Research Article

Interior BCK/BCI-Algebras

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ABSTRACT

The notions of interior BCK/BCI-algebras, positive implicative interior BCK-algebras, (weak) interior ideals, positive implicative interior ideals, a positive implicative weak interior ideal of type 1, type 2, and type 3 are introduced, and related properties are investigated. A mapping is provided to the set of all involutions of a bounded BCK-algebra in relation to any interior BCK-algebra so that the set of all involutions of a bounded BCK-algebra can be an interior BCK-algebra. The relationship between interior ideals, weak interior ideals, and positive implicative interior ideals is established. The conditions under which a weak interior ideal can change to an interior ideal are founded. The conditions for an interior ideal to be a positive implicative interior ideal are provided. The scalability for a positive implicative interior ideal is discussed. The relationship between type 1, type 2, and type 3 on positive implicative weak interior ideals are investigated. The relationship between weak interior ideal and positive implicative weak interior ideal of type 1, type 2, and type 3 are established.

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1. INTRODUCTION

Interior operators are mainly dealt with in topology and category theory. The notion of an interior operator in an arbitrary category was formally introduced by Vorster [1], and it was successfully used in [2,3] to study notions of connectedness and disconnectedness in topology. Sgro presented a model theory of the interior operator on product topologies with continuous functions (see [4]). An interior algebra is a certain type of algebraic structure that encodes the idea of the topological interior of a set. Interior algebras are to topology and modal logic.

In this article, we introduce the notions of interior BCK/BCI-algebras, positive implicative interior BCK-algebras, (weak) interior ideals, positive implicative interior ideals, a positive implicative weak interior ideal of type 1, type 2, and type 3. We present examples that support these notions and look at the relevant properties. Using the set of all interior BCK/BCI-algebras, we make a BCK/BCI-algebra. We suggest the conditions under which the composition of two interior BCK/BCI-algebras can be an interior BCK/BCI-algebra. We provide a mapping to the set of all involutions of a bounded BCK-algebra in relation to any interior BCK-algebra so that the set of all involutions of a bounded BCK-algebra can be an interior BCK-algebra. We show that the intersection of all (weak) interior ideals in an interior BCK/BCI-algebra is also a (weak) interior ideal in the same interior BCK/BCI-algebra. We discuss the

relationship between interior ideals, weak interior ideals, and positive implicative interior ideals. We look for conditions under which a weak interior ideal can be an interior ideal. We provide conditions under which an interior ideal can be a positive implicative interior ideal. We discuss the scalability for a positive implicative interior ideal. We consider the relationship between type 1, type 2, and type 3 on positive implicative weak interior ideals. We establish the relationship between weak interior ideal and positive implicative weak interior ideal of type 1, type 2, and type 3.

As far as the authors know, this is the first article that investigates the interior operations in BCK/BCI-algebras.

2. PRELIMINARIES

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [5,6]) and was extensively investigated by several researchers. Please refer to [7–13]

We recall the definitions and basic results required in this paper. See the books [14,15] for further information regarding BCK/BCI-algebras.

If a set X has a special element 0 and a binary operation $*$ satisfying the conditions:

- I) $(\forall u, v, w \in X) (((u * v) * (u * w)) * (w * v) = 0),$
- II) $(\forall u, v \in X) ((u * (u * v)) * v = 0),$

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- III) $(\forall u \in X)(u * u = 0)$,
- IV) $(\forall u, v \in X)(u * v = 0, v * u = 0 \Rightarrow u = v)$,
then we say that *BCK-algebra* is a *X*. If a BCI-algebra *X* satisfies the following identity:
- V) $(\forall u \in X)(0 * u = 0)$,
then *X* is called a *BCK-algebra*.

The order relation “ $\leq$ ” in a BCK/BCI-algebra *X* is defined as follows:

$$(\forall x, y \in X) (x \leq y \Leftrightarrow x * y = 0). \quad (1)$$

Every BCK/BCI-algebra *X* satisfies the following conditions:

$$(\forall u \in X) (u * 0 = u), \quad (2)$$

$$(\forall u, v, w \in X)(u \leq v \Rightarrow u * w \leq v * w, w * v \leq w * u), \quad (3)$$

$$(\forall u, v, w \in X)((u * v) * w = (u * w) * v) \quad (4)$$

where $u \leq v$ if and only if $u * v = 0$.

A BCK-algebra *X* is said to be

- *bounded* (see [15]) if there exist an element 1 in *X* such that $x \leq 1$ for all $x \in X$. In a bounded BCK-algebra *X*, we denote $1 * x$ by $\neg x$.
- *positive implicative* (see [15]) if it satisfies

$$(\forall x, y, z \in X)((x * z) * (y * z) = (x * y) * z). \quad (5)$$

Every bounded BCK-algebra *X* satisfies

$$(\forall x, y \in X)(x \leq y \Rightarrow \neg y \leq \neg x), \quad (6)$$

$$(\forall x \in X)(\neg \neg x = x). \quad (7)$$

If an element *x* of a bounded BCK-algebra *X* satisfies $\neg \neg x = x$, then *x* is called an involution of *X* (see [15]). The set of all involutions of a bounded BCK-algebra *X* is denoted by $Iv(X)$. Note that if *X* is a bounded BCK-algebra, then $Iv(X)$ is a bounded subalgebra of *X*.

A nonempty subset *S* of a BCK/BCI-algebra *X* is called a *subalgebra* of *X* (see [15]) if $x * y \in S$ for all $x, y \in S$. A subset *S* of a BCK/BCI-algebra *X* is called an *ideal* of *X* (see [15]) if it satisfies

$$0 \in A, \quad (8)$$

$$(\forall x \in X)(\forall y \in A)(x * y \in A \Rightarrow x \in A). \quad (9)$$

Every ideal *A* of a BCK/BCI-algebra *X* satisfies the next assertion.

$$(\forall x, y \in X)(x \leq y, y \in A \Rightarrow x \in A). \quad (10)$$

A subset *A* of a BCK-algebra *X* is called a *positive implicative ideal* of *X* (see [15]) if it satisfies (8) and

$$(\forall u, v, w \in X)((u * v) * w \in A, v * w \in A \Rightarrow u * w \in A). \quad (11)$$

3. INTERIOR BCK/BCI-ALGEBRAS

Definition 3.1. An *interior BCK/BCI-algebra* is defined to be a pair (X, ℓ) in which *X* is a BCK/BCI-algebra and ℓ is a self-map on *X* such that

$$(\forall x \in X)(\ell(x) \leq x), \quad (12)$$

$$(\forall x \in X)(\ell^2(x) = \ell(x)), \quad (13)$$

$$(\forall x, y \in X)(x \leq y \Rightarrow \ell(x) \leq \ell(y)). \quad (14)$$

Example 3.2

If *X* is a BCK/BCI-algebra, then is an interior BCK/BCI-algebra where (X, i) is the identity self-map on *X*.

Example 3.3

Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	2
3	3	3	2	0	3
4	4	4	4	4	0

Define a mapping $\ell : X \rightarrow X$ by $\ell(0) = 0, \ell(1) = 1, \ell(2) = 1, \ell(3) = 1$ and $\ell(4) = 4$. Then (X, ℓ) is an interior BCK-algebra.

Example 3.4

Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	2	1	0
4	4	4	4	4	0

Then *X* is bounded BCK-algebra with 4 as a greatest element. If we define a mapping $\ell : X \rightarrow X$ by $\ell(0) = 0, \ell(1) = 1, \ell(2) = 0, \ell(3) = 1, \ell(4) = 4$, then (X, ℓ) is an interior BCK-algebra. If we

define a mapping $\mathcal{J} : X \rightarrow X$ by $\mathcal{J}(0) = 0, \mathcal{J}(1) = 1, \mathcal{J}(2) = 2, \mathcal{J}(3) = 0$ and $\mathcal{J}(4) = 4$, then $(X, \mathcal{J})$ is not an interior BCK-algebra since $2 \leq 3$ and $(2) \not\leq \mathcal{J}(3)$.

Given a BCK/BCI-algebra X , let $\ell(X)$ be the set of all interior BCK/BCI-algebras, that is, $\ell(X) = \{(X, \ell_i) \mid \ell_i \text{ is a self-map on } X \text{ that satisfies conditions (12), (13) and (14)}\}$.

We consider a binary operation $\ast$ on $\ell(X)$ as follows:

$$(\forall (X, \ell_i), (X, \ell_j) \in \ell(X)) ((X, \ell_i) \ast (X, \ell_j) = (X, \ell_i \otimes \ell_j)) \quad (15)$$

in which $(\ell_i \otimes \ell_j)(x) = \ell_i(x) \ast \ell_j(x)$ for all $x \in X$.

Theorem 3.5. *If X is a BCK/BCI-algebra, then $(\ell(X), \ast, (X, \ell_0))$ is a BCK/BCI-algebra where $\ell_0(x) = 0$ for all $x \in X$.*

Proof. It is straightforward. $\square$

The following example describes Theorem 3.5.

Example 3.6

Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the following Cayley table.

$\ast$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	2
3	3	3	3	0

The set of all interior BCK/BCI-algebras is $\ell(X) = \{(X, \ell_0), (X, \ell_1), \dots, (X, \ell_7)\}$ where the self-maps $\ell_0, \ell_1, \dots, \ell_7$ are given by the Table 1.

By routine verification we can check that $(\ell(X), \ast, (X, \ell_0))$ is a BCK/BCI-algebra.

Question 3.7. If (X, ℓ) and $(X, \mathcal{J})$ are interior BCK/BCI-algebras, then

- i) Is the composition of (X, ℓ) and $(X, \mathcal{J})$ an interior BCK/BCI-algebra?

Table 1 Table for the self-maps $\ell_0, \ell_1, \dots, \ell_7$.

x	0	1	2	3
$\ell_0(x)$	0	0	0	0
$\ell_1(x)$	0	0	0	3
$\ell_2(x)$	0	0	2	0
$\ell_3(x)$	0	0	2	3
$\ell_4(x)$	0	1	1	1
$\ell_5(x)$	0	1	2	1
$\ell_6(x)$	0	1	1	3
$\ell_7(x)$	0	1	2	3

- ii) Is $\ell \circ \mathcal{J} = \mathcal{J} \circ \ell$?

We look at the example below to see that the answer to the above question is negative.

Example 3.8

Let X be the BCK-algebra and (X, ℓ) be the interior BCK-algebra in Example 3.3. Let $\mathcal{J}$ be the self-map on X which is defined by $\mathcal{J}(0) = 0, \mathcal{J}(1) = 0, \mathcal{J}(2) = 2, \mathcal{J}(3) = 2$ and $\mathcal{J}(4) = 4$. Then $(X, \mathcal{J})$ is an interior BCK-algebra. The compositions $\ell \circ \mathcal{J}$ and $\mathcal{J} \circ \ell$ of ℓ and $\mathcal{J}$ are given by Table 2.

Then $(X, \mathcal{J} \circ \ell)$ is an interior BCK-algebra, but $(X, \ell \circ \mathcal{J})$ is not an interior BCK-algebra since $(\ell \circ \mathcal{J})^2(2) = (\ell \circ \mathcal{J})(1) = 0 \neq 1 = (\ell \circ \mathcal{J})(2)$. According to Table 2, $\ell \circ \mathcal{J} \neq \mathcal{J} \circ \ell$ can be immediately identified.

Example 3.9

Consider a BCI-algebra $X = \{0, 1, 2, a, b\}$ with the following Cayley table.

$\ast$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	0	a	a
2	2	2	0	b	a
a	a	a	a	0	0
b	b	b	a	2	0

Let ℓ and $\mathcal{J}$ be self-maps on X defined by $\ell(0) = \ell(1) = 0, \ell(2) = 2, \ell(a) = a$ and $\ell(b) = a$, while $\mathcal{J}(0) = 0, \mathcal{J}(1) = \mathcal{J}(2) = 1, \mathcal{J}(a) = a$ and $\mathcal{J}(b) = b$. Then (X, ℓ) and $(X, \mathcal{J})$ are interior BCI-algebras. The compositions $\ell \circ \mathcal{J}$ and $\mathcal{J} \circ \ell$ of ℓ and $\mathcal{J}$ are given by Table 3.

Then $(X, \ell \circ \mathcal{J})$ is an interior BCI-algebra, but $(X, \mathcal{J} \circ \ell)$ is not an interior BCI-algebra since $(\mathcal{J} \circ \ell)^2(2) = (\mathcal{J} \circ \ell)(1) = 0 \neq 1 = (\mathcal{J} \circ \ell)(2)$. According to Table 3, $\ell \circ \mathcal{J} \neq \mathcal{J} \circ \ell$ can be immediately identified.

We suggest the conditions under which the composition of two interior BCK/BCI-algebras can be an interior BCK/BCI-algebra.

Theorem 3.10. *If (X, ℓ) and $(X, \mathcal{J})$ are interior BCK/BCI-algebras such that $\ell \circ \mathcal{J} = \mathcal{J} \circ \ell$, then $(X, \ell \circ \mathcal{J})$ is an interior BCK/BCI-algebra.*

Table 2 Table for the compositions $\ell \circ \mathcal{J}$ and $\mathcal{J} \circ \ell$.

x	0	1	2	3	4
$(\ell \circ \mathcal{J})(x)$	0	0	1	1	4
$(\mathcal{J} \circ \ell)(x)$	0	0	0	0	4

Table 3 Table for the compositions $\ell \circ \mathcal{J}$ and $\mathcal{J} \circ \ell$.

x	0	1	2	a	b
$(\ell \circ \mathcal{J})(x)$	0	0	0	a	a
$(\mathcal{J} \circ \ell)(x)$	0	0	1	a	a

Proof. Assume that $\ell \circ \mathcal{J} = \mathcal{J} \circ \ell$. Using (12), we have $(\ell \circ \mathcal{J})(x) = \ell(\mathcal{J}(x)) \leq \mathcal{J}(x) \leq x$ for all $x \in X$, and so $\ell \circ \mathcal{J}$ satisfies the condition (12). Also,

$$\begin{aligned} (\ell \circ \mathcal{J})^2(x) &= ((\ell \circ \mathcal{J}) \circ (\ell \circ \mathcal{J}))(x) \\ &= ((\ell \circ \ell) \circ (\mathcal{J} \circ \mathcal{J}))(x) \\ &= (\ell \circ \ell)(\mathcal{J} \circ \mathcal{J}(x)) \\ &= (\ell \circ \ell)(\mathcal{J}(x)) \\ &= \ell(\mathcal{J}(x)) = (\ell \circ \mathcal{J})(x) \end{aligned}$$

for all $x \in X$, which shows that $\ell \circ \mathcal{J}$ satisfies the condition (13). For every $x, y \in X$, if $x \leq y$ then $\mathcal{J}(x) \leq \mathcal{J}(y)$, and so $(\ell \circ \mathcal{J})(x) = \ell(\mathcal{J}(x)) \leq \ell(\mathcal{J}(y)) = (\ell \circ \mathcal{J})(y)$. Therefore $(X, \ell \circ \mathcal{J})$ is an interior BCK/BCI-algebra. $\square$

Given interior BCK/BCI-algebras (X, ℓ) and $(X, \mathcal{J})$, we define

$$\ell \leq \mathcal{J} \Leftrightarrow (\forall x \in X)(\ell(x) \leq \mathcal{J}(x)). \quad (16)$$

Given a self-map ℓ on X , the set

$$I(\ell) := \{x \in X \mid \ell(x) = x\}$$

is called the *identity part* of ℓ in X , and the set

$$\ker(\ell) := \{x \in X \mid \ell(x) = 0\}$$

is called the *kernel* of ℓ .

Proposition 3.11. If (X, ℓ) and $(X, \mathcal{J})$ are interior BCK/BCI-algebras, then

- i) $\ell \leq \mathcal{J} \Leftrightarrow \ell \circ \mathcal{J} = \ell$,
- ii) $\mathcal{J}(\ell) = \mathcal{J}(\mathcal{J}) \Leftrightarrow \ell = \mathcal{J}$.

Proof.

- i) If $\ell \leq \mathcal{J}$, then $\ell(x) \leq \mathcal{J}(x)$ for all $x \in X$, and so $\ell(x) = \ell^2(x) \leq (\ell \circ \mathcal{J})(x)$. Also $(\ell \circ \mathcal{J})(x) = \ell(\mathcal{J}(x)) \leq \mathcal{J}(\mathcal{J}(x)) = \mathcal{J}(x) \leq x$ which implies that $(\ell \circ \mathcal{J})(x) = \ell(\mathcal{J}(x)) = \ell(\ell(\mathcal{J}(x))) \leq \ell(x)$. Hence $(\ell \circ \mathcal{J})(x) = \ell(x)$ for all $x \in X$. Therefore $\ell \circ \mathcal{J} = \ell$. Conversely, assume that $\ell \circ \mathcal{J} = \ell$. Then $\ell(x) = (\ell \circ \mathcal{J})(x) \leq \mathcal{J}(x)$ for all $x \in X$. Thus $\ell \leq \mathcal{J}$.
- ii) It is clear that if $\ell = \mathcal{J}$, then $\mathcal{J}(\ell) = \mathcal{J}(\mathcal{J})$. Suppose that $\mathcal{J}(\ell) = \mathcal{J}(\mathcal{J})$. The condition (13) induces $\ell(x) \in \mathcal{J}(\ell) = \mathcal{J}(\mathcal{J})$, and so $\mathcal{J}(\ell(x)) = \ell(x)$ for all $x \in X$. Hence $\mathcal{J} \circ \ell = \ell$. Similarly $\ell \circ \mathcal{J} = \mathcal{J}$. Using (12) and (14), we have $\ell(x) = (\mathcal{J} \circ \ell)(x) \leq \mathcal{J}(x)$ and $\mathcal{J}(x) = (\ell \circ \mathcal{J})(x) \leq \ell(x)$. Thus $\ell(x) = \mathcal{J}(x)$ for all $x \in X$. Therefore $\ell = \mathcal{J}$. $\square$

Proposition 3.12. If (X, ℓ) is an interior BCK/BCI-algebra, then

- i) $\ell(x) * y \leq x * \ell(y)$,
- ii) $\ell(x * y) \leq x * \ell(y)$,

for all $x, y \in X$.

Proof. Let $x, y \in X$. Using (12) and (3), we have $\ell(x) * y \leq \ell(x) * \ell(y) \leq x * \ell(y)$ which proves (i). Since $\ell(x * y) \leq x * y$ and $\ell(y) \leq y$, it follows from (3) that $\ell(x * y) \leq x * y \leq x * \ell(y)$. Hence (ii) is valid. $\square$

Proposition 3.13. If (X, ℓ) is an interior BCK-algebra, then

- i) $\ell(0) = 0$. This is also true in an interior BCI-algebra (X, ℓ) ,
- ii) $\ell((x * (x * y)) * (y * x)) \leq \ell(x * (x * (y * (y * x))))$,
- iii) $\ell(x * y) * \ell(z) \leq (x * \ell(z)) * y$ for all $x, y \in X$. Moreover, if X is bounded, then
- iv) $\ell(\neg x) \leq \neg \ell(x)$,
- v) $\ell(\neg x * \neg y) \leq y * x$, for all $x, y \in X$.

Proof.

- i) Using (12), we have $\ell(0) \leq 0$, and so $\ell(0) = \ell(0) * 0 = 0$ by (2).
- ii) For every $x, y, z \in X$, we have

$$\begin{aligned} &((x * (x * y)) * (y * x)) * (x * (x * (y * (y * x)))) \\ &= ((x * (x * y)) * (x * (x * (y * (y * x))))) * (y * x) \\ &= ((x * (x * (x * (y * (y * x))))) * (x * y)) * (y * x) \\ &= ((x * (y * (y * x))) * (x * y)) * (y * x) \\ &\leq (y * (y * (y * x))) * (y * x) \\ &= (y * (y * x)) * (y * (y * x)) = 0 \end{aligned}$$

by (I), (III) and (4). Since $0 \leq x$ for all $x \in X$, it follows from (IV) that

$$((x * (x * y)) * (y * x)) * (x * (x * (y * (y * x)))) = 0,$$

i.e., $(x * (x * y)) * (y * x) \leq x * (x * (y * (y * x)))$. Hence $\ell((x * (x * y)) * (y * x)) \leq \ell(x * (x * (y * (y * x))))$ for all $x, y, z \in X$ by (14).

- iii) Let $x, y, z \in X$. Using (3), (4) and (12), we have

$$\ell(x * y) * \ell(z) \leq (x * y) * \ell(z) = (x * \ell(z)) * y.$$

Assume that X is bounded. If we put $x = 1$ and $y = x$ in Proposition 3.12(ii), then $\ell(\neg x) = \ell(1 * x) \leq 1 * \ell(x) = \neg \ell(x)$ for all $x \in X$. Hence (iv) is valid.

- iv) If we take $u = 1, v = x$ and $w = y$ in (I), then $\neg x * \neg y = (1 * x) * (1 * y) \leq y * x$. It follows from (12) that $\ell(\neg x * \neg y) \leq \neg x * \neg y \leq y * x$. $\square$

Theorem 3.14. If (X, ℓ) is an interior BCK-algebra, then $(Iv(X), \tilde{\ell})$ is an interior BCK-algebra where $\tilde{\ell}$ is a self-map on $Iv(X)$ which is defined by $\tilde{\ell}(x) = \neg \neg \ell(x)$ for all $x \in Iv(X)$.

Proof. Let $x \in Iv(X)$. Since $\ell(x) \leq x$, it follows from (6) that

$$\tilde{\ell}(x) = \neg \neg \ell(x) \leq \neg \neg x = x,$$

that is, $\tilde{\ell}$ satisfies the condition (12). For every $x \in Iv(X)$, we have

$$\tilde{\ell}^2(x) = \tilde{\ell}(\neg\neg\ell(x)) = \neg\neg\neg\neg\ell(x) = \neg\neg\ell(x) = \tilde{\ell}(x)$$

by (7). Hence $\tilde{\ell}$ satisfies the condition (13). Let $x, y \in Iv(X)$ be such that $x \leq y$. Then $\ell(x) \leq \ell(y)$ by (14), and so

$$\tilde{\ell}(x) = \neg\neg\ell(x) \leq \neg\neg\ell(y) = \tilde{\ell}(y)$$

by (6). This shows that $\tilde{\ell}$ satisfies the condition (14). $\square$

Question 3.15. If (X, ℓ) is an interior BCK-algebra, will the following items be established?

$$(\forall x, y, z \in X)(x * y \leq z \Rightarrow \ell(x * z) \leq \ell(y * z)). \quad (17)$$

$$(\forall x, y \in X)(x * y \leq y \Rightarrow \ell(x * y) = \ell(0)). \quad (18)$$

$$(\forall x, y \in X)(\ell((x * y) * y) = \ell(x * y) = \ell((x * y) * (x * (x * y)))). \quad (19)$$

$$(\forall x, y, z \in X)(\ell((x * y) * z) = \ell((x * z) * (y * z))). \quad (20)$$

The following example shows that the answer to the above question is negative.

Example 3.16

Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	1	0	1	2
3	3	1	1	0	3
4	4	4	4	4	0

Define a mapping $\ell : X \rightarrow X$ by $\ell(0) = \ell(4) = 0$, $\ell(1) = \ell(2) = 1$ and $\ell(3) = 3$. Then (X, ℓ) is an interior BCK-algebra. For elements $2, 3 \in X$, we have $2 * 3 = 1 \leq 3$. But $\ell(2 * 3) = \ell(1) = 1 \not\leq 0 = \ell(0) = \ell(3 * 3)$ and $\ell(2 * 3) \neq \ell(0)$. Hence (17) and (18) are not true in general. Since $\ell((3 * 2) * 2) = 0 = \ell((3 * 2) * (3 * (3 * 2)))$ and $\ell(3 * 2) = 1$, we know that (19) is not true. The equality (20) is not true since

$$\ell((2 * 1) * 3) = \ell(0) = 0 \neq 1 = \ell(1) = \ell((2 * 3) * (1 + 3)).$$

Definition 3.17. An interior BCK-algebra (X, ℓ) is said to be *positive implicative* if the condition (20) is established.

Example 3.18

Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	2
3	3	2	1	0	2
4	4	4	4	4	0

Define a mapping $\ell : X \rightarrow X$ by $\ell(0) = \ell(2) = 0$, $\ell(1) = \ell(4) = 1$ and $\ell(3) = 3$. Then (X, ℓ) is a positive implicative interior BCK-algebra.

Theorem 3.19. If (X, ℓ) is a positive implicative BCK-algebra, then the interior BCK-algebra (X, ℓ) is positive implicative.

Proof. It is straightforward. $\square$

In the following example, we can see that the converse of Theorem 3.19 is not established.

Example 3.20

Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Let ℓ be a self-map on X defined by $\ell(0) = \ell(1) = 0$, $\ell(2) = 2$ and $\ell(3) = 3$. Then (X, ℓ) is a positive implicative interior BCK-algebra while X is not a positive implicative BCK-algebra since $(2 * 1) * 1 \neq (2 * 1) * (1 * 1)$.

Proposition 3.21. If (X, ℓ) is a positive implicative interior BCK-algebra, then the conditions (17), (18), (19) and (20) are established.

Proof. Assume that (X, ℓ) is a positive implicative interior BCK-algebra. Let $x, y, z \in X$ be such that $x * y \leq z$. Then $(x * z) * (y * z) = 0$, i.e., $x * z \leq y * z$ since X is a positive implicative BCK-algebra. It follows from (14) that $\ell(x * z) \leq \ell(y * z)$. Hence (17) is valid. Let $x, y \in X$ be such that $x * y \leq y$. Then $x * y = (x * y) * (y * y) = 0$ by (III), (2) and (5). Hence $\ell(x * y) = \ell(0)$, and so (18) holds. Since X is a positive implicative BCK-algebra, we have $x * y = (x * y) * y = (x * y) * (x * (x * y))$ for all $x, y \in X$. Hence $\ell(x * y) = \ell((x * y) * y) = \ell((x * y) * (x * (x * y)))$, i.e., (19) is true. It is clear that (20) is valid. $\square$

4. INTERIOR IDEALS IN INTERIOR BCK/BCI-ALGEBRAS

Definition 4.1. Let (X, ℓ) be an interior BCK/BCI-algebra. Then a subset A of X is called an *interior ideal* in (X, ℓ) if A is an ideal of X which satisfies the following assertion:

$$(\forall x \in X)(\ell(x) \in A \Rightarrow x \in A). \quad (21)$$

It is clear that $\{0\}$ and X are interior ideals in every interior BCK/BCI-algebra (X, ℓ) .

Example 4.2

- i) Consider the interior BCK-algebra (X, ℓ) in Example 3.16. Then $A := \{0, 4\}$ is an interior ideal in $B := \{0, 1, 2, 3\}$, and (X, ℓ) is an ideal of X which is not an interior ideal in (X, ℓ) since $\ell(4) = 0 \in B$ but $4 \notin B$.
- ii) Consider the interior BCI-algebra (X, ℓ) in Example 3.16. Then $A := \{0, 1, 2\}$ is an interior ideal in (X, ℓ) and $B := \{0, 1\}$ is an ideal of (X, ℓ) which is not an interior ideal in X since $j(2) = 1 \in B$ but $2 \notin B$.

Theorem 4.3. The intersection of all interior ideals in an interior BCK/BCI-algebra (X, ℓ) is also an interior ideal in (X, ℓ) .

Proof. Let $\{A_i \mid i \in \Lambda\}$ be the set of all interior ideals in an interior BCK/BCI-algebra (X, ℓ) . It is clear that $\bigcap_{i \in \Lambda} A_i$ is an ideal of X . Let $x \in X$ be such that $\ell(x) \in \bigcap_{i \in \Lambda} A_i$. Then $\ell(x) \in A_i$ for all $i \in \Lambda$. Since A_i is an interior ideal in (X, ℓ) for all $i \in \Lambda$, it follows from (21) that $x \in A_i$ for all $i \in \Lambda$. Therefore $x \in \bigcap_{i \in \Lambda} A_i$, and so $\bigcap_{i \in \Lambda} A_i$ is an interior ideal in (X, ℓ) . $\square$

In the following example, we know that the union of interior ideals in an interior BCK/BCI-algebra (X, ℓ) may not be an interior ideal in (X, ℓ) .

Example 4.4

Consider the BCK-algebra $X = \{0, 1, 2, 3, 4\}$ in Example 3.18. Let ℓ be a self-map on X given by $\ell(0) = 0, \ell(1) = \ell(4) = 1, \ell(2) = 2$ and $\ell(3) = 3$. Then (X, ℓ) is an interior BCK-algebra. Put $A := \{0, 1, 4\}$ and $B := \{0, 2\}$. By routine verification we can check that A and B are interior ideals in (X, ℓ) . Since the union $A \cup B = \{0, 1, 2, 4\}$ of A and B is not an ideal of X , and so it can't be an interior ideal in (X, ℓ) .

Definition 4.5. Let (X, ℓ) be an interior BCK/BCI-algebra. Then a subset A of X is called a *weak interior ideal* in (X, ℓ) if it satisfies (8) and

$$(\forall x, y \in X)(\ell(x) * y \in A, \ell(y) \in A \Rightarrow x \in A). \quad (22)$$

Example 4.6

Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	0
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Let ℓ be a self-map on X given by $\ell(0) = \ell(2) = 0, \ell(1) = \ell(4) = 1$ and $\ell(3) = 3$. Then (X, ℓ) is an interior BCK-algebra and $A := \{0, 2, 4\}$ is a weak interior ideal in (X, ℓ) .

Example 4.7

Consider a BCI-algebra $X = \{0, 1, 2, a, b\}$ with the following Cayley table.

*	0	1	2	a	b
0	0	0	0	b	a
1	1	0	0	b	a
2	2	1	0	b	a
a	a	a	a	0	b
b	b	b	b	a	0

Define a mapping $\ell : X \rightarrow X$ by $\ell(0) = 0, \ell(1) = \ell(2) = 1, \ell(a) = a$ and $\ell(b) = b$. Then (X, ℓ) is an interior BCI-algebra and $A := \{0, 1, 2\}$ is a weak interior ideal in (X, ℓ) .

Note that the interior $A := \{0, 1, 2\}$ of (X, ℓ) in Example 4.7 is an ideal of X . But the interior $A := \{0, 2, 4\}$ of (X, ℓ) in Example 4.6 is not an ideal of X . This shows that any weak interior ideal in an interior BCK/BCI-algebra (X, ℓ) may not be an ideal of X in general.

Theorem 4.8. The intersection of all weak interior ideals in an interior BCK/BCI-algebra (X, ℓ) is also a weak interior ideal in (X, ℓ) .

Proof. Let $\{A_i \mid i \in \Lambda\}$ be the set of all weak interior ideals in an interior BCK/BCI-algebra (X, ℓ) . It is clear that $0 \in \bigcap_{i \in \Lambda} A_i$. Let $x, y \in X$ be such that $\ell(x) * y \in \bigcap_{i \in \Lambda} A_i$ and $\ell(y) \in \bigcap_{i \in \Lambda} A_i$.

Then $\ell(x) * y \in A_i$ and $\ell(y) \in A_i$ for all $i \in \Lambda$. Since A_i is a weak interior ideal in (X, ℓ) for all $i \in \Lambda$, it follows from (22) that $x \in A_i$ for all $i \in \Lambda$. Hence $x \in \bigcap_{i \in \Lambda} A_i$, and therefore $\bigcap_{i \in \Lambda} A_i$ is a weak interior ideal in (X, ℓ) . $\square$

In the following example, we know that the union of weak interior ideals in an interior BCK/BCI-algebra (X, ℓ) may not be a weak interior ideal in (X, ℓ) .

Example 4.9

Consider the BCK-algebra $X=\{0,1,2,3,4\}$ in Example 3.18. Let ℓ be a self-map on X given by $\ell(0) = 0, \ell(1) = \ell(4) = 1, \ell(2) = 2$ and $\ell(3) = 3$. Then (X, ℓ) is an interior BCK-algebra. Routine calculations show that $A_1:=\{0,2\}$ and $A_2 := \{0, 1, 4\}$ are weak interior ideals in (X, ℓ) . But the union $A_1 \cup A_2 = \{0, 1, 2, 4\}$ is not a weak interior ideal in (X, ℓ) since $\ell(3) * 2 = 3 * 2 = 1 \in A_1 \cup A_2$ and $\ell(2) = 2 \in A_1 \cup A_2$, but $3 \notin A_1 \cup A_2$.

Theorem 4.10. *In an interior BCK/BCI-algebra (X, ℓ) , every interior ideal is a weak interior ideal.*

Proof. Let A be an interior ideal in an interior BCK/BCI-algebra (X, ℓ) . Let $x, y \in X$ be such that $\ell(x) * y \in A$ and $\ell(y) \in A$. Then $y \in A$ by (21). Using (9), we have $\ell(x) \in A$ which implies from (21) that $x \in A$. Therefore A is a weak interior ideal in (X, ℓ) . $\square$

The following example shows that the converse of Theorem 4.10 is not established.

Example 4.11

Consider the weak interior ideal A in Example 4.6. Then A is not an ideal of X since $1 * 4 \in A$ and $4 \in A$ but $1 \notin A$. Hence A is not an interior ideal.

In the following theorem, we present a condition under which the converse of Theorem 4.10 can be established in an interior BCK-algebra.

Theorem 4.12. *If A is a weak interior ideal in an interior BCK/BCI-algebra (X, ℓ) which is also an ideal of X , then A is an interior ideal in (X, ℓ) .*

Proof. Let A be a weak interior ideal in an interior BCK/BCI-algebra (X, ℓ) which is also an ideal of X . Let $x \in X$ be such that $\ell(x) \in A$. Then $\ell(x) * 0 = \ell(x) \in A$ and $\ell(0) = 0 \in A$. It follows from (22) that $x \in A$. Therefore A is an interior ideal in (X, ℓ) . $\square$

Let (X, ℓ) be an interior BCK-algebra. Given a nonempty subset L of X , the interior ideal in (X, ℓ) generated by L is defined to be the smallest interior ideal in (X, ℓ) containing L , and it is denoted by $\langle L \rangle_\ell$.

Example 4.13

Consider a BCK-algebra $X=\{0,1,2,3,4\}$ with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Define a mapping $\ell : X \rightarrow X$ by $\ell(0) = 0, \ell(1) = \ell(3) = 1$ and $\ell(2) = \ell(4) = 2$. Then (X, ℓ) is an interior BCK-algebra and all interior ideals are $A_0 = \{0\}, A_1 = \{0, 1\}, A_2 = \{0, 2\}, A_3 =$

$\{0, 1, 3\}, A_4 = \{0, 2, 4\}$ and $A_5 = X$. If we take $L = \{0, 3\}$, then the interior ideal in (X, ℓ) generated by L is $\langle L \rangle_\ell = \{0, 1, 3\}$.

We present a question about the interior ideal generated by a set as follows:

Question 4.14. How is the interior ideal generated by subset of an interior BCK-algebra described?

Definition 4.15. Let (X, ℓ) be an interior BCK-algebra. Then a subset A of X is called a *positive implicative interior ideal* in (X, ℓ) if A is a positive implicative ideal of X which satisfies the condition (21).

Example 4.16

Let X be the BCK-algebra in Example 3.3 and let ℓ be a self-map on X given by $\ell(0) = 0, \ell(1) = \ell(4) = 1$ and $\ell(2) = \ell(3) = 2$. Then (X, ℓ) is an interior BCK-algebra and the set $A:=\{0,1,3\}$ is a positive implicative interior ideal in (X, ℓ) . The set $B:=\{0,1,4\}$ is an interior ideal in (X, ℓ) , but it is not a positive implicative interior ideal in (X, ℓ) since it is not a positive implicative ideal of X .

Theorem 4.17. *In an interior BCK-algebra (X, ℓ) , every positive implicative interior ideal is an interior ideal.*

Proof. It is straightforward because every positive implicative ideal is an ideal in a BCK-algebra. $\square$

In general, the converse of Theorem 4.17 is not true in general as seen in Example 4.16. So we need to find condition(s) for an interior ideal to be a positive implicative interior ideal.

Given an ideal A and an element w in an interior BCK-algebra (X, ℓ) , consider the set

$$A_w := \{x \in X \mid x * w \in A\}. \quad (23)$$

It is clear that $0 \in A_w$. The following example shows that A_w is neither an interior ideal nor a weak interior ideal.

Example 4.18

Consider a BCK-algebra $X=\{0,1,2,3,4\}$ with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Define a mapping $\ell : X \rightarrow X$ by $\ell(0) = 0, \ell(1) = \ell(3) = 1$ and $\ell(2) = \ell(4) = 2$. Then (X, ℓ) is an interior BCK-algebra and $A = \{0, 2\}$ is an ideal of X . If we take $w = 1$, then $A_w = \{0, 1, 2\}$ which is not an ideal of X . Also, A_w does not satisfy (21) since $\ell(4) = 2 \in A_w$ but $4 \notin A_w$. Hence A_w is not an interior ideal in (X, ℓ) . Since $\ell(3) * 1 = 0 \in A_w$ and $\ell(1) = 1 \in A_w$ but $3 \notin A_w$, we know that A_w is not a weak interior ideal in (X, ℓ) .

Question 4.19. Under what conditions can the set A_w be a (weak) interior ideal in (X, ℓ) ?

We consider conditions for an interior ideal to be a positive implicative interior ideal in an interior BCK-algebra.

Theorem 4.20. Let A be an interior ideal in an interior BCK-algebra (X, ℓ) and suppose that A_w is an ideal of X for all $w \in X$. Then A is a positive implicative interior ideal in (X, ℓ) .

Proof. It is sufficient to show that A is a positive implicative ideal of X . Let $x, y, z \in X$ be such that $(x * y) * z \in A$ and $y * z \in A$. Then $x * y \in A_z$ and $y \in A_z$. Since A_z is an ideal of X , we have $x \in A_z$ and so $x * z \in A$. This shows that A is a positive implicative ideal of X , and hence A is a positive implicative interior ideal in (X, ℓ) . $\square$

Given a subset A and an element w in an interior BCK-algebra (X, ℓ) , we present conditions for the set A_w to be an interior ideal.

Theorem 4.21. If A is a positive implicative interior ideal in an interior BCK-algebra (X, ℓ) such that

$$(\forall u \in X) (\ell^2(u) = u), \quad (24)$$

$$(\forall u, v \in X)(u \leq v, u \in A \Rightarrow v \in A), \quad (25)$$

$$(\forall u \in X)(u \in A \Rightarrow \ell(u) \in A), \quad (26)$$

then A_w is an interior ideal in (X, ℓ) for $w \in X$.

Proof. Let $x, y \in X$ be such that $x * y \in A_w$ and $y \in A_w$. Then $(x * y) * w \in A$ and $y * w \in A$. Since A is a positive implicative ideal of X , it follows that $x * w \in A$, that is, $x \in A_w$. Hence A_w is an ideal of X . Let $x \in X$ be such that $\ell(x) \in A_w$. Then $\ell(x) * w \in A$, and so $\ell(\ell(x) * w) \in A$ by (26). Using Proposition 3.12 and (24), we have

$$\ell(\ell(x) * w) \leq \ell(x) * \ell(w) \leq x * \ell^2(w) = x * w.$$

It follows from (25) that $x * w \in A$, that is, $x \in A_w$. Therefore A_w is an interior ideal in (X, ℓ) for $w \in X$. $\square$

Theorem 4.22. Let A be a subset of X in an interior BCK-algebra (X, ℓ) satisfying the condition (21) and

$$0 \in A, \quad (27)$$

$$(\forall x, y \in X)(\forall z \in A) ((x * y) * y \in A_z \Rightarrow x * y \in A), \quad (28)$$

then A is a positive implicative interior ideal in (X, ℓ) .

Proof. Let $x, y \in A$ be such that $x * y \in A$ and $y \in A$. Then $(x * 0) * 0 = x \in A$, and so $x = x * 0 \in A$ by (28). Hence A is an interior ideal in (X, ℓ) . Let $x, y, z \in X$ be such that $(x * y) * z \in A$ and $y * z \in A$. Note that $((x * z) * z) * (y * z) \leq (x * z) * y = (x * y) * z \in A$. Since A is an ideal of X , it follows that $(x * z) * z \in A_0$. Hence $x * z \in A$ by (28). Therefore A is a positive implicative interior ideal in (X, ℓ) . $\square$

In the following theorem, we discuss the scalability for positive implicative interior ideal.

Theorem 4.23. Let A and B be interior ideals in an interior BCK-algebra (X, ℓ) . If A is contained in B and A is a positive implicative interior ideal in (X, ℓ) , then so is B .

Proof. Assume that $(x * y) * z \in B$ for all $x, y, z \in X$. Then

$$((x * ((x * y) * z)) * y) * z = ((x * y_z) * ((x * y) * z)) * z = 0 \in A.$$

Note that

$$\begin{aligned} & (((x * ((x * y) * z)) * (y * z)) * z) * z \\ &= (((x * ((x * y) * z)) * z) * (y * z)) * z \\ &\leq ((x * ((x * y) * z)) * y) * z. \end{aligned}$$

Hence $((x * ((x * y) * z)) * (y * z)) * z \in A$. Since A is a positive implicative ideal of X , it follows that

$$\begin{aligned} & ((x * z) * (y * z)) * ((x * y) * z) = ((x * ((x * y) * z)) * z) * (y * z) \\ &= ((x * ((x * y) * z)) * (y * z)) * z \in A \subseteq B. \end{aligned}$$

Thus $(x * z) * (y * z) \in B$ since B is an ideal of X . Assume that $(x * y) * y \in B_z$ for $z \in B$. Then $((x * z) * y) * y \in B$, which implies that $(x * y) * z = ((x * z) * y) * (y * y) \in B$. Since $z \in B$ and B is an ideal of X , it follows that $x * y \in B$. Therefore B is a positive implicative interior ideal in (X, ℓ) by Theorem 4.22. $\square$

Definition 4.24. Let (X, ℓ) be an interior BCK-algebra and let A be a subset of X which satisfies (8). Then A is called

- a *positive implicative weak interior ideal of type 1* in (X, ℓ) if it satisfies

$$(\forall x, y, z \in X)((\ell(x) * y) * z \in A, \ell(y * z) \in A \Rightarrow x * z \in A). \quad (29)$$

- a *positive implicative weak interior ideal of type 2* in (X, ℓ) if it satisfies

$$(\forall x, y, z \in X)((\ell(x * y) * z \in A, \ell(y * z) \in A \Rightarrow x * \ell(z) \in A). \quad (30)$$

- a *positive implicative weak interior ideal of type 3* in (X, ℓ) if it satisfies

$$(\forall x, y, z \in X)((\ell(x) * y) * z \in A, \ell(y * z) \in A \Rightarrow \ell(x) * z \in A). \quad (31)$$

Example 4.25

Consider the BCK-algebra X in Example 3.20. Define a mapping $\ell : X \rightarrow X$ by $\ell(0) = \ell(1) = 0, \ell(2) = 2$ and $\ell(3) = 3$. Then (X, ℓ) is an interior BCK-algebra and $A := \{0, 1, 2\}$ is a positive implicative weak interior ideal of type 1 in (X, ℓ) .

Example 4.26

Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	3	3	0	3
4	4	4	4	4	0

Define a mapping $\ell : X \rightarrow X$ by $\ell(0) = 0, \ell(1) = \ell(2) = 1, \ell(3) = 3$ and $\ell(4) = 4$. Then (X, ℓ) is an interior BCK-algebra and $A := \{0, 1, 2\}$ is a positive implicative weak interior ideal of type 2 in (X, ℓ) .

Example 4.27

Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Define a mapping $\ell : X \rightarrow X$ by $\ell(0) = \ell(1) = \ell(3) = 0$ and $\ell(2) = \ell(4) = 2$. Then (X, ℓ) is an interior BCK-algebra and $A := \{0, 2\}$ is a positive implicative weak interior ideal of type 3 in (X, ℓ) .

It is generally well known that every positive implicative ideal is an ideal in BCK-algebras. In considering the relationship between a positive implicative weak interior ideal of type 1 (respectively, type 2 and type 3) and an ideal, we pose the following question:

Question 4.28. In an interior BCK-algebra (X, ℓ) , if A is a positive implicative weak interior ideal of type 1 (respectively, type 2 and type 3) in (X, ℓ) , then is A an ideal of A ?

The answer to the Question 4.28 is negative as shown in the following example:

Example 4.29

Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

*	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

Define a mapping $\ell : X \rightarrow X$ by $\ell(0) = 0$ and $\ell(1) = \ell(2) = 1$. Then (X, ℓ) is an interior BCK-algebra and $A := \{0, 2\}$ is a positive

implicative weak interior ideal of type 1 and type 2 in (X, ℓ) which is not an ideal of X . Let (X, ℓ) be an interior BCK-algebra in Example 4.27. Then $A = \{0, 1, 2\}$ is a positive implicative weak interior ideal of type 3 which is not an ideal of X .

We discuss relationship between type 1, type 2, and type 3 on positive implicative weak interior ideal.

Theorem 4.30. In an interior BCK-algebra X , every positive implicative weak interior ideal A of type 2 which satisfies the condition (9) is a positive implicative weak interior ideal of type 1.

Proof. Let A be a positive implicative weak interior ideal of type 2 which satisfies the condition (9) in an interior BCK-algebra (X, ℓ) . Let $x, y, z \in X$ be such that $(\ell(x) * y) * z \in A$ and $\ell(y * z) \in A$. Then $x * \ell(z) \in A$ by (30). Using (12) and (3), we have $x * z \leq x * \ell(z)$. It follows from (10) that $x * z \in A$. Hence A is a positive implicative weak interior ideal of type 1. $\square$

The following example shows that any positive implicative weak interior ideal of type 1 is not a positive implicative weak interior ideal of type 2.

Example 4.31

Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	1	3	0	3
4	4	4	4	4	0

Define a mapping $\ell : X \rightarrow X$ by $\ell(0) = 0, \ell(1) = 1, \ell(2) = \ell(3) = 2$ and $\ell(4) = 4$. Then (X, ℓ) is an interior BCK-algebra and $A := \{0, 3\}$ is a positive implicative weak interior ideal of type 1 in (X, ℓ) but it is not a positive implicative weak interior ideal of type 2 since $(\ell(1) * 0) * 3 = 0 \in A$ and $\ell(0 * 3) = 0 \in A$ but $1 * \ell(3) = 1 * 2 = 1 \notin A$.

Theorem 4.32. In an interior BCK-algebra X , every positive implicative weak interior ideal A of type 1 which satisfies the condition (9) is a positive implicative weak interior ideal of type 3.

Proof. Let A be a positive implicative weak interior ideal of type 1 which satisfies the condition (9) in an interior BCK-algebra (X, ℓ) . Let $x, y, z \in X$ be such that $(\ell(x) * y) * z \in A$ and $\ell(y * z) \in A$. Then $x * z \in A$ by (29). Using (12) and (3), we have $\ell(x) * z \leq x * z$. It follows from (10) that $\ell(x) * z \in A$. Hence A is a positive implicative weak interior ideal of type 3. $\square$

The following example shows that any positive implicative weak interior ideal of type 3 is not a positive implicative weak interior ideal of type 1:

Example 4.33

Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the following Cayley table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Define a mapping $\ell : X \rightarrow X$ by $\ell(0) = 0, \ell(1) = \ell(2) = 1$ and $\ell(3) = 0$. Then (X, ℓ) is an interior BCK-algebra and $A := \{0, 1, 2\}$ is a positive implicative weak interior ideal of type 3 in (X, ℓ) but it is not a positive implicative weak interior ideal of type 1 since $(\ell(3) * 3) * 0 = 0 \in A$ and $\ell(3 * 0) = 0 \in A$ but $3 * 0 = 3 \notin A$.

Corollary 4.34. *In an interior BCK-algebra X , every positive implicative weak interior ideal A of type 2 which satisfies the condition (9) is a positive implicative weak interior ideal of type 3.*

The following example shows that any positive implicative weak interior ideal of type 3 is not a positive implicative weak interior ideal of type 2:

Example 4.35

Consider the BCK-algebra X in Example 4.33. Then the set $A := \{0, 1, 2\}$ is a positive implicative weak interior ideal of type 3 in (X, ℓ) but it is not a positive implicative weak interior ideal of type 2 since $(\ell(3) * 3) * 0 = 0 \in A$ and $\ell(3 * 0) = 0 \in A$ but $3 * \ell(0) = 3 \notin A$.

We establish the relationship between weak interior ideal and positive implicative weak interior ideal of type 1, type 2, and type 3.

Theorem 4.36. *In an interior BCK-algebra, every positive implicative weak interior ideal of type 1 is a weak interior ideal.*

Proof. Let A be a positive implicative weak interior ideal of type 1 in an interior BCK-algebra (X, ℓ) . If we take $z = 0$ in (29) and use (2), then $\ell(x) * y = (\ell(x) * y) * 0 \in A$ and $\ell(y) = \ell(y * 0) \in A$ which imply that $x = x * 0 \in A$. Hence A is a weak interior ideal in (X, ℓ) . $\square$

The converse in Theorem 4.36 is not true in general as shown in the next example.

Example 4.37

Consider the BCK-algebra X in Example 3.20. Define a mapping $\ell : X \rightarrow X$ by $\ell(0) = \ell(3) = 0$ and $\ell(1) = \ell(2) = 1$. Then (X, ℓ) is an interior BCK-algebra and $A := \{0, 3\}$ is a weak interior ideal which is not a positive implicative weak interior ideal of type 1 since $(\ell(2) * 1) * 1 = 0 \in A, \ell(1 * 1) = 0 \in A$ but $2 * 1 = 1 \notin A$.

Theorem 4.38. *In an interior BCK-algebra, every positive implicative weak interior ideal of type 2 is a weak interior ideal.*

Proof. Let A be a positive implicative weak interior ideal of type 2 in an interior BCK-algebra (X, ℓ) . If we take $z = 0$ in (30), then

$\ell(x) * y = (\ell(x) * y) * 0 \in A$ and $\ell(y) = \ell(y * 0) \in A$ which imply from (2), Proposition 3.13(i) and (30) that $x = x * 0 = x * \ell(0) \in A$. Hence A is a weak interior ideal in (X, ℓ) . $\square$

The converse in Theorem 4.38 is not true in general as shown in the next example.

Example 4.39

The weak interior ideal $A := \{0, 3\}$ in Example 4.37 is not a positive implicative weak interior ideal of type 2 since $(\ell(2) * 1) * 1 = 0 \in A, \ell(1 * 1) = 0 \in A$ but $2 * \ell(1) = 1 \notin A$.

Theorem 4.40. *In an interior BCK-algebra, every positive implicative interior ideal is a positive implicative weak interior ideal of type 3.*

Proof. Let A be a positive implicative interior ideal in an interior BCK-algebra (X, ℓ) . Let $x, y, z \in X$ be such that $(\ell(x) * y) * z \in A$ and $\ell(y * z) \in A$. Then $y * z \in A$ by (21). Since A is a positive implicative ideal of X , it follows from (11) that $\ell(x) * z \in A$. Therefore A is a positive implicative weak interior ideal of type 3 in (X, ℓ) . $\square$

Corollary 4.41. *In an interior BCK-algebra, every positive implicative interior ideal is a weak interior ideal.*

Theorem 4.42. *The intersection of all positive implicative weak interior ideals of type 1 (respectively, type 2 and type 3) in an interior BCK-algebra (X, ℓ) is also a positive implicative weak interior ideal of type 1 (resp., type 2 and type 3) in (X, ℓ) .*

Proof. Let $\{A_i \mid i \in \Lambda\}$ be the set of all positive implicative weak interior ideals of type 1 in an interior BCK-algebra (X, ℓ) . It is clear that $0 \in \bigcap_{i \in \Lambda} A_i$. Let $x, y, z \in X$ be such that $(\ell(x) * y) * z \in \bigcap_{i \in \Lambda} A_i$ and $\ell(y * z) \in \bigcap_{i \in \Lambda} A_i$. Then $(\ell(x) * y) * z \in A_i$ and $\ell(y * z) \in A_i$ for all $i \in \Lambda$. Since A_i is a positive implicative weak interior ideal of type 1, it follows that $x * z \in A_i$ all $i \in \Lambda$. Thus $x * z \in \bigcap_{i \in \Lambda} A_i$.

Therefore $\bigcap_{i \in \Lambda} A_i$ is a positive implicative weak interior ideal of type 1 in (X, ℓ) . $\square$

5. CONCLUSIONS

In this paper, we introduce the notion of interior BCK/BCI-algebras, positive implicative interior BCK-algebras, (weak) interior ideals, positive implicative interior ideals, a positive implicative weak interior ideal of type 1, type 2, and type 3, and present some properties and examples related to them. After investigating some results concerning the interior BCK/BCI-algebra, our study has focused on the relationship between bounded BCK-algebras and interior BCK-algebras. We have studied interior ideals, weak interior ideals, and positive implicative interior ideals and have looked for conditions under which a weak interior ideal can be an interior ideal and conditions under which an interior ideal can be a positive implicative interior ideal. Finally, we consider the relationship between type 1, type 2, and type 3 on positive implicative weak interior ideals.

Our future work will include new results regarding the different kinds of interior BCK-algebras and interior ideas. In particular, we will investigate whether interior BCK-algebras have some properties such as commutative, (positive) implicative, and quasi-commutative or not. Moreover, for interior ideals, we will investigate the maximal, (positive) implicative, commutative, prime, and irreducible properties.

COMPLIANCE WITH ETHICAL STANDARDS

Ethical approval: This article does not contain any studies with human participants or animals performed by the author.

Informed consent: Informed consent was obtained from all individual participants included in the study.

CONFLICT OF INTEREST

The authors declare they have no conflicts of interest.

AUTHORS' CONTRIBUTIONS

All authors contributed equally and significantly to the study and preparation of the manuscript. They have read and approved the final article.

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