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F^1-transform in Fuzzy Fredholm Integral Equations

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Abstract

The goal of this study is to solve fuzzy Fredholm integral equations of the second kind. In order to solve these equations with respect to fuzzy valued functions, we propose a very powerful and relatively simple technique called fuzzy transform. This approach allows the transformation of a fuzzy Fredholm integral equation to a system of algebraic equations. A solution to this algebraic system gives the appropriate parameters of the inverse F1-transform. Hence, we can estimate the approximate solution to the original problem. The existence and uniqueness of the exact solution and approximate solution are also discussed.

Keywords: F1-transform, Fredholm equation, Fuzzy integral equation.

1 Introduction

Many researchers from AI community are interested by Integral equations with fuzzy-valued parameters, see [1, 2, 5, 9, 10]. For that reason, a fuzzy-valued function is a suitable model. Fuzzy numbers and arithmetic operations over them was first proposed by Lotfi A. Zadeh in [11], and further elaborated by D.Dubois and H.Prade in [4].

Our research considers fuzzy Fredholm integral equations of the second kind. The presence of fuzziness makes these equations more complicated then their classical versions. We can solve the equations by iterative computation [3, 12] or we can approximate the solution by simple functions [9, 10]. In this paper, we use the second method.

First, the fuzzy-valued functions in the original problem are replaced by their inverse F^1-transforms. By this, we transform the original problem to its approximation version, and we create an auxiliary problem with ordinary vector functions. Hence, we get an approximate model of the problem.

The success of theory of F^1-transforms is in combining fuzzy and conventional methods. This theory is a good example of modern artificial intelligence.

The general form of the Fredholm integral equation of the second kind is as follows:

$$y(t) = f(t) + \int_0^T k(t,s)y(s)ds, \qquad (1)$$

where k is a given kernel with domain $D = [0,T] \times [0,T]$, f is a given function with domain $D = [0,T]$. Our purpose is finding unknown function y.

In our research, we consider the Fredholm equation where the given function f and the solution y are fuzzy-number-valued functions. In next section, we will introduce the related basic concepts.

2 Preliminaries

2.1 Fuzzy number

Definition 2.1. *Fuzzy number u is a pair $u = (\underline{u}(r), \overline{u}(r))$, where $\underline{u}, \overline{u}$ are two real functions have domain is $[0,1]$ and satisfy three conditions:*

1. *Both $\underline{u}(r), \overline{u}(r)$ are left continuous on $(0,1]$ and right continuous at 0.*
2. *$\underline{u}(r)$ is a bounded monotonically increasing and $\overline{u}(r)$ is a bounded monotonically decreasing.*
3. *$\underline{u}(1) \leq \overline{u}(1)$.*

The collection of all fuzzy numbers on $\mathbb{R}$ is called $\mathbb{E}$. A fuzzy-number-valued (fnv) function f is function from interval $[0,T]$ to space $\mathbb{E}$. More specially, fnv-function f has parametric from $(\underline{f}(t,r), \overline{f}(t,r))$ on $(t,r) \in [0,T] \times [0,1]$ where $\underline{f}(t,r)$, $\overline{f}(t,r)$ are fuzzy numbers corresponding to r. And the definition of integral of fnv-function f is

$$\int_0^T f(t,r)dt = \left(\int_0^T \underline{f}(t,r)dt, \int_0^T \overline{f}(t,r)dt \right).$$

2.2 The fuzzy Fredholm integral equations

In our research, we consider the fuzzy version (FFIE) of the Fredholm integral equation (1) where functions f and y are fnv-functions. Therefore, the problem we consider can be written as follows:

$$(\underline{y}(t,r), \overline{y}(t,r))^T = (\underline{f}(t,r), \overline{f}(t,r))^T + \int_0^T k(t,s)(\underline{y}(t,r), \overline{y}(t,r))^T dt$$

This (vectorial) equation can be further rewritten into the system below:

$$\begin{aligned} &\underline{y}(t,r) = \\ &\underline{f}(t,r) + \int_0^T \Big(k_+(s,t)\underline{y}(t,r) - k_-(s,t)\overline{y}(t,r) \Big) dt, \\ &\overline{y}(t,r) = \\ &\overline{f}(t,r) + \int_0^T \Big(k_+(s,t)\overline{y}(t,r) - k_-(s,t)\underline{y}(t,r) \Big) dt, \quad (2) \end{aligned}$$

where

$$k_+(s,t) = \begin{cases} k(s,t), & k(s,t) \geq 0 \\ 0, & otherwise, \end{cases}$$

and

$$k_-(s,t) = \begin{cases} -k(s,t), & k(s,t) \leq 0 \\ 0, & otherwise. \end{cases}$$

Then we rearrange (2) into a new form

$$y(t,r) = f(t,r) + \int_0^T \mathbf{k}(t,s) y(s,r) ds, \qquad (3)$$

using the following vector-functions:

$$y(t,r) = [\underline{y}(t,r), \overline{y}(t,r)]^T, \; f(t,r) = [\underline{f}(t,r), \overline{f}(t,r)]^T,$$

and

$$\mathbf{k}(t,s) = \begin{pmatrix} k_+(t,s) & -k_-(t,s) \\ -k_-(t,s) & k_+(t,s) \end{pmatrix}.$$

It is important to remark that the system (3) includes ordinary real-valued functions. The method we use to solve (3) is the F^1-transform. We will discuss about the concept of this method in next subsection.

2.3 F^1-transform

For arbitrary natural number $n > 2$, interval $[a,b]$ and $h = \frac{b-a}{n}$, we set up nodes $x_0, x_1, ..., x_n$ be h-equidistant on $[a,b]$ such that $x_0 = a$, $x_n = b$. Base on the definitions of h-uniform fuzzy partition $A_0, A_1, ..., A_n$ of interval $[a,b]$ and its generation function A mentioned in [7], we consider the follow definition of F^1-transform below

Definition 2.2 (F^1-transform)**.** *Let $f \in C[a,b]$ and $A_0(x), A_1(x), ..., A_n(x)$ be h-uniform partition of $[a,b]$. Then (see [6]), the expression $\sum_{k=0}^n (c_{k,0} + c_{k,1}(x - x_k))A_k(x)$, where the coefficients are the F^1-transform components such that for all $1 \leq k \leq n-1$,*

$$c_{k,0} = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)A_k(x)dx}{h},$$
$$c_{k,1} = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)(x-x_k)A_k(x)dx}{\int_{x_{k-1}}^{x_{k+1}} (x-x_k)^2 A_k(x)dx}$$

is known as the inverse F^1-transform of f denoted as $\hat{f}_n$. The inverse F^1-transform $\hat{f}_n$ approximate function f, the proof is mentioned in [7], then

$$f(x) \approx \hat{f}_n = \sum_{k=1}^{n-1} (c_{k,0} + c_{k,1}(x - x_k))A_k(x). \qquad (4)$$

Below we repeat [6] to get the extension of the F^1-transform to functions with two variables. Let $f(x,y) \in C([a,b] \times [c,d])$, two arbitrary natural numbers $m, n > 2$ and rectangle $[a,b] \times [c,d] \subset \mathbb{R}^2$. Let us denote $x_0, x_1, ..., x_n$ are h_1-equidistant nodes of $[a,b]$ where $x_0 = a, x_n = b$ and $h_1 = \frac{b-a}{n}$ and $y_0, y_1, ..., y_m$ are h_2-equidistant nodes of $[c,d]$ where $y_0 = c, y_m = d$ and $h_2 = \frac{d-c}{m}$.
Let $A_0(x), A_1(x), A_2(x), ..., A_n(x)$ be h_1-uniform fuzzy partition with respect to variable x, and $B_0(y), B_1(y), B_2(y), ..., B_m(y)$ be h_2-uniform fuzzy partition respect to variable y. Then the inverse F^1-transform $\hat{f}_{n,m}(x,y)$ of $f(x,y)$ approximates $f(x,y)$, and is expressed by

$$\hat{f}_{n,m}(x,y) = \sum_{k=1}^{n-1} \sum_{l=1}^{m-1} F_k^l A_k(x) B_l(y), \qquad (5)$$

where for all $k = 1, ..., n-1$ and $l = 1, ..., m-1$,

$$F_k^l = c_{k,0}^{l,0} + c_{k,1}^{l,0}(x - x_k) + c_{k,0}^{l,1}(y - y_l) + c_{k,1}^{l,1}(x - x_k)(y - y_l)$$

where

$$c_{k,0}^{l,0} = \frac{\int_{x_{k-1}}^{x_{k+1}} \int_{y_{l-1}}^{y_{l+1}} f(x,y)A_k(x)B_l(y)dydx}{h_1 h_2},$$
$$c_{k,1}^{l,0} = \frac{\int_{x_{k-1}}^{x_{k+1}} \int_{y_{l-1}}^{y_{l+1}} f(x,y)(x-x_k)A_k(x)B_l(y)dydx}{h_2 \int_{x_{k-1}}^{x_{k+1}} (x-x_k)^2 A_k(x)dx},$$
$$c_{k,0}^{l,1} = \frac{\int_{x_{k-1}}^{x_{k+1}} \int_{y_{l-1}}^{y_{l+1}} f(x,y)(y-y_l)A_k(x)B_l(y)dydx}{h_1 \int_{x_{k-1}}^{x_{k+1}} (y-y_l)^2 B_l(y)dy},$$
$$c_{k,1}^{l,1} = \frac{\int_{x_{k-1}}^{x_{k+1}} \int_{y_{l-1}}^{y_{l+1}} f(x,y)(x-x_k)(y-y_l)A_k(x)B_l(y)dydx}{\left(\int_{x_{k-1}}^{x_{k+1}} (x-x_k)^2 A_k(x)dx\right)\left(\int_{y_{l-1}}^{y_{l+1}} (y-y_l)^2 B_l(y)dy\right)}.$$

Let $\varphi(x) = [A_1(x), A_1(x)(x-x_1), A_2(x), A_2(x)(x-x_2), \ldots, A_{n-1}(x), A_{n-1}(x)(x-x_{n-1})]^T$,
$\psi(y) = [B_1(y), B_1(y)(y-y_1), B_2(y), B_2(y)(y-y_2), \ldots, B_{m-1}(y), B_{m-1}(y)(y-y_{m-1})]^T$ and

$$F = \begin{pmatrix} c_{1,0}^{1,0} & c_{1,0}^{1,1} & \ldots & c_{1,0}^{m-1,0} & c_{1,0}^{m-1,1} \\ c_{1,1}^{1,0} & c_{1,1}^{1,1} & \ldots & c_{1,1}^{m-1,0} & c_{1,1}^{m-1,1} \\ c_{2,0}^{1,0} & c_{2,0}^{1,1} & \ldots & c_{2,0}^{m-1,0} & c_{2,0}^{m-1,1} \\ c_{2,1}^{1,0} & c_{2,1}^{1,1} & \ldots & c_{2,1}^{m-1,0} & c_{2,1}^{m-1,1} \\ \ldots & & & & \\ c_{n-1,0}^{1,0} & c_{n-1,0}^{1,1} & \ldots & c_{n-1,0}^{m-1,0} & c_{n-1,0}^{m-1,1} \\ c_{n-1,1}^{1,0} & c_{n-1,1}^{1,1} & \ldots & c_{n-1,1}^{m-1,0} & c_{n-1,1}^{m-1,1} \end{pmatrix}$$

be a real matrix of size $(2n-2)\times(2m-2)$, then

$$f(x,y) \approx \hat{f}_{n,m}(x,y) = \varphi^T(x)F\psi(y). \tag{6}$$

3 Function approximation

According to (2), we need to use F^1-transform of functions of two variables to approximate $y(t,r) = [\underline{y}(t,r), \overline{y}(t,r)]$, $f[t,r] = [\underline{f}(t,r), \overline{f}(t,r)]$ and $k_+(t,s), k_-(t,s)$.

First we consider variables $t,s \in [0,T]$. For natural number $n > 2$, $h_1 = \frac{T}{n}$, let us denote $t_0, t_1, \ldots, t_n$ be h_1-equidistant nodes of $[0,T]$ satisfy $t_0 = 0, t_n = T$. Let $A_0(t), A_1(t), \ldots, A_n(t)$ be h_1-uniform partition with respect to variable t and $A : [-1,1] \to [0,1]$ is its generation function. We denote $\varphi(t) = [A_1(t), A_1(t)(t-t_1), A_2(t), A_2(t)(t-t_2), \ldots, A_{n-1}(t), A_{n-1}(t)(t-t_{n-1})]^T$.

Then, we consider variable $r \in [0,1]$. We establish notes $r_0, r_1, \ldots, r_m$ on $[0,1]$ as an h_2-equidistant such that $r_0 = 0, r_m = 1$. Let fuzzy partition $B_0(r), B_1(r), \ldots, B_m(r)$ be h_2-uniform with respect to variable r and the generating function $B : [-1,1] \to [0,T]$ is its generation function. We denote $\psi(r) = [B_1(r), B_1(r)(r-r_1), B_2(r), B_2(r)(r-r_2), \ldots, B_{m-1}(r), B_{m-1}(r)(r-r_{m-1})]^T$.

Using (6), we obtain the following approximations for y, f, k_+, k_-. For $t,s \in [0,T]$ and $r \in [0,1]$

$$\begin{cases} y(t,r) \approx [\varphi^T(t)\underline{Y}\psi(r), \varphi^T(t)\overline{Y}\psi(r)], \\ f(t,r) \approx [\varphi^T(t)\underline{F}\psi(r), \varphi^T(t)\overline{F}\psi(r)], \\ k_+(t,s) \approx \varphi^T(t)K_1\varphi(s), \\ k_-(t,s) \approx \varphi^T(t)K_2\varphi(s). \end{cases} \tag{7}$$

where $\underline{Y}, \overline{Y}, \underline{F}, \overline{F}$ are $(2n-2)\times(2m-2)$ real matrices and K_1, K_2 are $(2n-2)\times(2n-2)$ real matrices.

3.1 Some preliminary properties of φ and ψ

Theorem 3.1. *Let φ be defined as above. Let the $(2n-2)\times(2n-2)$ matrix P be defined by*

$$P := \int_0^T \varphi(t)\varphi^T(t)dt. \tag{8}$$

Let us denote

$$\alpha_1 = \int_{-1}^1 A^2(t)dt, \quad \alpha_2 = \int_{-1}^1 t^2 A^2(t)dt,$$
$$\beta_1 = \int_0^1 A(t)A(1-t)dt, \quad \beta_2 = \int_0^1 tA(t)A(1-t)dt,$$
$$\beta_3 = \int_{-1}^0 tA(t)A(1+t)dt, \quad \beta_4 = \int_0^1 t(t-1)A(t)A(1-t)dt.$$

Then, the matrix elements can be determinded by:
For all $i = 1, \ldots, n-1$

$$\begin{cases} p_{2i-2,2i-2} = h_1\alpha_1, \\ p_{2i-1,2i-1} = h_1^3\alpha_2, \end{cases}$$

And for all $i = 1, \ldots, n-2$

$$\begin{cases} p_{2i-2,2i-1} = p_{2i-1,2i-2} = 0, \\ p_{2i-2,2i} = p_{2i,2i-2} = h_1\beta_1, \\ p_{2i-2,2i+1} = p_{2i+1,2i-2} = h_1^2\beta_3, \\ p_{2i-1,2i} = p_{2i,2i-1} = h_1^2\beta_2, \\ p_{2i-1,2i+1} = p_{2i+1,2i_1} = h_1^3\beta_4. \end{cases}$$

Now, we will build some important functions. For $k = 1, \ldots, n-1$, let us denote

$\omega_k : \mathbb{R} \to \mathbb{R}$

$$t \mapsto \omega_k(t) = \begin{cases} e^{-\frac{1}{t_k - t}} & t_{k-1} + \frac{h_1}{2} < t < t_k \\ -e^{-\frac{1}{t - t_{k-1}}} & t_{k-1} < t < t_{k-1} + \frac{h_1}{2} \\ 0 & \text{otherwise} \end{cases}. \tag{9}$$

For $l = 1, ..., m-1$, let us denote

$$\zeta_l : \mathbb{R} \to \mathbb{R}$$

$$r \mapsto \zeta_l(r) = \begin{cases} e^{-\frac{1}{r_l - r}} & r_{l-1} + \frac{h_2}{2} < r < r_l \\ -e^{-\frac{1}{r - r_{l-1}}} & r_{l-1} < r < r_{l-1} + \frac{h_2}{2} \\ 0 & \text{otherwise} \end{cases}. \quad (10)$$

Lemma 3.1. *Assume that $A_0, \dots, A_n$ is a h_1-uniform fuzzy partition of $[0,T]$ which is denoted above and ω_k is as in (9). Then for all $0 \le k \le n-2$,*

$$\int_{t_k}^{t_{k+1}} \frac{\omega_{k+1}(t)}{t - t_{k+1}} A_k(t) A_{k+1}(t) dt \neq 0,$$

$$\int_{t_k}^{t_{k+1}} \omega_{k+1}(t)(t - t_{k+1}) A_k(t) A_{k+1}(t) dt \neq 0,$$

$$\int_{t_k}^{t_{k+1}} \omega_{k+1}(t) A_k(t) A_{k+1}(t) dt = 0.$$

Theorem 3.2. *Let us establish vector*

$$\omega(t) = \left(\frac{\omega_1(t)}{t - t_1} A_0(t), \omega_1(t) A_0(t), \dots, \frac{\omega_{n-1}(t)}{t - t_{n-1}} A_{n-2}(t), \omega_{n-1}(t) A_{n-2}(t) \right)^T,$$

where ω_k is taken from (9). Then $Q = \int_{\mathbb{R}} \omega(t)\varphi^T(t)dt$ is a lower triangular matrix with non-zero diagonal.

Lemma 3.2. *Assume that $B_0, \dots, B_m$ is h_2-uniform fuzzy partition of $[0,1]$ which is denoted above and ζ_l is as in (10). Then for all $0 \le l \le m-2$,*

$$\int_{r_l}^{r_{l+1}} \frac{\zeta_{l+1}(r)}{r - r_{l+1}} B_l(r) B_{l+1}(r) dr \neq 0,$$

$$\int_{r_l}^{r_{l+1}} \zeta_{l+1}(r)(r - r_{l+1}) B_l(r) B_{l+1}(r) dr \neq 0,$$

$$\int_{r_l}^{r_{l+1}} \zeta_{l+1}(r) B_l(r) B_{l+1}(r) dr = 0.$$

Theorem 3.3. *Similar to Theorem 3.2, let us establish vector*

$$\zeta(r) = \left(\frac{\zeta_1(r)}{r - r_1} B_0(r), \zeta_1(r) B_0(r), \dots, \frac{\zeta_{m-1}(r)}{r - r_{m-1}} B_{m-2}(r), \zeta_{m-1}(r) B_{m-2}(r) \right)^T,$$

where ζ_l is taken from (10). Then $\hat{Q} = \int_{\mathbb{R}} \psi(r)\zeta^T(r)dr$ is a lower triangular matrix with non-zero diagonal.

Then, we compute determinants of Q and $\hat{Q}$

$$|Q| = \prod_{k=0}^{2n-3} Q_{i,i} \neq 0, \quad \text{and} \quad |\hat{Q}| = \prod_{k=0}^{2n-3} \hat{Q}_{i,i} \neq 0$$

Therefore, Q and $\hat{Q}$ are invertible.

4 General scheme of the proposed method

As we mentioned in section introduction, we approximate system (2) by replacing its functions by (7). Then we have

$$\begin{pmatrix} \varphi^T(t)\underline{Y}\psi(r) \\ \varphi^T(t)\overline{Y}\psi(r) \end{pmatrix} = \begin{pmatrix} \varphi^T(t)\underline{F}\psi(r) \\ \varphi^T(t)\overline{F}\psi(r) \end{pmatrix} + \int_0^T \begin{pmatrix} \varphi^T(t)K_1\varphi(s) & -\varphi^T(t)K_2\varphi(s) \\ -\varphi^T(t)K_2\varphi(s) & \varphi^T(t)K_1\varphi(s) \end{pmatrix} \begin{pmatrix} \varphi^T(s)\underline{Y}\psi(r) \\ \varphi^T(s)\overline{Y}\psi(r) \end{pmatrix} ds. \quad (11)$$

For the first row of (11), we have

$$\begin{aligned} \varphi^T(t)\underline{Y}\psi(r) =& \varphi^T(t)\underline{F}\psi(r) + \int_0^T \Big[\varphi^T(t)K_1\varphi(s)\varphi^T(s)\underline{Y}\psi(r) \\ & - \varphi^T(t)K_2\varphi(s)\varphi^T(s)\overline{Y}\psi(r) \Big] ds \\ =& \varphi^T(t) \Bigg(\underline{F} + K_1 \Big(\int_0^T \varphi(s)\varphi^T(s)ds \Big) \underline{Y} \\ & - K_2 \Big(\int_0^T \varphi(s)\varphi^T(s)ds \Big) \overline{Y} \Bigg) \psi(r). \end{aligned}$$

Then we have

$$\varphi^T(t)\underline{Y}\psi(r) = \varphi^T(t) \left(\underline{F} + (K_1P\underline{Y} - K_2P\overline{Y}) \right) \psi(r). \quad (12)$$

Multiplying (12) by $\omega(t)$ from the left then integrating with respect to t and by $\zeta^T(r)$ from the right. Then integrating with respect to t and r, we have

$$Q\underline{Y}\hat{Q} = Q \left(\underline{F} + (K_1P\underline{Y} - K_2P\overline{Y}) \right) \hat{Q} \quad (13)$$

Since Theorem 3.2 and Theorem 3.3, we know that Q^{-1} and $\hat{Q}^{-1}$ exist. Multiplying (13) by Q^{-1} from the left and $\hat{Q}^{-1}$ from the right and we have

$$\underline{Y} = \underline{F} + (K_1P\underline{Y} - K_2P\overline{Y}).$$

Repeating the same procedure with the second row of (11), we have

$$\begin{pmatrix} \underline{Y} \\ \overline{Y} \end{pmatrix} = \begin{pmatrix} \underline{F} \\ \overline{F} \end{pmatrix} + \begin{pmatrix} K_1P & -K_2P \\ -K_2P & K_1P \end{pmatrix} \begin{pmatrix} \underline{Y} \\ \overline{Y} \end{pmatrix}.$$

Then,

$$\begin{pmatrix} I-K_1P & K_2P \\ K_2P & I-K_1P \end{pmatrix} \begin{pmatrix} \underline{Y} \\ \overline{Y} \end{pmatrix} = \begin{pmatrix} \underline{F} \\ \overline{F} \end{pmatrix}. \quad (14)$$

4.1 Existence of a unique solution to system (2)

Function $g \in C[0,T]$ is called Dini-Lipschitz if and only if

$$\gamma(\delta, g)\log(\delta) \to 0, \quad \text{provided that } \delta \to 0, \quad (15)$$

where $\gamma(\delta, g)$ is the modulus of continuity of g with respect to δ. And collection of all Dini-Lipschitz functions is called $C_{DL}[0,T]$.

Let us denote identity operator $\mathscr{I} : (C_{DL}[0,T])^2 \to (C_{DL}[0,T])^2$ and operator $\mathscr{R} : (C_{DL}[0,T])^2 \to (C_{DL}[0,T])^2$ satisfy

$$\mathscr{R}(y(t,r)) := \int_0^T \mathbf{k}(t,s)y(s,r)ds. \quad (16)$$

Then, the fuzzy Fredholm integral equation (3) can be rewritten as

$$(\mathscr{I} - \mathscr{R})y(t,r) = f(t,r), \quad (17)$$

According to [9], $(\mathscr{I} - \mathscr{R})^{-1}$ exists and is bounded if $\mathscr{R}$ is bounded and $\|\mathscr{R}\|_\infty < 1$. This assumption leads to the existence and uniqueness of solution to (17) and therefore, to (3), respectively to (2).

4.2 Existence of a fuzzy approximate solution

In this subsection, we show that the system (14) is solvability. We denote operator $\mathscr{R}_n(y(t,r))$ as below

$$\int_0^T \begin{pmatrix} \varphi^T(t)K_1\varphi(s) & -\varphi^T(t)K_2\varphi(s) \\ -\varphi^T(t)K_2\varphi(s) & \varphi^T(t)K_1\varphi(s) \end{pmatrix} \begin{pmatrix} \underline{y}(s,r) \\ \overline{y}(s,r) \end{pmatrix} ds,$$

where K_1, K_2 are denoted in (7). We can easily to see that operator $\mathscr{R}_n$ is an approximate version of (16) and the lemma below will show that the distant between $\mathscr{R}$ and $\mathscr{R}_n$ convergence to 0 for all sufficiently large n.

The equation (11) can be rewritten as

$$(\mathscr{I} - \mathscr{R}_n)y(t,r) = \varphi^T F \psi, \quad (18)$$

where $F = [\underline{F}, \overline{F}]$, and by (7), $\varphi^T F \psi = [\varphi^T(t)\underline{F}\psi(r), \varphi^T(t)\overline{F}\psi(r)] = \hat{f}_{m,n}(t,r) \approx f(t,r)$.

As we mentioned before, equation (11) is approximate form of equation (3). Thus, equation (18) is approximate form of equation (17).

Let us recall the following general theorem.

Theorem 4.1 ([9]). *Let $\mathscr{R} : X \to X$ be a bounded linear operator in a Banach space X and let $\mathscr{I} - \mathscr{R}$ be injective. Assume $\mathscr{R}_n$ is a sequence of bounded operators with*

$$\|\mathscr{R} - \mathscr{R}_n\| \to 0, \quad (19)$$

as $n \to \infty$.
Then for all sufficiently large $n > n_0$, the inverse operators $(\mathscr{I} - \mathscr{R}_n)^{-1}$ exists and is bounded in accordance with

$$\|(\mathscr{I} - \mathscr{R}_n)^{-1}\| \le \frac{\|(\mathscr{I} - \mathscr{R})^{-1}\|}{1 - \|(\mathscr{I} - \mathscr{R})^{-1}(\mathscr{R} - \mathscr{R}_n)\|}. \quad (20)$$

By using Theorem 4.1, we will prove that operators $\mathscr{R}_n$ in (18) fulfill assumption (19).

Lemma 4.1. *Let $k \in C([0,T]^2)$ and $f \in (C_{DL}([0,T] \times [0,1]))^2$. Denote*

$$M_{1,n} = \sup_{(s,t)\in[0,T]^2} |\varphi^T(t)K_1\varphi(s) - k_+(s,t)|,$$

and

$$M_{2,n} = \sup_{(s,t)\in[0,T]^2} |\varphi^T(t)K_2\varphi(s) - k_-(s,t)|.$$

We claim that

$$M_{1,n} \to 0 \quad as \quad n \to \infty,$$

$$M_{2,n} \to 0 \quad as \quad n \to \infty.$$

Using Lemma 4.1, we can easily prove the assumption (19) of Theorem 4.1.

$$\|\mathscr{R} - \mathscr{R}_n\|_\infty = \sup_{\|y\|_\infty \le 1} \|(\mathscr{R} - \mathscr{R}_n)y\| \le (M_{1,n} + M_{2,n})T\|y\|_\infty \to 0 \quad \text{as } n \to \infty, \quad (21)$$

and the assumption (19) is confirmed.

We continue the analysis of the solution to (14), using the same reasoning as in [9].

Theorem 4.2. *Let $k \in C([0,T]^2)$, $f \in (C_{DL}([0,T] \times [0,1]))^2$ and $\|\mathscr{R}\|_\infty < 1$. Then, for all sufficiently large $n > n_0$, the solution Y_n of the system* (14) *exists and it approximates the solution of (2).*

Conclusion

In this contribution, we proposed a new numerical method for solving fuzzy Fredholm integral equation of the second kind based on the based on the F^1-transforms. We proposed conditions that guarantee the existence and uniqueness of both exact and approximate fuzzy solutions. We observed that the convergence rate of approximate solution to exact solution is proportional to the quality of approximation of the kernel using inverse F^1-transforms.

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