aligned} \mathcal{F}(\vartheta) &\leq \mathcal{F}(u) \Rightarrow \\ \mathcal{F}^*(u), \partial(\vartheta, u) \tilde{+} \Omega \partial(\vartheta, u)^p &\leq \tilde{0}, \forall u, \vartheta \in K_\partial. \end{aligned} \quad (17)$$

If  $\Omega = 0$ , then HOS” pseudo invex fuzzy mapping is known as quasi invex fuzzy mapping w.r.t.  $\partial$ .

If  $\partial(\vartheta, u) = -\partial(u, \vartheta)$ , then Definitions 4.1–4.3 reduce to known ones. All these definitions may play important role in fuzzy optimization problem and mathematical programming.

## Example 4.2.

We consider the fuzzy mappings  $\mathcal{F} : (0, \infty) \rightarrow \mathfrak{F}_0$  defined by,  $\mathcal{F}_\gamma(u) = [\gamma u, (5 - 4\gamma)u]$ , then  $\mathcal{F}(u)$  is HOS-pseudo invex fuzzy mapping w.r.t. bifunction  $\partial(\vartheta, u) = \vartheta - u$ , with  $0 \leq \Omega$  and  $p \geq 1$ , where  $u \leq \vartheta$ . We have  $\mathcal{F}_*(u, \gamma) = \gamma u$  and  $\mathcal{F}^*(u, \gamma) = (5 - 4\gamma)u$ . Now we computing the following:

$$\begin{aligned} \langle \mathcal{F}_*(u, \gamma), \partial(\vartheta, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p \\ = \gamma(\vartheta - u) + \Omega \|\vartheta - u\|^p \geq 0, \end{aligned}$$

for all  $u, \vartheta \in K_\partial$  and  $\gamma \in [0, 1]$  with  $u \leq \vartheta$ ,  $0 \leq \Omega$  and  $p \geq 1$ ; which implies that

$$\begin{aligned} \mathcal{F}_*(\vartheta, \gamma) &= \gamma\vartheta \geq \gamma u = \mathcal{F}_*(u, \gamma), \\ \mathcal{F}_*(\vartheta, \gamma) &\geq \mathcal{F}_*(u, \gamma), \end{aligned} \quad (18)$$

Similarly, it can be easily show that

$$\begin{aligned} \langle \mathcal{F}^*(u, \gamma), \partial(\vartheta, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p \\ = (5 - 4\gamma)(\vartheta - u) + \Omega \|\vartheta - u\|^p \geq 0, \end{aligned}$$

for all  $u, \vartheta \in K_\partial$  and  $\gamma \in [0, 1]$  with  $u \leq \vartheta$ ,  $0 \leq \Omega$  and  $p \geq 1$ ; that means

$$\mathcal{F}^*(\vartheta, \gamma) = (5 - 4\gamma)\vartheta \geq \gamma u = \mathcal{F}^*(u, \gamma).$$

From which, It follows that

$$\mathcal{F}^*(\vartheta, \gamma) \geq \mathcal{F}^*(u, \gamma). \quad (19)$$

Hence, the fuzzy mapping  $\mathcal{F}_\gamma(u) = [\gamma u, (5 - 4\gamma)u]$  is HOS-pseudo invex fuzzy mapping w.r.t.  $\partial(\vartheta, u) = \vartheta - u$ , with  $0 \leq \Omega$  and  $p \geq 1$ , where  $u \leq \vartheta$ . it can be easily note that  $\mathcal{F}(u)$  is neither pseudo invex fuzzy mapping nor quasi invex fuzzy mapping w.r.t.  $\partial$ .

**Theorem 4.1.** Let  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  be a G-differentiable HOS-preinvex fuzzy mapping on invex set  $K_\partial$  and let condition C hold. Then  $\mathcal{F}(u)$  is HOS-preinvex fuzzy mapping, if and only if  $\mathcal{F}(u)$  is a HOS-invex fuzzy mapping.

**Proof.** Let  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  be G-differentiable HOS-preinvex fuzzy mapping. Since  $\mathcal{F}(u)$  is HOS-preinvex fuzzy mapping then there exist a constant  $\Omega > 0$ , for all  $u, \vartheta \in K_\partial$  and  $\tau \in [0, 1]$ , we have

$$\begin{aligned} \mathcal{F}(u + \tau\partial(\vartheta, u)) \\ \leq (1 - \tau)\mathcal{F}(u) \tilde{+} \tau\mathcal{F}(\vartheta) \\ \tilde{-} \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \\ \leq \mathcal{F}(u) \tilde{+} \tau(\mathcal{F}(\vartheta) \tilde{-} \mathcal{F}(u)) \\ \tilde{-} \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned}$$

Therefore, for every  $\gamma \in [0, 1]$ , we have

$$\begin{aligned} \mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) \\ \leq \mathcal{F}_*(u, \gamma) + \tau(\mathcal{F}_*(\vartheta, \gamma) - \mathcal{F}_*(u, \gamma)) \\ - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \\ \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) \\ \leq \mathcal{F}^*(u, \gamma) + \tau(\mathcal{F}^*(\vartheta, \gamma) - \mathcal{F}^*(u, \gamma)) \\ - \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned}$$

which implies that

$$\begin{aligned} \tau(\mathcal{F}_*(\vartheta, \gamma) - \mathcal{F}_*(u, \gamma)) &\geq \mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) \\ - \mathcal{F}_*(u, \gamma) + \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \\ \tau(\mathcal{F}^*(\vartheta, \gamma) - \mathcal{F}^*(u, \gamma)) &\geq \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) \\ - \mathcal{F}^*(u, \gamma) + \Omega \{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned}$$

$$\begin{aligned} \mathcal{F}_*(\vartheta, \gamma) - \mathcal{F}_*(u, \gamma) \\ \geq \frac{\mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) - \mathcal{F}_*(u, \gamma)}{\tau} \\ + \Omega \{ \tau^{p-1}(1 - \tau) + (1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \\ \mathcal{F}^*(\vartheta, \gamma) - \mathcal{F}^*(u, \gamma) \\ \geq \frac{\mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) - \mathcal{F}^*(u, \gamma)}{\tau} \\ + \Omega \{ \tau^{p-1}(1 - \tau) + (1 - \tau)^p \} \|\partial(\vartheta, u)\|^p. \end{aligned}$$

Taking limit in the above inequality as  $\tau \rightarrow 0$ , we have

$$\mathcal{F}_*(\vartheta, \gamma) - \mathcal{F}_*(u, \gamma) \geq \langle \mathcal{F}_*(u, \gamma), \partial(\vartheta, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p,$$

$$\mathcal{F}^*(\vartheta, \gamma) - \mathcal{F}^*(u, \gamma) \geq \langle \mathcal{F}^*(u, \gamma), \partial(\vartheta, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p,$$

i.e.,

$$\mathcal{F}(\vartheta) \tilde{-} \mathcal{F}(u) \geq \langle \mathcal{F}^*(u), \partial(\vartheta, u) \rangle \tilde{+} \Omega \|\partial(\vartheta, u)\|^p. \quad (20)$$

Conversely, assume that  $\mathcal{F}(u)$  is a HOS-invex fuzzy mapping. Since  $K_\partial$  is an invex set then, we have,  $\vartheta_\tau = u + \tau\partial(\vartheta, u) \in K_\partial$ , for all  $u, \vartheta \in K_\partial$  and  $\tau \in [0, 1]$ . Taking  $\vartheta = \vartheta_\tau$  in (20), we get

$$\begin{aligned} \mathcal{F}_*(\vartheta_\tau, \gamma) - \mathcal{F}_*(u, \gamma) &\geq \langle \mathcal{F}_*(u, \gamma), \partial(\vartheta_\tau, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p, \\ \mathcal{F}^*(\vartheta_\tau, \gamma) - \mathcal{F}^*(u, \gamma) &\geq \langle \mathcal{F}^*(u, \gamma), \partial(\vartheta_\tau, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p. \end{aligned}$$



Using Condition C, we have

$$\begin{aligned}\mathcal{F}_*(\vartheta_\tau, \gamma) - \mathcal{F}_*(u, \gamma) &\geq (1 - \tau) \langle \mathcal{F}_*^*(u, \gamma), \partial(\vartheta, u) \rangle \\ &\quad + \Omega(1 - \tau)^p \|\partial(\vartheta, u)\|^p, \\ \mathcal{F}^*(\vartheta_\tau, \gamma) - \mathcal{F}^*(u, \gamma) &\geq (1 - \tau) \langle \mathcal{F}^{*,*}(u, \gamma), \partial(\vartheta, u) \rangle \\ &\quad + \Omega(1 - \tau)^p \|\partial(\vartheta, u)\|^p.\end{aligned}\quad (21)$$

In a similar way, we have

$$\begin{aligned}\mathcal{F}_*(u, \gamma) - \mathcal{F}_*(\vartheta_\tau, \gamma) &\geq -\tau \langle \mathcal{F}_*^*(u, \gamma), \partial(\vartheta, u) \rangle \\ &\quad + \Omega\tau^p \|\partial(\vartheta, u)\|^p, \\ \mathcal{F}^*(u, \gamma) - \mathcal{F}^*(\vartheta_\tau, \gamma) &\geq -\tau \langle \mathcal{F}^{*,*}(u, \gamma), \partial(\vartheta, u) \rangle \\ &\quad + \Omega\tau^p \|\partial(\vartheta, u)\|^p.\end{aligned}\quad (22)$$

Multiplying (21) by  $\tau$  and (22) by  $(1 - \tau)$ , and adding the resultant, we have

$$\begin{aligned}\mathcal{F}_*(\vartheta_\tau, \gamma) &\leq (1 - \tau) \mathcal{F}_*(u, \gamma) + \tau \mathcal{F}_*(\vartheta, \gamma) \\ &\quad - \Omega \{ \tau^p (1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \\ \mathcal{F}^*(\vartheta_\tau, \gamma) &\leq (1 - \tau) \mathcal{F}^*(u, \gamma) + \tau \mathcal{F}^*(\vartheta, \gamma) \\ &\quad - \Omega \{ \tau^p (1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p,\end{aligned}$$

which implies that

$$\begin{aligned}\mathcal{F}(u + \tau\partial(\vartheta, u), \gamma) \\ \leq (1 - \tau) \mathcal{F}(u, \gamma) + \tau \mathcal{F}(\vartheta, \gamma) \\ \leq \Omega \{ \tau^p (1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p.\end{aligned}$$

Hence,  $\mathcal{F}(u)$  is HOS-preinvex fuzzy mapping w.r.t.  $\partial$ .

As special case of Theorem 4.2, when  $\Omega = 0$ , we have the following:

**Corollary 4.1.** Let  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  be a G-differentiable fuzzy mapping on invex set  $K_\partial$  and let condition C hold. Then  $\mathcal{F}(u)$  is preinvex fuzzy mapping, if and only if  $\mathcal{F}(u)$  is a invex fuzzy mapping.

**Theorem 4.2.** Let  $\mathcal{F}(u)$  be a G-differentiable HOS-preinvex fuzzy mapping on  $K_\partial$  and Condition C hold. If  $\mathcal{F}(u)$  is a HOS-invex fuzzy mapping, then

$$\begin{aligned}\langle \mathcal{F}^*(u), \partial(\vartheta, u) \rangle + \langle \mathcal{F}^*(\vartheta), \partial(u, \vartheta) \rangle \\ \leq \Omega (\|\partial(\vartheta, u)\|^p + \|\partial(u, \vartheta)\|^p), \forall u, \vartheta \in K_\partial.\end{aligned}\quad (23)$$

**Proof.** Assume that  $\mathcal{F}(u)$  is a HOS-invex fuzzy mapping. Then, for every  $\gamma \in [0, 1]$ , we have

$$\begin{aligned}\mathcal{F}_*(\vartheta, \gamma) - \mathcal{F}_*(u, \gamma) &\geq \langle \mathcal{F}_*^*(u, \gamma), \partial(\vartheta, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p, \\ \mathcal{F}^*(\vartheta, \gamma) - \mathcal{F}^*(u, \gamma) &\geq \langle \mathcal{F}^{*,*}(u, \gamma), \partial(\vartheta, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p.\end{aligned}\quad (24)$$

Replacing  $\vartheta$  by  $u$  and  $u$  by  $\vartheta$  in (24), we have

$$\begin{aligned}\mathcal{F}_*(u, \gamma) - \mathcal{F}_*(\vartheta, \gamma) &\geq \langle \mathcal{F}_*^*(\vartheta, \gamma), \partial(u, \vartheta) \rangle + \Omega \|\partial(u, \vartheta)\|^p, \\ \mathcal{F}^*(u, \gamma) - \mathcal{F}^*(\vartheta, \gamma) &\geq \langle \mathcal{F}^{*,*}(\vartheta, \gamma), \partial(u, \vartheta) \rangle + \Omega \|\partial(u, \vartheta)\|^p.\end{aligned}\quad (25)$$

Adding (24) and (25), we have

$$\begin{aligned}\langle \mathcal{F}_*^*(u, \gamma), \partial(\vartheta, u) \rangle + \langle \mathcal{F}_*^*(\vartheta, \gamma), \partial(u, \vartheta) \rangle \\ \leq -\Omega (\|\partial(\vartheta, u)\|^p + \|\partial(u, \vartheta)\|^p), \\ \langle \mathcal{F}^{*,*}(u, \gamma), \partial(\vartheta, u) \rangle + \langle \mathcal{F}^{*,*}(\vartheta, \gamma), \partial(u, \vartheta) \rangle \\ \leq -\Omega (\|\partial(\vartheta, u)\|^p + \|\partial(u, \vartheta)\|^p),\end{aligned}$$

i.e.,

$$\begin{aligned}\langle \mathcal{F}^*(u), \partial(\vartheta, u) \rangle + \langle \mathcal{F}^*(\vartheta), \partial(u, \vartheta) \rangle \\ \leq \Omega (\|\partial(\vartheta, u)\|^p + \|\partial(u, \vartheta)\|^p).\end{aligned}$$

Hence, the required result.

It can be easily noted that converse of above Theorem 4.2, is true only for  $p = 2$ .

Theorems 4.1 and 4.2, enable us to define the followings new definitions.

**Definition 4.4.** A G-differentiable fuzzy mapping  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  is said to be

- fuzzy HOS-monotone w.r.t.  $\partial$  if and only if, there exist a constant  $\Omega > 0$  such that

$$\begin{aligned}\langle \mathcal{F}^*(u), \partial(\vartheta, u) \rangle + \langle \mathcal{F}^*(\vartheta), \partial(u, \vartheta) \rangle \\ \leq \Omega (\|\partial(\vartheta, u)\|^p + \|\partial(u, \vartheta)\|^p), \forall u, \vartheta \in K_\partial.\end{aligned}\quad (26)$$

- fuzzy monotone w.r.t. bi-function  $\partial$  if and only if,

$$\langle \mathcal{F}^*(u), \partial(\vartheta, u) \rangle + \langle \mathcal{F}^*(\vartheta), \partial(u, \vartheta) \rangle \leq \bar{0}, \forall u, \vartheta \in K_\partial. \quad (27)$$

- fuzzy HOS-pseudomonotone w.r.t.  $\partial$  if and only if, there exist a constant  $\Omega > 0$  such that

$$\begin{aligned}\langle \mathcal{F}^*(u), \partial(\vartheta, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p &\geq \bar{0} \\ \Rightarrow \langle \mathcal{F}^*(\vartheta), \partial(u, \vartheta) \rangle &\geq \bar{0}, \forall u, \vartheta \in K_\partial.\end{aligned}\quad (28)$$

- Strictly fuzzy monotone w.r.t. bi-function  $\partial$  if and only if,

$$\langle \mathcal{F}^*(u), \partial(\vartheta, u) \rangle + \langle \mathcal{F}^*(\vartheta), \partial(u, \vartheta) \rangle < \bar{0}, \forall u, \vartheta \in K_\partial. \quad (29)$$

- fuzzy pseudomonotone w.r.t. bi-function  $\partial$  if and only if,

$$\begin{aligned}\langle \mathcal{F}^*(u), \partial(\vartheta, u) \rangle &\geq \bar{0} \\ \Rightarrow \langle \mathcal{F}^*(\vartheta), \partial(u, \vartheta) \rangle &\leq \bar{0}, \forall u, \vartheta \in K_\partial.\end{aligned}\quad (30)$$

- fuzzy quasimonotone w.r.t. bi-function  $\partial$  if and only if,

$$\begin{aligned}\langle \mathcal{F}^*(u), \partial(\vartheta, u) \rangle &> \bar{0} \\ \Rightarrow \langle \mathcal{F}^*(\vartheta), \partial(u, \vartheta) \rangle &\leq \bar{0}, \forall u, \vartheta \in K_\partial.\end{aligned}\quad (31)$$

- strictly fuzzy pseudomonotone w.r.t. bi-function  $\partial$  if and only if,

$$\begin{aligned}\langle \mathcal{F}^*(u), \partial(\vartheta, u) \rangle &\geq \bar{0} \\ \Rightarrow \langle \mathcal{F}^*(\vartheta), \partial(u, \vartheta) \rangle &< \bar{0}, \forall u, \vartheta \in K_\partial.\end{aligned}\quad (32)$$

If  $\partial(\vartheta, u) = -\partial(u, \vartheta)$ , then Definition 4.4, reduces to new ones.

### Example 4.3.

We consider the fuzzy mappings  $\mathcal{F} : (0, \infty) \rightarrow \mathfrak{F}_0$  defined by

$$\mathcal{F}(u)(\sigma) = \begin{cases} \frac{\sigma}{2u^2}, & \sigma \in [0, 2u^2] \\ \frac{5u^2 - \sigma}{3u^2}, & \sigma \in (2u^2, 5u^2] \\ 0, & \text{otherwise.} \end{cases}$$

Then, for each  $\gamma \in [0, 1]$ , we have  $\mathcal{F}_\gamma(u) = [2\gamma u^2, (5 - 3\gamma)u^2]$ ,  $\mathcal{F}(u)$  is fuzzy HOS-pseudomonotone w.r.t. bifunction  $\partial(\vartheta, u) = u - \vartheta$ , with  $1 \leq \Omega$  and  $p \geq 1$ , where  $\vartheta \leq u$ . We have  $\mathcal{F}_*(u, \gamma) = 2\gamma u^2$  and  $\mathcal{F}^*(u, \gamma) = (5 - 3\gamma)u^2$ . Now we computing the following:

$$\begin{aligned} & \langle \mathcal{F}_*(u, \gamma), \partial(\vartheta, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p \\ &= 4\gamma u(u - \vartheta) + \Omega \|u - \vartheta\|^p \geq 0, \end{aligned}$$

for all  $u, \vartheta \in K_\partial$  and  $\gamma \in [0, 1]$  with  $\vartheta \leq u$ ,  $1 \leq \Omega$  and  $p \geq 1$ ; which implies that

$$\begin{aligned} & -\langle \mathcal{F}_*(\vartheta, \gamma), \partial(u, \vartheta) \rangle \\ &= -4\gamma u(\vartheta - u) = 4\gamma \vartheta(u - \vartheta), \geq 0, \forall u, \vartheta \in K_\partial, \end{aligned}$$

Similarly, it can be easily show that

$$\begin{aligned} & \langle \mathcal{F}^*(u, \gamma), \partial(\vartheta, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p \\ &= 2(5 - 3\gamma)u(u - \vartheta) + \Omega \|u - \vartheta\|^p \geq 0, \end{aligned}$$

for all  $u, \vartheta \in K_\partial$  and  $\gamma \in [0, 1]$  with  $\vartheta \leq u$ ,  $1 \leq \Omega$  and  $p \geq 1$ ; that means

$$\begin{aligned} -\langle \mathcal{F}^*(\vartheta, \gamma), \partial(u, \vartheta) \rangle &= -2(5 - 3\gamma)u(\vartheta - u) \\ &= 2(5 - 3\gamma)\vartheta(u - \vartheta) \\ &\geq 0, \forall u, \vartheta \in K_\partial, \end{aligned}$$

From which, it follows that

$$-\langle \mathcal{F}^*(\vartheta, \gamma), \partial(u, \vartheta) \rangle \geq 0.$$

Hence, the G-differentiable fuzzy mapping  $\mathcal{F}_\gamma(u) = [\gamma u, (5 - 4\gamma)u]$  is fuzzy HOS-pseudo monotone w.r.t.  $\partial(\vartheta, u) = u - \vartheta$ , with  $1 \leq \Omega$  and  $p \geq 1$ , where  $\vartheta \leq u$ . it can be easily note that  $\mathcal{F}_*(u)$  is neither fuzzy pseudomonotone mapping nor fuzzy quasimonotone w.r.t.  $\partial$ .

**Theorem 4.3.** Let  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  be fuzzy mapping on  $K_\partial$  w.r.t.  $\partial$  and Condition C hold. Let  $\mathcal{F}(u)$  is G-differentiable on  $K_\partial$  with following conditions:

- (i)  $\mathcal{F}(u + \tau \partial(\vartheta, u)) \leq \mathcal{F}(\vartheta)$ .
- (ii)  $\mathcal{F}_*(\cdot)$  is a fuzzy HOS-monotone.

Then

$$\mathcal{F}(\vartheta) \preceq \langle \mathcal{F}(u) \succeq \mathcal{F}_*(u), \partial(\vartheta, u) \rangle \mp \frac{2}{p} \Omega \|\partial(\vartheta, u)\|^p, \quad (33)$$

$$\forall u, \vartheta \in K_\partial.$$

**Proof.** Let  $\mathcal{F}_*(\cdot)$  is fuzzy HOS-monotone. Then, from (26), we have

$$\begin{aligned} & \langle \mathcal{F}_*(\vartheta), \partial(u, \vartheta) \rangle \\ & \leq \preceq \langle \mathcal{F}_*(u), \partial(\vartheta, u) \rangle \preceq \Omega (\|\partial(\vartheta, u)\|^p + \|\partial(u, \vartheta)\|^p). \end{aligned}$$

Therefore, for every  $\gamma \in [0, 1]$ , we have

$$\begin{aligned} & \langle \mathcal{F}_*(\vartheta, \gamma), \partial(u, \vartheta) \rangle \leq -\langle \mathcal{F}_*(u, \gamma), \partial(\vartheta, u) \rangle \\ & \quad -\Omega (\|\partial(\vartheta, u)\|^p + \|\partial(u, \vartheta)\|^p), \\ & \langle \mathcal{F}^*(\vartheta, \gamma), \partial(u, \vartheta) \rangle \leq -\langle \mathcal{F}^*(u, \gamma), \partial(\vartheta, u) \rangle \\ & \quad -\Omega (\|\partial(\vartheta, u)\|^p + \|\partial(u, \vartheta)\|^p), \end{aligned} \quad (34)$$

Since  $K_\partial$  is an invex set so we have,  $\vartheta_\tau = u + \tau \partial(\vartheta, u) \in K_\partial$  for all  $u, \vartheta \in K_\partial$  and  $\tau \in [0, 1]$ . Taking  $\vartheta = \vartheta_\tau$  in (34), we get

$$\begin{aligned} & \langle \mathcal{F}_*(u + \tau \partial(\vartheta, u), \gamma), \partial(u, u + \tau \partial(\vartheta, u)) \rangle \\ & \leq -\langle \mathcal{F}_*(u, \gamma), \partial(u + \tau \partial(\vartheta, u), u) \rangle \\ & \quad -\Omega \left( \|\partial(u + \tau \partial(\vartheta, u), u)\|^p + \|\partial(u, u + \tau \partial(\vartheta, u))\|^p \right), \\ & \langle \mathcal{F}^*(u + \tau \partial(\vartheta, u), \gamma), \partial(u, u + \tau \partial(\vartheta, u)) \rangle \\ & \leq -\langle \mathcal{F}^*(u, \gamma), \partial(u + \tau \partial(\vartheta, u), u) \rangle \\ & \quad -\Omega \left( \|\partial(u + \tau \partial(\vartheta, u), u)\|^p + \|\partial(u, u + \tau \partial(\vartheta, u))\|^p \right), \end{aligned}$$

by using Condition C, we have

$$\begin{aligned} & \langle \mathcal{F}_*(u + \tau \partial(\vartheta, u), \gamma), \tau \partial(\vartheta, u) \rangle \\ & \geq \langle \mathcal{F}_*(u, \gamma), \tau \partial(\vartheta, u) \rangle + 2\Omega \tau^p \|\partial(\vartheta, u)\|^p, \\ & \langle \mathcal{F}^*(u + \tau \partial(\vartheta, u), \gamma), \tau \partial(\vartheta, u) \rangle \\ & \geq \langle \mathcal{F}^*(u, \gamma), \tau \partial(\vartheta, u) \rangle + 2\Omega \tau^p \|\partial(\vartheta, u)\|^p, \\ & \langle \mathcal{F}_*(u + \tau \partial(\vartheta, u), \gamma), \partial(\vartheta, u) \rangle \\ & \geq \langle \mathcal{F}_*(u, \gamma), \partial(\vartheta, u) \rangle + 2\Omega \tau^{p-1} \|\partial(\vartheta, u)\|^p, \\ & \langle \mathcal{F}^*(u + \tau \partial(\vartheta, u), \gamma), \partial(\vartheta, u) \rangle \\ & \geq \mathcal{F}^*(u, \gamma), \partial(\vartheta, u) + 2\Omega \tau^{p-1} \|\partial(\vartheta, u)\|^p, \end{aligned} \quad (35)$$

Let

$$\begin{aligned} H_*(\tau) &= \mathcal{F}_*(u + \tau \partial(\vartheta, u), \gamma), \\ H^*(\tau) &= \mathcal{F}^*(u + \tau \partial(\vartheta, u), \gamma), \end{aligned}$$

taking G-derivative w.r.t.  $\tau$ , we get

$$\begin{aligned} H_*'(\tau) &= \mathcal{F}_*(u + \tau \partial(\vartheta, u), \gamma) \cdot \partial(\vartheta, u) \\ &= \langle \mathcal{F}_*(u + \tau \partial(\vartheta, u), \gamma), \partial(\vartheta, u) \rangle, \\ H^*(\tau) &= \mathcal{F}^*(u + \tau \partial(\vartheta, u), \gamma) \cdot \partial(\vartheta, u) \\ &= \langle \mathcal{F}^*(u + \tau \partial(\vartheta, u), \gamma), \partial(\vartheta, u) \rangle, \end{aligned}$$

from which, using (35), we have

$$\begin{aligned} H_*'(\tau) &\geq \langle \mathcal{F}_*(u, \gamma), \partial(\vartheta, u) \rangle + 2\Omega \tau^{p-1} \|\partial(\vartheta, u)\|^p, \\ H^*(\tau) &\geq \langle \mathcal{F}^*(u, \gamma), \partial(\vartheta, u) \rangle + 2\Omega \tau^{p-1} \|\partial(\vartheta, u)\|^p. \end{aligned} \quad (36)$$

Integrating (36) over  $[0, 1]$  w.r.t.  $\tau$ , we get

$$\begin{aligned} H_*(1) - H_*(0) &\geq \langle \mathcal{F}_*^*(u, \gamma), \partial(\vartheta, u) \rangle + \frac{2}{p} \Omega \|\partial(\vartheta, u)\|^p, \\ H^*(1) - H^*(0) &\geq \langle \mathcal{F}^*(u, \gamma), \partial(\vartheta, u) \rangle + \frac{2}{p} \Omega \|\partial(\vartheta, u)\|^p. \end{aligned}$$

$$\begin{aligned} &\mathcal{F}_*(u + \partial(\vartheta, u), \gamma) - \mathcal{F}_*(u, \gamma) \\ &\geq \langle \mathcal{F}_*^*(u, \gamma), \partial(\vartheta, u) \rangle + \frac{2}{p} \Omega \|\partial(\vartheta, u)\|^p, \\ &\mathcal{F}^*(u + \partial(\vartheta, u), \gamma) - \mathcal{F}^*(u, \gamma) \\ &\geq \langle \mathcal{F}^*(u, \gamma), \partial(\vartheta, u) \rangle + \frac{2}{p} \Omega \|\partial(\vartheta, u)\|^p. \end{aligned}$$

From condition (i), we have

$$\begin{aligned} \mathcal{F}_*(\vartheta, \gamma) - \mathcal{F}_*(u, \gamma) &\geq \langle \mathcal{F}_*^*(u, \gamma), \partial(\vartheta, u) \rangle + \frac{2}{p} \Omega \|\partial(\vartheta, u)\|^p, \\ \mathcal{F}^*(\vartheta, \gamma) - \mathcal{F}^*(u, \gamma) &\geq \langle \mathcal{F}^*(u, \gamma), \partial(\vartheta, u) \rangle + \frac{2}{p} \Omega \|\partial(\vartheta, u)\|^p. \end{aligned}$$

i.e.,

$$\begin{aligned} \mathcal{F}(\vartheta) \dot{\succeq} \mathcal{F}(u) &\geq \langle \mathcal{F}^*(u), \partial(\vartheta, u) \rangle + \frac{2}{p} \Omega \|\partial(\vartheta, u)\|^p, \\ \forall u, \vartheta \in K_\partial. \end{aligned}$$

**Theorem 4.4.** Let  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  be fuzzy mapping on  $K_\partial$  w.r.t.  $\partial$  and Condition C hold. Let  $\mathcal{F}(u)$  is G-differentiable on  $K_\partial$  with following conditions:

- (i)  $\mathcal{F}(u + \tau \partial(\vartheta, u)) \preceq \mathcal{F}(\vartheta)$ .
- (ii)  $\mathcal{F}^*(.)$  is a fuzzy HOS-pseudomonotone.

Then  $\mathcal{F}$  is a HOS-pseudo invex fuzzy mapping.

**Proof.** Let  $\mathcal{F}^*$  be a fuzzy HOS-pseudomonotone. Then for all  $u, \vartheta \in K_\partial$ , we have

$$\langle \mathcal{F}^*(u), \partial(\vartheta, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p \geq \tilde{0}.$$

Therefore, for every  $\gamma \in [0, 1]$ , we have

$$\begin{aligned} \langle \mathcal{F}_*^*(u, \gamma), \partial(\vartheta, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p &\geq 0, \\ \langle \mathcal{F}^*(u, \gamma), \partial(\vartheta, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p &\geq 0. \end{aligned}$$

which implies that

$$\begin{aligned} -\langle \mathcal{F}_*^*(\vartheta, \gamma), \partial(u, \vartheta) \rangle &\geq 0, \\ -\langle \mathcal{F}^*(\vartheta, \gamma), \partial(u, \vartheta) \rangle &\geq 0. \end{aligned} \quad (37)$$

Since  $K_\partial$  is an invex set so we have,  $\vartheta_\tau = u + \tau \partial(\vartheta, u) \in K_\partial$  for all  $u, \vartheta \in K_\partial$  and  $\tau \in [0, 1]$ . Taking  $\vartheta = \vartheta_\tau$  in (37), we get

$$\begin{aligned} -\langle \mathcal{F}_*^*(u + \tau \partial(\vartheta, u), \gamma), \partial(u, u + \tau \partial(\vartheta, u)) \rangle &\geq 0, \\ -\langle \mathcal{F}^*(u + \tau \partial(\vartheta, u), \gamma), \partial(u, u + \tau \partial(\vartheta, u)) \rangle &\geq 0. \end{aligned}$$

by using Condition C, we have

$$\begin{aligned} \langle \mathcal{F}_*^*(u + \tau \partial(\vartheta, u), \gamma), \partial(\vartheta, u) \rangle &\geq 0, \\ \langle \mathcal{F}^*(u + \tau \partial(\vartheta, u), \gamma), \partial(\vartheta, u) \rangle &\geq 0. \end{aligned} \quad (38)$$

Assume that

$$\begin{aligned} H_*(\tau) &= \mathcal{F}_*(u + \tau \partial(\vartheta, u), \gamma), \\ H^*(\tau) &= \mathcal{F}^*(u + \tau \partial(\vartheta, u), \gamma), \end{aligned}$$

taking G-derivative w.r.t.  $\tau$ , then using (38), we have

$$\begin{aligned} H_*'(\tau) &= \langle \mathcal{F}_*^*(u + \tau \partial(\vartheta, u), \gamma), \partial(\vartheta, u) \rangle \geq 0, \\ H^*(\tau) &= \langle \mathcal{F}^*(u + \tau \partial(\vartheta, u), \gamma), \partial(\vartheta, u) \rangle \geq 0, \end{aligned} \quad (39)$$

Integrating (39) over  $[0, 1]$  w.r.t.  $\tau$ , we get

$$\begin{aligned} H_*(1) - H_*(0) &\geq 0, \\ H^*(1) - H^*(0) &\geq 0, \end{aligned}$$

which implies that

$$\begin{aligned} \mathcal{F}_*(u + \partial(\vartheta, u), \gamma) - \mathcal{F}_*(u, \gamma) &\geq 0, \\ \mathcal{F}^*(u + \partial(\vartheta, u), \gamma) - \mathcal{F}^*(u, \gamma) &\geq 0. \end{aligned}$$

From condition (i), we have

$$\begin{aligned} \mathcal{F}_*(\vartheta, \gamma) - \mathcal{F}_*(u, \gamma) &\geq 0, \\ \mathcal{F}^*(\vartheta, \gamma) - \mathcal{F}^*(u, \gamma) &\geq 0, \end{aligned}$$

i.e.,

$$\mathcal{F}(\vartheta) \dot{\succeq} \mathcal{F}(u) \geq \tilde{0}, \forall u, \vartheta \in K_\partial.$$

Hence,  $\mathcal{F}(u)$  is a HOS-pseudo invex fuzzy mapping.

This result finds the necessary condition for HOS-pseudo invex fuzzy mapping.

As special case of Theorems 4.3 and 4.4, we have the following:

**Corollary 4.2.** Let  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  be fuzzy mapping on  $K_\partial$  w.r.t.  $\partial$  and Condition C hold. Let  $\mathcal{F}(u)$  is G-differentiable on  $K_\partial$  with following conditions:

- (i)  $\mathcal{F}(u + \tau \partial(\vartheta, u)) \preceq \mathcal{F}(\vartheta)$ .
- (ii)  $\mathcal{F}^*(.)$  is a fuzzy pseudomonotone.

Then  $\mathcal{F}$  is a pseudo invex fuzzy mapping.

**Corollary 4.3.** Let  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  be fuzzy mapping on  $K_\partial$  w.r.t.  $\partial$  and Condition C hold. Let  $\mathcal{F}(u)$  is G-differentiable on  $K_\partial$  with following conditions

- (i)  $\mathcal{F}(u + \tau \partial(\vartheta, u)) \preceq \mathcal{F}(\vartheta)$ .
- (ii)  $\mathcal{F}^*(.)$  is a fuzzy quasimonotone.

Then  $\mathcal{F}$  is a quasi invex fuzzy mapping.

We now discuss the fuzzy optimality condition for G-differentiable HOS-preinvex fuzzy mappings, which is main motivation of our results.

## 5. HIGHER-ORDER STRONGLY FUZZY MIXED VARIATIONAL-LIKE INEQUALITIES

A familiar reality in mathematical programming is that fuzzy variational inequality theory and complementary problem theory established strong relationship with mathematical problems.

**Theorem 5.1.** Let  $\mathcal{F}$  be a  $G$ -differentiable HOS-preinvex fuzzy mapping with modulus  $\Omega > 0$ . If  $u \in K_\partial$  is the minimum of the mapping  $\mathcal{F}$ , then

$$\mathcal{F}(\vartheta) \succeq \mathcal{F}(u) \succeq \Omega \|\partial(\vartheta, u)\|^p, \forall u, \vartheta \in K_\partial. \quad (40)$$

**Proof:** Let  $u \in K_\partial$  be a minimum of  $\mathcal{F}$ . Then

$$\mathcal{F}(u) \leq \mathcal{F}(\vartheta), \forall \vartheta \in K_\partial.$$

Therefore, for every  $\gamma \in [0, 1]$ , we have

$$\begin{aligned} \mathcal{F}_*(u, \gamma) &\leq \mathcal{F}_*(\vartheta, \gamma), \\ \mathcal{F}^*(u, \gamma) &\leq \mathcal{F}^*(\vartheta, \gamma). \end{aligned} \quad (41)$$

Since  $K_\partial$  is an invex set, for all  $u, \vartheta \in K_\partial$ ,  $\tau \in [0, 1]$ ,  $\vartheta_\tau = u + \tau\partial(\vartheta, u) \in K_\partial$ . Taking  $\vartheta = \vartheta_\tau$  in (41), we get

$$\begin{aligned} 0 &\leq \frac{\mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) - \mathcal{F}_*(u, \gamma)}{\tau}, \\ 0 &\leq \frac{\mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) - \mathcal{F}^*(u, \gamma)}{\tau}. \end{aligned}$$

Taking limit in the above inequality as  $\tau \rightarrow 0$ , we get

$$\begin{aligned} 0 &\leq \langle \mathcal{F}_*'(u, \gamma), \partial(\vartheta, u) \rangle, \\ 0 &\leq \langle \mathcal{F}^{*'}(u, \gamma), \partial(\vartheta, u) \rangle. \end{aligned} \quad (42)$$

Since  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  is a  $G$ -differentiable HOS-preinvex fuzzy mapping, so

$$\begin{aligned} &\mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) \\ &\leq (1 - \tau)\mathcal{F}_*(u, \gamma) + \tau\mathcal{F}_*(\vartheta, \gamma) \\ &\quad - \Omega \{ \tau^p (1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \\ &\mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) \\ &\leq (1 - \tau)\mathcal{F}^*(u, \gamma) + \tau\mathcal{F}^*(\vartheta, \gamma) \\ &\quad - \Omega \{ \tau^p (1 - \tau) + \tau(1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned}$$

$$\begin{aligned} &\mathcal{F}_*(\vartheta, \gamma) - \mathcal{F}_*(u, \gamma) \\ &\geq \frac{\mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) - \mathcal{F}_*(u, \gamma)}{\tau} \\ &\quad + \Omega \{ \tau^{p-1} (1 - \tau) + (1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \\ &\mathcal{F}^*(\vartheta, \gamma) - \mathcal{F}^*(u, \gamma) \\ &\geq \frac{\mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) - \mathcal{F}^*(u, \gamma)}{\tau} \\ &\quad + \Omega \{ \tau^{p-1} (1 - \tau) + (1 - \tau)^p \} \|\partial(\vartheta, u)\|^p, \end{aligned}$$

again taking limit in the above inequality as  $\tau \rightarrow 0$ , we get

$$\begin{aligned} \mathcal{F}_*(\vartheta, \gamma) - \mathcal{F}_*(u, \gamma) &\geq \langle \mathcal{F}_*'(u, \gamma), \partial(\vartheta, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p, \\ \mathcal{F}^*(\vartheta, \gamma) - \mathcal{F}^*(u, \gamma) &\geq \langle \mathcal{F}^{*'}(u, \gamma), \partial(\vartheta, u) \rangle + \Omega \|\partial(\vartheta, u)\|^p, \end{aligned}$$

from which, using (42), we have

$$\begin{aligned} \mathcal{F}_*(\vartheta, \gamma) - \mathcal{F}_*(u, \gamma) &\geq \Omega \|\partial(\vartheta, u)\|^p, \\ \mathcal{F}^*(\vartheta, \gamma) - \mathcal{F}^*(u, \gamma) &\geq \Omega \|\partial(\vartheta, u)\|^p, \end{aligned}$$

i.e.,

$$\mathcal{F}(\vartheta) \succeq \mathcal{F}(u) \succeq \Omega \|\partial(\vartheta, u)\|^p.$$

Hence, the result follows.

**Remark 5.1.** If  $\mathcal{F}(u)$  be a  $G$ -differentiable HOS-preinvex fuzzy mapping modulus  $\Omega > 0$ , and

$$\langle \mathcal{F}'(u), \partial(\vartheta, u) \rangle \mp \Omega \|\partial(\vartheta, u)\|^p \geq \tilde{0}, \forall u, \vartheta \in K_\partial, \quad (43)$$

then  $u \in K_\partial$  is the minimum of the mapping  $\mathcal{F}(u)$ . The inequality of the type (43) is called HOS-fuzzy variational-like inequality. It is very important to note that the optimality condition of preinvex fuzzy mappings can't be obtained with the help of (43). So this idea inspires us to introduce a more general form of fuzzy variational-like inequality of which (43) is a special case. To be more unambiguous, for given fuzzy mapping  $Q$ , bi function  $\partial(\cdot, \cdot)$  and a  $\Omega > 0$ , consider the problem of finding  $u \in K_\partial$ , such that

$$\langle Q(u), \partial(\vartheta, u) \rangle \mp \Omega \|\partial(\vartheta, u)\|^p \geq \tilde{0}, \forall \vartheta \in K_\partial, p \geq 1. \quad (44)$$

This inequality is called HOS-fuzzy variational-like inequality.

We consider the functional  $I(\vartheta)$ , defined as

$$I(\vartheta) = \mathcal{F}(\vartheta) \mp \mathcal{J}(\vartheta), \forall \vartheta \in \mathbb{R}, \quad (45)$$

where  $\mathcal{F}(u)$  is a  $G$ -differentiable preinvex fuzzy mapping and  $\mathcal{J}(u)$  is a non  $G$ -differentiable HOS-preinvex fuzzy mapping.

We know show that the minimum of the functional  $I(\vartheta)$ , can be characterized by a class of variational-like inequalities.

**Theorem 5.2.** Let  $\mathcal{F} : K_\partial \rightarrow \mathfrak{F}_0$  be a  $G$ -differentiable preinvex fuzzy mapping and  $\mathcal{J} : K_\partial \rightarrow \mathfrak{F}_0$  be a non- $G$ -differentiable HOS-preinvex fuzzy mapping. Then the functional  $I(\vartheta)$  has minimum  $u \in K_\partial$ , if and only if  $u \in K_\partial$  satisfies

$$\langle \mathcal{F}'(u), \partial(\vartheta, u) \rangle \mp \mathcal{J}'(\vartheta) \succeq \mathcal{J}(u) \succeq \Omega \|\partial(\vartheta, u)\|^p, \forall \vartheta \in K_\partial. \quad (46)$$

**Proof:** Let  $u \in K_\partial$  be the minimum of  $I$  then by definition, for all  $\vartheta \in K_\partial$  we have

$$I(u) \leq I(\vartheta).$$

Therefore, for every  $\gamma \in [0, 1]$ , we have

$$\begin{aligned} I_*(u, \gamma) &\leq I_*(\vartheta, \gamma), \\ I^*(u, \gamma) &\leq I^*(\vartheta, \gamma). \end{aligned} \quad (47)$$

Since  $K_\partial$  is an invex set so  $\vartheta_\tau = u + \tau\partial(\vartheta, u)$ , for all  $u, \vartheta \in K_\partial$  and  $\tau \in [0, 1]$ . Replacing  $\vartheta$  by  $\vartheta_\tau$  in (47), we get

$$\begin{aligned} I_*(u, \gamma) &\leq I_*(u + \tau\partial(\vartheta, u), \gamma), \\ I^*(u, \gamma) &\leq I^*(u + \tau\partial(\vartheta, u), \gamma). \end{aligned}$$

which implies that, using (45)

$$\begin{aligned} \mathcal{F}_*(u, \gamma) + \mathcal{J}_*(u, \gamma) &\leq \mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) \\ &\quad + \mathcal{J}_*(u + \tau\partial(\vartheta, u), \gamma), \\ \mathcal{F}^*(u, \gamma) + \mathcal{J}^*(u, \gamma) &\leq \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) \\ &\quad + \mathcal{J}^*(u + \tau\partial(\vartheta, u), \gamma). \end{aligned}$$

Since  $\mathcal{J}$  is HOS-preinvex fuzzy mapping then,

$$\begin{aligned} \mathcal{F}_*(u, \gamma) + \mathcal{J}_*(u, \gamma) &\leq \mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) + (1 - \tau)\mathcal{J}_*(u, \gamma) + \tau\mathcal{J}_*(\vartheta, \gamma) \\ &\quad - \Omega \left\{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \right\} \|\partial(\vartheta, u)\|^p, \\ \mathcal{F}^*(u, \gamma) + \mathcal{J}^*(u, \gamma) &\leq \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) + (1 - \tau)\mathcal{J}^*(u, \gamma) + \tau\mathcal{J}^*(\vartheta, \gamma) \\ &\quad - \Omega \left\{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \right\} \|\partial(\vartheta, u)\|^p, \end{aligned}$$

$$\begin{aligned} 0 &\leq \mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) - \mathcal{F}_*(u, \gamma) + \tau \left( \begin{array}{c} \mathcal{J}_*(\vartheta, \gamma) \\ -\mathcal{J}_*(u, \gamma) \end{array} \right) \\ &\quad - \Omega \left\{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \right\} \|\partial(\vartheta, u)\|^p, \\ 0 &\leq \mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) - \mathcal{F}^*(u, \gamma) + \tau \left( \begin{array}{c} \mathcal{J}^*(\vartheta, \gamma) \\ -\mathcal{J}^*(u, \gamma) \end{array} \right) \\ &\quad - \Omega \left\{ \tau^p(1 - \tau) + \tau(1 - \tau)^p \right\} \|\partial(\vartheta, u)\|^p, \end{aligned}$$

Now dividing by “ $\tau$ ” and taking  $\lim_{\tau \rightarrow 0}$ , we have

$$\begin{aligned} 0 &\leq \lim_{\tau \rightarrow 0} \left\{ \begin{array}{c} \frac{\mathcal{F}_*(u + \tau\partial(\vartheta, u), \gamma) - \mathcal{F}_*(u, \gamma)}{\tau} + \mathcal{J}_*(\vartheta, \gamma) \\ -\mathcal{J}_*(u, \gamma) - \Omega \left\{ \begin{array}{c} \tau^{p-1}(1 - \tau) \\ +(1 - \tau)^p \end{array} \right\} \|\partial(\vartheta, u)\|^p \end{array} \right\}, \\ 0 &\leq \lim_{\tau \rightarrow 0} \left\{ \begin{array}{c} \frac{\mathcal{F}^*(u + \tau\partial(\vartheta, u), \gamma) - \mathcal{F}^*(u, \gamma)}{\tau} + \mathcal{J}^*(\vartheta, \gamma) \\ -\mathcal{J}^*(u, \gamma) - \Omega \left\{ \begin{array}{c} \tau^{p-1}(1 - \tau) \\ +(1 - \tau)^p \end{array} \right\} \|\partial(\vartheta, u)\|^p \end{array} \right\}, \end{aligned}$$

then

$$\begin{aligned} 0 &\leq \langle \mathcal{F}_*(u, \gamma), \partial(\vartheta, u) \rangle + \mathcal{J}_*(\vartheta, \gamma) - \mathcal{J}_*(u, \gamma) \\ &\quad - \Omega \|\partial(\vartheta, u)\|^p, \\ 0 &\leq \langle \mathcal{F}^*(u, \gamma), \partial(\vartheta, u) \rangle + \mathcal{J}^*(\vartheta, \gamma) - \mathcal{J}^*(u, \gamma) \\ &\quad - \Omega \|\partial(\vartheta, u)\|^p, \end{aligned}$$

i.e.,

$$\Omega \|\partial(\vartheta, u)\|^p \leq \mathcal{F}_*(u, \gamma) + \mathcal{J}_*(\vartheta, \gamma) - \mathcal{J}_*(u, \gamma).$$

Conversely, let (46) be satisfy to prove  $u \in K_\partial$  is a minimum of  $I$ . Assume that for all  $\vartheta \in K_\partial$  we have

$$\begin{aligned} I(u) \preceq I(\vartheta) &= \mathcal{F}(u) \tilde{+} \mathcal{J}(u) \preceq \mathcal{F}(\vartheta) \preceq \mathcal{J}(\vartheta), \\ &= \mathcal{F}(u) \preceq \mathcal{F}(\vartheta) \tilde{+} \mathcal{J}(u) \preceq \mathcal{J}(\vartheta), \end{aligned}$$

Therefore, for every  $\gamma \in [0, 1]$ , we have

$$\begin{aligned} I_*(u, \gamma) - I_*(\vartheta, \gamma) &= \mathcal{F}_*(u, \gamma) - \mathcal{F}_*(\vartheta, \gamma) \\ &\quad + \mathcal{J}_*(u, \gamma) - \mathcal{J}_*(\vartheta, \gamma), \\ I^*(u, \gamma) - I^*(\vartheta, \gamma) &= \mathcal{F}^*(u, \gamma) - \mathcal{F}^*(\vartheta, \gamma) \\ &\quad + \mathcal{J}^*(u, \gamma) - \mathcal{J}^*(\vartheta, \gamma). \end{aligned}$$

by Corollary 4.1, we have

$$\begin{aligned} I_*(u, \gamma) - I_*(\vartheta, \gamma) &\leq - \left[ \begin{array}{c} \langle \mathcal{F}_*(u, \gamma), \partial(\vartheta, u) \rangle \\ + \mathcal{J}_*(\vartheta, \gamma) - \mathcal{J}_*(u, \gamma) \end{array} \right], \\ I^*(u, \gamma) - I^*(\vartheta, \gamma) &\leq - \left[ \begin{array}{c} \langle \mathcal{F}^*(u, \gamma), \partial(\vartheta, u) \rangle \\ + \mathcal{J}^*(\vartheta, \gamma) - \mathcal{J}^*(u, \gamma) \end{array} \right]. \end{aligned}$$

from which, using (46), we have

$$\begin{aligned} I_*(u, \gamma) - I_*(\vartheta, \gamma) &\leq -\Omega \|\partial(\vartheta, u)\|^p \leq 0, \\ I^*(u, \gamma) - I^*(\vartheta, \gamma) &\leq -\Omega \|\partial(\vartheta, u)\|^p \leq 0, \end{aligned}$$

i.e.,

$$I(u) \preceq I(\vartheta) \preceq -\Omega \|\partial(\vartheta, u)\|^p \preceq \tilde{0},$$

hence,  $I(u) \preceq I(\vartheta)$ .

Note that the (46) is called HOS-fuzzy mixed variational-like inequalities. This result shows that the minimum of fuzzy functional  $I(\vartheta)$  can be characterized by HOS-fuzzy mixed variational-like inequality. It is very important to observe that optimality conditions of preinvex fuzzy mappings and HOS-preinvex fuzzy mappings can't be obtained with the help of (46). This idea encourage us to introduce a more general type of fuzzy variational-like inequality of which (46) is a particular case. In order to be more precise, for given fuzzy mappings  $Q, \mathcal{T}$ , bi function  $\partial(\cdot, \cdot)$  and a  $\Omega > 0$ , consider problem of finding  $u \in K_\partial$ , such that

$$\begin{aligned} \langle Q(u), \partial(\vartheta, u) \rangle \tilde{+} \mathcal{T}(\vartheta) \preceq \mathcal{T}(u) \tilde{+} \Omega \|\partial(\vartheta, u)\|^p \preceq \tilde{0}, \\ \forall \vartheta \in K_\partial, p \geq 1. \end{aligned} \quad (48)$$

This inequality is called HOS-fuzzy mixed variational-like inequality.

Now we discuss some special cases of HOS-fuzzy mixed variational-like inequalities:

If  $p = 2$ , then (48) is called Strongly fuzzy mixed variational-like inequality such as

$$\langle Q(u), \partial(\vartheta, u) \rangle \tilde{+} \mathcal{T}(\vartheta) \preceq \mathcal{T}(u) \tilde{+} \Omega \|\partial(\vartheta, u)\|^2 \preceq \tilde{0}, \forall \vartheta \in K_\partial.$$

If  $\partial(\vartheta, u) = \vartheta - u$ , then (48) is called strongly fuzzy mixed variational inequality such as

$$\langle Q(u), \vartheta - u \rangle \tilde{+} \mathcal{T}(\vartheta) \preceq \mathcal{T}(u) \tilde{+} \Omega \|\vartheta - u\|^2 \preceq \tilde{0}, \forall \vartheta \in K_\partial.$$



Similarly, we can obtain fuzzy variational inequality and variational-like inequality as special cases of (48). In a similar way, some special cases of HOS-fuzzy variational-like inequality can also be discussed.

**Remark 5.2.** The inequalities (44) and (48), shows that the variational-like inequalities arise naturally in connection with the minimization of the G-differentiable preinvex fuzzy mappings subject to certain constraints.

## 6. CONCLUSION

Convex and nonconvex fuzzy mappings play an important role in fuzzy optimization. Therefore, by the importance of nonconvex fuzzy mappings, we introduced and consider a family of classes of nonconvex fuzzy mappings is called HOS-preinvex fuzzy mappings. It is illustrated that classical convexity and nonconvexity are special cases of HOS-preinvex fuzzy mappings. We have also introduced the notions of quasi-preinvex and log-preinvex fuzzy mappings and investigated some properties. Some relations among the HOS-preinvex fuzzy mappings, HOS-invex fuzzy mappings, and fuzzy HOS-monotonicities are derived under some mild conditions. We have proved that optimality conditions of G-differentiable HOS-preinvex fuzzy mappings and for the sum of G-differentiable preinvex fuzzy mappings and non G-differentiable HOS-preinvex fuzzy mappings can be characterized by HOS-fuzzy variational-like inequalities and HOS-fuzzy mixed variational-like inequalities, respectively. The inequalities (44) and (48) are the interesting outcome of our main results. It is itself an engaging problem to flourish some well-organized some numerical methods for solving HOS-fuzzy variational-like inequalities and HOS-fuzzy mixed variational-like inequalities together with applications in applied and pure sciences. In the future, we try to investigate the applications of HOS-fuzzy variational-like inequalities and HOS-fuzzy mixed variational-like inequalities in existence theory. We hope that these concepts and applications will be helpful for other authors to pay their roles in different fields of sciences.

## CONFLICTS OF INTEREST

The authors declare that they have no competing interests.

## AUTHORS' CONTRIBUTIONS

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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