Fuzzy Partitioned Discrete-event Systems Under Controllability

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Abstract

This paper aims to study the fuzzy partitioned discrete-event systems and its supervisory control theory under full observation. Specifically, we introduce the concept of a fuzzy partitioned automaton corresponding to a given automaton. Further, we introduce the model of a fuzzy partitioned discrete-event system, which is described by using the concept of a fuzzy partitioned automaton. Furthermore, we introduce the model of a controlled system of a fuzzy partitioned discrete-event system under (fully observable) fuzzy supervisor. Interestingly, we underlay the relationships between the fuzzy languages generated/marked by controlled system and controllable, $\mathscr{L}_m(\mathscr{P})$-closed fuzzy languages. Moreover, we talk about the supremal and infimal controllable fuzzy languages.

Keywords: Fuzzy partition, Fuzzy partitioned automaton, Fuzzy partitioned discrete-event system, Supervisory control.

1 Introduction

The concept of a discrete-event system (DES) as a dynamic system with discrete states is well-known. DES exists in the studies such as computer operating systems and the control of complex multimodel processes. DES theory has been shown to be useful in areas such as concurrent program, semantics and monitoring, and control of complex systems. Over the last decades, in order to confine the behavior of a DES to a chosen range, supervisory control theory introduced by Ramadge and Wonham [23] has been studied in detail (cf., [4, 10]). In general, a DES is defined by using finite state automaton alongwith events as input alphabets. Further, the behavior of a DES is described by the language of an automaton. Most of the researchers of DES has been confined on systems modelled as deterministic automata and non-deterministic automata with ε-moves. A typical property of such models is that no uncertainty arises in the state transitions. However, there are some situations where the transitions of some systems are always uncertain, imprecise, and vague in nature. One such example is given in [14, 15] regarding the patient's condition, where change of condition from a state to another is imprecise, as it is typical to measure the changes precisely. Other examples can be found in chemical reactions, intelligent vehicle control [24], mobile robots in unstructured environments [2], waste water treatment [32], etc.

It is known that the specific properties of most of the complex systems are described by vagueness, imprecision and uncertainty, which can be handled by the concept of fuzzy sets introduced by Zadeh [33]. Fuzzy models such as fuzzy automata, fuzzy Petri nets, and fuzzy neural networks have attracted many researchers and found successful applications in numerous areas (c.f., [5, 21, 24, 32]). For details on fuzzy automata and languages, we can refer to the works done in [1, 7–9, 12, 13, 20, 22, 27–31]. It is to be pointed out here that various variants of fuzzy automata have been proposed in different applications and different generalizations (c.f., [12, 16–18]). In the way, new variants of fuzzy automata definitions presented, a tool for comparing these definitions and corresponding properties of automata was given in [17, 18], in the framework of category theory. Further, spaces with fuzzy partitions represent a new ground category for some mathematical constructions and applications. These structures generalize classical sets and sets with similarity relations, and it is natural to define automata in these structures. Such automata are known as fuzzy partitioned automata (cf., [17, 18, 26]) and play a significant role in our study. In order to capture the uncertainty appearing in states and state transitions of DES, Lin and Ying have used fuzzy set theory with DES and have generalized crisp DES to fuzzy DES by using the concept of a fuzzy finite automaton (c.f., [14, 15]). Further, they studied concepts such as generalized observability and optimal control problems. To handle uncertainty associated with control strategies, Lavrov [11] presented an improvement of industrial control by mingling discrete event supervisory control methods with fuzzy techniques. Inspired by the work done in [14, 15], in this paper, we use the concept of a fuzzy partitioned automaton to model the fuzzy partitioned DES and study its supervisory control theory. Such systems can generate the strings that occur with minimum membership values in I under max-min composition. Further, we introduce the concepts of controllability and $\mathscr{L}_m(\mathscr{P})$-closed of fuzzy languages, and establish the necessary and sufficient conditions for (fully observable) fuzzy supervisors synthesis.

Specifically,

- we introduce the concept of a fuzzy partitioned automaton corresponding to a given fuzzy automaton; and
- we propose the model of a fuzzy partitioned DES and its supervisory control theory under full observation, which is described by using the idea of a fuzzy partitioned automaton.

The structure of the paper is organized as following manner. Section 2 contains elementary knowledge about content of the article. In Section 3, we introduce the concept of a fuzzy partitioned automaton corresponding to a given fuzzy automaton. Next, in Section 4, we propose the model of a fuzzy partitioned DES and its supervisory control theory under full observation, which is described by using the idea of a fuzzy partitioned automaton. In addition, we introduce the concept of a (fully observable) fuzzy supervisor. Further, we introduce the model of a controlled system of a fuzzy partitioned DES under fuzzy supervisor. Furthermore, we introduce the concepts of controllable and $\mathscr{L}_m(\mathscr{P})$-closed fuzzy languages. Moreover, we underlay the relationships between the fuzzy languages generated/marked by controlled system and controllable, $\mathscr{L}_m(\mathscr{P})$-closed fuzzy languages.

2 Preliminaries

This section is divided into two subsections. In first subsection, we recall some basic concepts and notations related to fuzzy partitions; while that of fuzzy automata is recalled in second subsection.

Throughout this paper, I stands for $[0,1]$ and the fuzzy set in S is a map $\lambda : S \longrightarrow I$. For a nonempty set S, $\mathscr{F}(S)$ denotes the collection of all fuzzy sets in S. Further, for all $\lambda \in \mathscr{F}(S)$, the $supp(\lambda)$ and $core(\lambda)$ are sets of each elements $s \in S$ such that $\lambda(s) > 0$ and $\lambda(S) = 1$, respectively and if $core(\lambda) \neq \emptyset$, then λ is called a **normal fuzzy set**. Moreover, Ω denotes an indexed set. Furthermore, a **fuzzy relation** R on a set S is a map $R : S \times S \longrightarrow I$. R is called a **reflexive**; **symmetric**; and **transitive** fuzzy relation if for all $s_1, s_2, s_3 \in S$, $R(s_1, s_1) = 1$; $R(s_1, s_2) = R(s_2, s_1)$; and $R(s_1, s_2) \wedge R(s_2, s_3) \leqslant R(s_1, s_3)$, respectively. A reflexive, symmetric, and transitive fuzzy relation is called a fuzzy similarity relation. In sequel, we use $a \leftrightarrow b = (\vee\{c \in I : a \wedge c \leqslant b\}) \wedge (\vee\{d \in I : b \wedge d \leqslant a\})$, $\forall a, b \in I$.

2.1 Fuzzy partitions

Herein, we recall some essential concepts related to fuzzy partitions and fuzzy objects. We refer to the work done in [3, 6, 17–19, 26] for more details. We begin with the following.

Definition 2.1 *For a nonempty set S, a system* $\mathscr{A} = \{A_\omega : \omega \in \Omega\}$ *of normal fuzzy sets in S is called a* **fuzzy partition** *of S if* $\{core(A_\omega) : \omega \in \Omega\}$ *is a partition of S. A pair* $(S, \mathscr{A})$ *is called a* **space with a fuzzy partition**.

Definition 2.2 *Let* $(I, \mathscr{I})$ *be a space with a fuzzy partition, where* $\mathscr{I} = \{I_a : a \in I\}$ *is a fuzzy partition of I such that for all* $b \in I$, $I_a(b) = a \leftrightarrow b$. *Then a* **fuzzy object** *in a space with a fuzzy partition* $(S, \mathscr{A})$ *is a map* $(\lambda, \mu) : (S, \mathscr{A}) \longrightarrow (I, \mathscr{I})$ *such that*

(i) $\lambda : S \longrightarrow I$ *is a map;*

(ii) $\mu : \Omega \longrightarrow I$ *is a map; and*

(iii) for all $\omega \in \Omega$ *and* $s_1 \in S$, $A_\omega(s_1) \leqslant I_{\mu(\omega)}(\lambda(s_1)) = \mu(\omega) \leftrightarrow \lambda(s_1)$.

$T(S, \mathscr{A})$ denotes set of all fuzzy objects in $(S, \mathscr{A})$. Also, $T_1(S, \mathscr{A}) = \{\lambda : S \longrightarrow I : (\lambda, \mu) \in T(S, \mathscr{A})$, for some map $\mu : \Omega \longrightarrow I\}$.

2.2 Fuzzy automata

Herein, we recall the concepts related to fuzzy automata alongwith their languages. For details, we refer the works done in [4, 25]. We begin with following.

Definition 2.3 *A* **fuzzy automaton** *is a 5-tuple* $\mathscr{A} = (Q, E, \delta, \delta_0, \delta_m)$, *where Q is the* **state-set**; *E is the* **finite event-set**; $\delta : Q \times E \longrightarrow \mathscr{F}(Q)$ *is the* **fuzzy transition map**; $\delta_0 : Q \longrightarrow I$ *is the* **fuzzy set of initial**

states; *and* $\delta_m : Q \longrightarrow I$ *is the* **fuzzy set of marked states**.

Remark 2.1 *Let* $(E^*, \cdot, \varepsilon)$ *be the monoid of event-set* E *with identity element* $\varepsilon \in E^*$. *Then* δ *can be extended to a map* $\delta^* : Q \times E^* \longrightarrow \mathscr{F}(Q)$ *such that for all* $q_1, q_2 \in Q$, $v \in E^*$, *and* $a \in E$,

$$\delta^*(q_1, \varepsilon)(q_2) = \begin{cases} 1 & \text{if } q_1 = q_2 \\ 0 & \text{if } q_1 \neq q_2\text{; and} \end{cases}$$

$$\delta^*(q_1, v \cdot a)(q_2) = \vee\{\delta^*(q_1, v)(r) \wedge \delta(r, a)(q_2) : r \in Q\}.$$

Definition 2.4 *Let* $\mathscr{A} = (Q, E, \delta, \delta_0, \delta_m)$ *be a fuzzy automaton. Then*

(i) the **fuzzy language** $\mathscr{L}(\mathscr{A}) : E^* \longrightarrow I$ *is* **generated** *by* $\mathscr{A}$ *if* $\mathscr{L}(\mathscr{A})(v) = \vee\{\delta_0(q_1) \wedge \delta^*(q_1, v)(q_2) : q_1, q_2 \in Q\}$, $\forall v \in E^*$; *and*

(ii) the **fuzzy language** $\mathscr{L}_m(\mathscr{A}) : E^* \longrightarrow I$ *is* **marked** *by* $\mathscr{A}$ *if* $\mathscr{L}_m(\mathscr{A})(v) = \vee\{\delta_0(q_1) \wedge \delta^*(q_1, v)(q_2) \wedge \delta_m(q_2) : q_1, q_2 \in Q\}$, $\forall v \in E^*$.

Example 2.1 *Let* $\mathscr{A} = (Q, E, \delta, \delta_0, \delta_m)$ *be a fuzzy automaton, where* $Q = \{q_0, q_1\}$, $E = \{\sigma_1, \sigma_2\}$, $\delta : Q \times E \longrightarrow \mathscr{F}(Q)$ *is a fuzzy transition map such that* $\delta(q_0, \sigma_1)(q_1) = 0.8, \delta(q_1, \sigma_1)(q_1) = 0.7, \delta(q_0, \sigma_2)(q_1) = 0.9, \delta(q_1, \sigma_2)(q_1) = 0.6$, *and rest are* 0; $\delta_0 : Q \longrightarrow I$ *is a fuzzy set of initial states such that* $\delta_0(q_0) = 0.4$ *and* $\delta_0(q_1) = 0$; *and* $\delta_m : Q \longrightarrow I$ *is a fuzzy set of final states such that* $\delta_m(q_0) = 0$ *and* $\delta_m(q_1) = 0.3$. *Then it can be easily seen that* $\mathscr{L}(\mathscr{A})(\varepsilon) = \mathscr{L}(\mathscr{A})(v) = 0.4$ *and* $\mathscr{L}_m(\mathscr{A})(\varepsilon) = 0, \mathscr{L}_m(\mathscr{A})(v) = 0.3$, $\forall v \in E^+$.

Next, we establish the following result.

Proposition 2.1 *Let* $f : E^* \longrightarrow I$ *be a fuzzy language. Then there exists a fuzzy automaton* $\mathscr{A}$ *such that* $\mathscr{L}(\mathscr{A}) = f$ *iff* $f(v_1 \cdot v_2) \leqslant f(v_1)$, $\forall v_1, v_2 \in E^*$.

Proof: Let $\mathscr{A} = (Q, E, \delta, \delta_0, \delta_m)$ be a fuzzy automaton such that $\mathscr{L}(\mathscr{A}) = f$. Then for all $v_1, v_2 \in E^*$, $f(v_1 \cdot v_2) = \mathscr{L}(\mathscr{A})(v_1 \cdot v_2) = \vee\{\delta_0(q_1) \wedge \delta^*(q_1, v_1 \cdot v_2)(q_2) : q_1, q_2 \in Q\} \leqslant \vee\{\delta_0(q_1) \wedge \delta^*(q_1, v_1)(r) : q_1, r \in Q\} = \mathscr{L}(\mathscr{A})(v_1) = f(v_1)$. Thus $f(v_1 \cdot v_2) \leqslant f(v_1)$.

Conversely, let f be a fuzzy language such that for all $v_1, v_2 \in E^*$, $f(v_1 \cdot v_2) \leqslant f(v_1)$. Then we can construct a fuzzy automaton $\mathscr{A}_f = (Q_f, E, \delta_f, E, \delta_{0_f}, \delta_{m_f})$, where $Q_f = supp(f)$; $\delta_f(v_1, a)(v_2) = f(v_2)$, if $v_1 a = v_2$ and 0, otherwise; and $\delta_{0_f}(\varepsilon) = \delta_{m_f}(\varepsilon) = 1$ and 0, otherwise. Now, let $v \in E^*$. Then $\mathscr{L}(\mathscr{A}_f)(v) = \vee\{\delta_{0_f}(v_1) \wedge \delta_f^*(v_1, v)(v_2) : v_1, v_2 \in Q_f\} = \vee\{\delta_f^*(\varepsilon, v)(v_2) : v_2 \in Q_f\} = f(v)$. Thus $\mathscr{L}(\mathscr{A}_f) = f$.

3 The fuzzy partitioned automata

Herein, we introduce the concept of a fuzzy partitioned automaton corresponding to a given fuzzy automaton. Further, we underlay the relationships between the fuzzy languages generated/marked by fuzzy partitioned automaton and fuzzy automaton. We begin with the following definition from [26].

Definition 3.1 *A* **fuzzy automaton with a fuzzy partition** *is a 3-tuple* $((S, \mathscr{A}), (M, \cdot, \varepsilon), d)$, *where* S *is the state-set with a fuzzy partition* $\mathscr{A}$; $(M, \cdot, \varepsilon)$ *is a monoid of inputs; and* $d : S \times M \longrightarrow T_1(S, \mathscr{A})$ *is the fuzzy transition map such that for all* $s_1, s_2 \in S$, $s_1' \in core(A_\omega)$, *and* $m_1, m_2 \in M$,

(i) $d(s_1, \varepsilon)(s_2) = \delta_{S,\mathscr{A}}(s_1, s_2)$;

(ii) $d(s_1, m_1 \cdot m_2)(s_2) = \vee\{d(s_1, m_1)(s_3) \wedge d(s_3, m_2)(s_2) : s_3 \in S\}$;

(iii) $d(s_1, m_1)(s_2) \wedge A_\omega(s_1) \leqslant d(s_1', m_1)(s_2)$; *and*

(iv) From the condition (iii), for each $s_1, s_1' \in core(A_\omega)$, $d(s_1, m_1) = d(s_1', m_1)$.

In the remaining part of this paper, $(Q, \mathscr{Q})$ is a space with a fuzzy partition, where $\mathscr{Q} = \{Q_\omega : \omega \in \Omega\}$.

Now, we introduce the concept of fuzzy partitioned automaton.

Definition 3.2 *Let* $\mathscr{A} = (Q, E, \delta, \delta_0, \delta_m)$ *be a fuzzy automaton. Then the* **fuzzy partitioned automaton** *corresponding to* $\mathscr{A}$, *denoted by* $\mathscr{P}$, *is the 5-tuple* $\mathscr{P} = ((Q, \mathscr{Q}), (E^*, \cdot, \varepsilon), \delta_1, \delta_0, \delta_m)$, *where* Q *is the state-set with a fuzzy partition* $\mathscr{Q}$; $(E^*, \cdot, \varepsilon)$ *is the monoid of event-set* E; *and* $\delta_1 : Q \times E^* \longrightarrow T_1(Q, \mathscr{Q})$ *is the fuzzy transition map such that for all* $q_1, q_2 \in Q$ *and* $v \in E^*$, $\delta_1(q_1, v)(q_2) = \vee\{\delta_{Q,\mathscr{Q}}(q_1, r_1) \wedge \delta^*(r_1, v)(r_2) \wedge \delta_{Q,\mathscr{Q}}(r_2, q_2) : r_1, r_2 \in Q\}$, *where* $\delta_{Q,\mathscr{Q}}(q_1, q_2) = \rho_{Q,\mathscr{Q}}(\omega_1, \omega_2)$, $\forall \omega_1, \omega_2 \in \Omega$, $\forall q_1 \in core(Q_{\omega_1})$, *and* $\forall q_2 \in core(Q_{\omega_2})$.

Proposition 3.1 *Let* $\mathscr{P} = ((Q, \mathscr{Q}), (E^*, \cdot, \varepsilon), \delta_1, \delta_0, \delta_m)$ *be the fuzzy partitioned automaton corresponding to a fuzzy automaton* $\mathscr{A} = (Q, E, \delta, \delta_0, \delta_m)$. *Then* $\mathscr{P}$ *is a fuzzy automaton with a fuzzy partition.*

Proof: (i) Let $q_1, q_2 \in Q$, $q_2' \in core(Q_\omega)$, and $v \in E^*$. Then $\delta_1(q_1, v)(q_2) \leftrightarrow \delta_1(q_1, v)(q_2') = (\vee\{\delta_{Q,\mathscr{Q}}(q_1, r_1) \wedge \delta^*(r_1, v)(r_2) \wedge \delta_{Q,\mathscr{Q}}(r_2, q_2) : r_1, r_2 \in Q\}) \leftrightarrow (\vee\{\delta_{Q,\mathscr{Q}}(q_1, r_1) \wedge \delta^*(r_1, v)(r_2) \wedge \delta_{Q,\mathscr{Q}}(r_2, q_2') : r_1, r_2 \in Q\}) \geqslant (\delta_{Q,\mathscr{Q}}(q_1, q_1) \wedge \delta^*(q_1, v)(q_1) \wedge \delta_{Q,\mathscr{Q}}(q_1, q_2)) \leftrightarrow (\delta_{Q,\mathscr{Q}}(q_1, q_1) \wedge \delta^*(q_1, v)(q_1) \wedge \delta_{Q,\mathscr{Q}}(q_1, q_2')) \geqslant \delta_{Q,\mathscr{Q}}(q_1, q_2) \leftrightarrow \delta_{Q,\mathscr{Q}}(q_1, q_2') \geqslant Q_\omega(q_2)$. Thus $Q_\omega(q_2) \leqslant \delta_1(q_1, v)(q_2) \leftrightarrow \delta_1(q_1, v)(q_2')$, which implies that

$\delta_1(q_1,v) \in T_1(Q,\mathscr{Q})$.

(ii) Let $q_1,q_2 \in Q$. Then $\delta_1(q_1,\varepsilon)(q_2) = \vee\{\delta_{Q,\mathscr{Q}}(q_1, r_1) \wedge \delta^*(r_1,\varepsilon)(r_2) \wedge \delta_{Q,\mathscr{Q}}(r_2,q_2) : r_1,r_2 \in Q\} = \vee\{\delta_{Q,\mathscr{Q}}(q_1,r) \wedge \delta_{Q,\mathscr{Q}}(r,q_2) : r \in Q\} \leqslant \delta_{Q,\mathscr{Q}}(q_1,q_2)$. Thus $\delta_1(q_1,\varepsilon)(q_2) \leqslant \delta_{Q,\mathscr{Q}}(q_1,q_2)$.

Conversely, $\delta_1(q_1,\varepsilon)(q_2) = \vee\{\delta_{Q,\mathscr{Q}}(q_1,r_1) \wedge \delta^*(r_1,\varepsilon)(r_2) \wedge \delta_{Q,\mathscr{Q}}(r_2,q_2) : r_1,r_2 \in Q\} = \vee\{\delta_{Q,\mathscr{Q}}(q_1,r) \wedge \delta_{Q,\mathscr{Q}}(r,q_2) : r \in Q\} \geqslant \delta_{Q,\mathscr{Q}}(q_1,q_1) \wedge \delta_{Q,\mathscr{Q}}(q_1,q_2) = \delta_{Q,\mathscr{Q}}(q_1,q_2)$. Thus $\delta_1(q_1,\varepsilon)(q_2) \geqslant \delta_{Q,\mathscr{Q}}(q_1,q_2)$. Hence $\delta_1(q_1,\varepsilon)(q_2) = \delta_{Q,\mathscr{Q}}(q_1,q_2)$.

(iii) Let $q_1,q_2 \in Q$ and $v_1,v_2 \in E^*$. Then $\delta_1(q_1,v_1\cdot v_2)(q_2) = \vee\{\delta_{Q,\mathscr{Q}}(q_1,r_1) \wedge \delta^*(r_1,v_1\cdot v_2)(r_2) \wedge \delta_{Q,\mathscr{Q}}(r_2, q_2) : r_1,r_2 \in Q\} = \vee\{\delta_{Q,\mathscr{Q}}(q_1,r_1) \wedge (\vee\{\delta^*(r_1,v_1)(r) \wedge \delta^*(r,v_2)(r_2) : r \in Q\}) \wedge \delta_{Q,\mathscr{Q}}(r_2,q_2) : r_1,r_2 \in Q\} = \vee\{\delta_{Q,\mathscr{Q}}(q_1,r_1) \wedge \delta^*(r_1,v_1)(r) \wedge \delta^*(r,v_2)(r_2) \wedge \delta_{Q,\mathscr{Q}}(r_2,q_2) : r,r_1,r_2 \in Q\} = \vee\{\delta_{Q,\mathscr{Q}}(q_1,r_1) \wedge \delta^*(r_1,v_1)(r) \wedge \delta_{Q,\mathscr{Q}}(r,r) \wedge \delta_{Q,\mathscr{Q}}(r,r) \wedge \delta^*(r,v_2)(r_2) \wedge \delta_{Q,\mathscr{Q}}(r_2,q_2) : r,r_1,r_2 \in Q\} = \vee\{\delta_1(q_1,v_1)(r) \wedge \delta_1(r,v_2)(q_2) : r \in Q\}$. Thus $\delta_1(q_1,v_1\cdot v_2)(q_2) = \vee\{\delta_1(q_1,v_1)(r) \wedge \delta_1(r,v_2)(q_2) : r \in Q\}$.

(iv) Let $q_1,q_2 \in Q$, $q_2' \in core(Q_\omega)$, and $v \in E^*$. Then $\delta_1(q_1,v)(q_2) \wedge Q_\omega(p) = \vee\{\delta_{Q,\mathscr{Q}}(q_1,r_1) \wedge \delta^*(r_1,v)(r_2) \wedge \delta_{Q,\mathscr{Q}}(r_2,q_2) : r_1,r_2 \in Q\} \wedge Q_\omega(q_1) = \vee\{\delta_{Q,\mathscr{Q}}(q_1,r_1) \wedge \delta^*(r_1,v)(r_2) \wedge \delta_{Q,\mathscr{Q}}(r_2,q_2) \wedge Q_\omega(q_1) : r_1,r_2 \in Q\} \leqslant \vee\{\delta_{Q,\mathscr{Q}}(q_2',r_1) \wedge \delta^*(r_1,v)(r_2) \wedge \delta_{Q,\mathscr{Q}}(r_2,q_2) : r_1,r_2 \in Q\} = \delta_1(q_2',v)(q_2)$. Thus $\delta_1(q_1, v)(q_2) \wedge Q_\omega(q_1) \leqslant \delta_1(q_2',v)(q_2)(q_2)$.

(v) From condition (iv), $\delta_1(q_2,v) = \delta_1(q_2',v)$, $\forall q_2,q_2' \in core(Q_\omega)$ and $\forall v \in E^*$.

Hence $\mathscr{P} = ((Q,\mathscr{Q}),(E^*,\cdot,\varepsilon),\delta_1,\delta_0,\delta_m)$ is a fuzzy automaton with a fuzzy partition.

Example 3.1 *In continuation to Example 2.1, let $\mathscr{A}$ be a fuzzy automaton. In addition, let $(Q,\mathscr{Q})$ be a fuzzy partition space, where $\mathscr{Q} = \{Q_1,Q_2\}$ is a fuzzy partition of Q such that $Q_1(q_0) = 0, Q_1(q_1) = 0.4$ and $Q_2(q_0) = 0.3, Q_2(q_1) = 0$. Then the fuzzy partitioned automaton corresponding to $\mathscr{A}$ is $\mathscr{P} = ((Q,\mathscr{Q}),(E^*,\cdot,\varepsilon),\delta_1,\delta_0,\delta_m)$, where $\delta_1(q_0,\sigma_1)(q_0) = 0.4, \delta_1(q_0,\sigma_1)(q_1) = 0.8, \delta_1(q_1,\sigma_1)(q_0) = 0.4, \delta_1(q_1,\sigma_1)(q_1) = 0.7, \delta_1(q_0,\sigma_2)(q_0) = 0.4, \delta_1(q_0, \sigma_2)(q_1) = 0.9, \delta_1(q_1,\sigma_2)(q_0) = 0.4$, and $\delta_1(q_1,\sigma_2)(q_1) = 0.6$. Now, it can be easily seen that $\mathscr{L}(\mathscr{P})(\varepsilon) = \mathscr{L}(\mathscr{P})(v) = 0.4$ and $\mathscr{L}_m(\mathscr{P})(\varepsilon) = 0, \mathscr{L}_m(\mathscr{P})(v) = 0.3$, $\forall v \in E^+$.*

The following are to underlay the relationships between the fuzzy languages generated/marked by fuzzy partitioned automaton and fuzzy automaton.

Proposition 3.2 *Let $\mathscr{P} = ((Q,\mathscr{Q}),(E^*,\cdot,\varepsilon),\delta_1,\delta_0,\delta_m)$ be the fuzzy partitioned automaton corresponding to a fuzzy automaton $\mathscr{A} = (Q,E,\delta,\delta_0,\delta_m)$. Then $\mathscr{L}(\mathscr{P}) \supseteq \mathscr{L}(\mathscr{A})$.*

Proof: Let $v \in E^*$. Then $\mathscr{L}(\mathscr{P})(v) = \vee\{\delta_0(q_1) \wedge \delta_1(a_1,v)(q_2) : q_1,q_2 \in Q\} = \vee\{\delta_0(q_1) \wedge (\vee\{\delta_{Q,\mathscr{Q}}(q_1,r_1) \wedge \delta^*(r_1,v)(r_2) \wedge \delta_{Q,\mathscr{Q}}(r_2,q_2) : r_1,r_2 \in Q\}) : q_1,q_2 \in Q\} = \vee\{\delta_0(q_1) \wedge \delta_{Q,\mathscr{Q}}(q_1,r_1) \wedge \delta^*(r_1,v)(r_2) \wedge \delta_{Q,\mathscr{Q}}(r_2,q_2) : r,r_1,r_2,q_2 \in Q\} \geqslant \vee\{\delta_0(q_1) \wedge \delta_{Q,\mathscr{Q}}(q_1, q_1) \wedge \delta^*(q_1,v)(q_2) \wedge \delta_{Q,\mathscr{Q}}(q_2,q_2) : q_1,q_2 \in Q\} = \vee\{\delta_0(q_1) \wedge \delta^*(q_1,v)(q_2)\} = \mathscr{L}(\mathscr{A})(v)$. Thus $\mathscr{L}(\mathscr{P}) \supseteq \mathscr{L}(\mathscr{A})$.

Proposition 3.3 *Let $\mathscr{P} = ((Q,\mathscr{Q}),(E^*,\cdot,\varepsilon),\delta_1,\delta_0,\delta_m)$ be the fuzzy partitioned automaton corresponding to a fuzzy automaton $\mathscr{A} = (Q,E,\delta,\delta_0,\delta_m)$, where $\mathscr{Q} = \{Q_\omega : \omega \in \Omega\}$ is a fuzzy partition of Q such that for all $\omega \in \Omega$, there exists a unique $q_1 \in Q$ with condition that $Q_\omega(q_1) = 1$ and 0, otherwise. Then $\mathscr{L}(\mathscr{P}) = \mathscr{L}(\mathscr{A})$.*

Proof: The proof is straightforward.

Proposition 3.4 *Let $\mathscr{P} = ((Q,\mathscr{Q}),(E^*,\cdot,\varepsilon),\delta_1,\delta_0,\delta_m)$ be the fuzzy partitioned automaton corresponding to a fuzzy automaton $\mathscr{A} = (Q,E,\delta,\delta_0,\delta_m)$. Then $\mathscr{L}_m(\mathscr{P}) \supseteq \mathscr{L}_m(\mathscr{A})$.*

Proof: Similar to that Proposition 3.2.

Proposition 3.5 *Let $\mathscr{P} = ((Q,\mathscr{Q}),(E^*,\cdot,\varepsilon),\delta_1,\delta_0,\delta_m)$ be the fuzzy partitioned automaton corresponding to a fuzzy automaton $\mathscr{A} = (Q,E,\delta,\delta_0,\delta_m)$, where $\mathscr{Q} = \{Q_\omega : \omega \in \Omega\}$ is a fuzzy partition of Q such that for all $\omega \in \Omega$, there exists a unique $q_1 \in Q$ with condition that $Q_\omega(q_1) = 1$ and 0, otherwise. Then $\mathscr{L}_m(\mathscr{P}) = \mathscr{L}_m(\mathscr{A})$.*

Proof: Similar to Proposition 3.3.

Proposition 3.6 *Let $f : E^* \longrightarrow I$ be a fuzzy language. Then there exists a fuzzy partitioned automaton $\mathscr{P}$ such that $\mathscr{L}(\mathscr{P}) = f$ iff $f(v_1\cdot v_2) \leqslant f(v_1)$, $\forall v_1,v_2 \in E^*$.*

Proof: Follows from Propositions 2.1 and 3.3.

Before stating next, we recall the following definition from [15].

Definition 3.3 *The* **prefix closure** *of $L \subseteq E^*$, denoted by $\overline{L}$, is given as $\overline{L} = \{v_1 \in E^* : v_1\cdot v_2 \in L$, for some $v_2 \in E^*\}$. Further, L is called a* **prefix closed language** *if $L = \overline{L}$.*

Proposition 3.7 *Let $\mathscr{P} = ((Q,\mathscr{Q}),(E^*,\cdot,\varepsilon),\delta_1,\delta_0,\delta_m)$ be the fuzzy partitioned automaton corresponding to a fuzzy automaton $\mathscr{A} = (Q,E,\delta,\delta_0,\delta_m)$. Then $supp(\mathscr{L}(\mathscr{P}))$ is a prefix closed language.*

Proof: We only demonstrate that $\overline{supp(\mathscr{L}(\mathscr{P}))} \subseteq supp(\mathscr{L}(\mathscr{P}))$. For which, let $v_1 \in \overline{supp(\mathscr{L}(\mathscr{P}))}$. Then there exists $v_2 \in E^*$ such that $v_1 \cdot v_2 \in supp(\mathscr{L}(\mathscr{P}))$. Therefore $\mathscr{L}(\mathscr{P})(v_1 \cdot v_2) > 0$, or that $\mathscr{L}(\mathscr{P})(v_1) > 0$, i.e., $v_1 \in supp(\mathscr{L}(\mathscr{P}))$. Thus $\overline{supp(\mathscr{L}(\mathscr{P}))} \subseteq supp(\mathscr{L}(\mathscr{P}))$. Thus $supp(\mathscr{L}(\mathscr{P}))$ is a prefix closed language.

4 Controlled system of the fuzzy partitioned DES under full observation

In this section, we propose the model of a fuzzy partitioned DES and its supervisory control theory under full observation. We introduce the concept of a (fully observable) fuzzy supervisor. Further, we introduce the model of a controlled system of a fuzzy partitioned DES under fuzzy supervisor. Furthermore, we introduce the concepts of controllable and $\mathscr{L}_m(\mathscr{P})$-closed fuzzy languages. Finally, we underlay the relationships between the fuzzy languages generated/marked by controlled system and controllable, $\mathscr{L}_m(\mathscr{P})$-closed fuzzy languages. Throughout this section, $f \in \mathscr{F}(E^*)$ is a fuzzy language with condition that $f(v_1 \cdot v_2) \leqslant f(v_1)$, $\forall v_1, v_2 \in E^*$.

The fuzzy partitioned DES is described by a fuzzy partitioned automaton $\mathscr{P} = ((Q,\mathscr{Q}),(E^*,\cdot,\varepsilon),\delta_1,\delta_0,\delta_m)$ corresponding to a fuzzy automaton $\mathscr{A} = (Q,E,\delta,\delta_0,\delta_m)$. In the fuzzy partitioned DES's model, our concern is the strings of event-set that the system can generate. Thus the behavior of a fuzzy partitioned DES is modelled by fuzzy language $\mathscr{L}(\mathscr{P}) : E^* \longrightarrow I$ such that $\mathscr{L}(\mathscr{P})(v) = \vee\{\delta_0(q_1) \wedge \delta_1(q_1,v)(q_2) : q_1, q_2 \in Q\}$, $\forall v \in E^*$.

Similar to [15], we partition the event-set E into controllable event-set E_c and uncontrollable event-set E_{uc} such that $E_c \cap E_{uc} = \emptyset$, which are to be used in following consequences.

Definition 4.1 *Let $\mathscr{P} = ((Q,\mathscr{Q}),(E^*,\cdot,\varepsilon),\delta_1,\delta_0,\delta_m)$ be the fuzzy partitioned automaton corresponding to a fuzzy automaton $\mathscr{A} = (Q,E,\delta,\delta_0,\delta_m)$. Then the (fully observable)* **fuzzy supervisor** *for $\mathscr{P}$ is a map $\mathsf{S} : supp(\mathscr{L}(\mathscr{P})) \longrightarrow \mathscr{F}(E)$ such that for all $v \in supp(\mathscr{L}(\mathscr{P}))$ and $a \in E_{uc}$, $\mathsf{S}(v)(a) = t$, where $t \in I/\{0\}$.*

The controlled system of a fuzzy partitioned DES under a fuzzy supervisor S is denoted by $\mathsf{S}/\mathscr{P}$. The behaviors of $\mathsf{S}/\mathscr{P}$ are defined as follow.

Definition 4.2 *Let $\mathsf{S}/\mathscr{P}$ be the controlled system of a fuzzy partitioned DES under fuzzy supervisor S. Then*

(i) the **fuzzy language** *$\mathscr{L}^{\mathsf{S}} : E^* \longrightarrow I$ is* **generated** *by $\mathsf{S}/\mathscr{P}$ if*

(a) $\mathscr{L}^{\mathsf{S}}(\varepsilon) = t$, and

(b) $\mathscr{L}^{\mathsf{S}}(v \cdot a) = \mathscr{L}^{\mathsf{S}}(v) \wedge \mathsf{S}(v)(a) \wedge \mathscr{L}(\mathscr{P})(v \cdot a)$, $\forall v \in E^*$ and $\forall a \in E$.

(ii) the **fuzzy language** *$\mathscr{L}_m^{\mathsf{S}} : E^* \longrightarrow I$ is* **marked** *by $\mathsf{S}/\mathscr{P}$ if $\mathscr{L}_m^{\mathsf{S}} = \mathscr{L}^{\mathsf{S}} \cap \mathscr{L}_m(\mathscr{P})$.*

Definition 4.3 *Let $\mathscr{P} = ((Q,\mathscr{Q}),(E^*,\cdot,\varepsilon),\delta_1,\delta_0,\delta_m)$ be the fuzzy partitioned automaton corresponding to a fuzzy automaton $\mathscr{A} = (Q,E,\delta,\delta_0,\delta_m)$. Then the fuzzy supervisor S for $\mathscr{P}$ is* **nonblocking** *if $\mathscr{L}_m^{\mathsf{S}} = \mathscr{L}^{\mathsf{S}}$. Further, if $\mathscr{L}_m^{\mathsf{S}} \neq \mathscr{L}^{\mathsf{S}}$, then S is a* **blocking** *fuzzy supervisor.*

Example 4.1 *In continuation to example 3.1, let $\mathscr{P}$ be the fuzzy partitioned automaton corresponding to a fuzzy automaton $\mathscr{A}$. In addition, let $E_c = \sigma_1$, $E_o = \sigma_2$, and $\mathsf{S} : supp(\mathscr{L}(\mathscr{P})) \longrightarrow \mathscr{F}(E)$ be a fuzzy supervisor such that for all $v \in supp(\mathscr{L}(\mathscr{P}))$ and $a \in E$, $\mathsf{S}(v)(a) = 0.4$. Then for all $v \in E^*$ and $a \in E$, $\mathscr{L}^{\mathsf{S}}(v)(a) = 0.4$. Also, $\mathscr{L}_m^{\mathsf{S}}(v) = \mathscr{L}^{\mathsf{S}}(v) \wedge \mathscr{L}_m(\mathscr{P})(v) = 0.3$, which implies that $\mathscr{L}_m^{\mathsf{S}} \neq \mathscr{L}^{\mathsf{S}}$. Thus S is a blocking fuzzy supervisor.*

Definition 4.4 *Let $\mathscr{P} = ((Q,\mathscr{Q}),(E^*,\cdot,\varepsilon),\delta_1,\delta_0,\delta_m)$ be the fuzzy partitioned automaton corresponding to a fuzzy automaton $\mathscr{A} = (Q,E,\delta,\delta_0,\delta_m)$. Then the fuzzy language $\mathscr{K} \subseteq \mathscr{L}(\mathscr{P})$ is called a* **controllable fuzzy language** *(with respect to $\mathscr{L}(\mathscr{P})$ and E_{uc}) if $\mathscr{K}\mathscr{E}_{uc} \cap \mathscr{L}(\mathscr{P}) \subseteq \mathscr{K}$, where $\mathscr{E}_{uc} : E^* \longrightarrow I$ is a map such that $\forall v \in E^*$,*

$$\mathscr{E}_{uc}(v) = \begin{cases} \mathscr{K}(\varepsilon) & \text{if } v \in E_{uc} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.5 *Let $\mathscr{P} = ((Q,\mathscr{Q}),(E^*,\cdot,\varepsilon),\delta_1,\delta_0,\delta_m)$ be the fuzzy partitioned automaton corresponding to a fuzzy automaton $\mathscr{A} = (Q,E,\delta,\delta_0,\delta_m)$. Then the fuzzy language $\mathscr{K} \subseteq \mathscr{L}(\mathscr{P})$ is called an* **$\mathscr{L}_m(\mathscr{P})$-closed fuzzy language** *if $\mathscr{K} = \mathscr{K} \cap \mathscr{L}_m(\mathscr{P})$.*

Proposition 4.1 *Let $\mathscr{P} = ((Q,\mathscr{Q}),(E^*,\cdot,\varepsilon),\delta_1,\delta_0,\delta_m)$ be the fuzzy partitioned automaton corresponding to a fuzzy automaton $\mathscr{A} = (Q,E,\delta,\delta_0,\delta_m)$ and $\mathscr{K} \subseteq \mathscr{L}(\mathscr{P})$ be a fuzzy language. Then $\mathscr{K}$ is a controllable fuzzy language iff $\mathscr{K}(v \cdot a) = \mathscr{K}(v) \wedge \mathscr{K}(\varepsilon) \wedge \mathscr{L}(\mathscr{P})(v \cdot a)$, $\forall v \in E^*$ and $\forall a \in E_{uc}$.*

Proof: Let $\mathscr{K}$ be a controllable fuzzy language. Then for all $v \in E^*$ and $a \in E_{uc}$, $(\mathscr{K}\mathscr{E}_{uc} \cap \mathscr{L}(\mathscr{P}))(v \cdot a) \leqslant \mathscr{K}(v \cdot a)$. Now, $(\mathscr{K}\mathscr{E}_{uc} \cap \mathscr{L}(\mathscr{P}))(v \cdot a) = \mathscr{K}(v) \wedge \mathscr{E}_{uc}(a) \wedge \mathscr{L}(\mathscr{P})(v \cdot a) = \mathscr{K}(v) \wedge \mathscr{K}(\varepsilon) \wedge \mathscr{L}(\mathscr{P})(v \cdot a)$. Thus $\mathscr{K}(v) \wedge \mathscr{K}(\varepsilon) \wedge \mathscr{L}(\mathscr{P})(v \cdot a) \leqslant \mathscr{K}(v \cdot a)$. On the other hand, as $\mathscr{K}(v \cdot a) \leqslant \mathscr{K}(v)$, $\mathscr{K}(v \cdot a) \leqslant \mathscr{K}(\varepsilon)$, and $\mathscr{K}(v \cdot a) \leqslant \mathscr{L}(\mathscr{P})(v \cdot a)$,

$\mathscr{K}(v \cdot a) \leqslant \mathscr{K}(v) \wedge \mathscr{K}(\varepsilon) \wedge \mathscr{L}(\mathscr{P})(v \cdot a)$. Thus $\mathscr{K}(v \cdot a) = \mathscr{K}(v) \wedge \mathscr{K}(\varepsilon) \wedge \mathscr{L}(\mathscr{P})(v \cdot a)$.

Conversely, let $\mathscr{K}(v \cdot a) = \mathscr{K}(v) \wedge \mathscr{K}(\varepsilon) \wedge \mathscr{L}(\mathscr{P})(v \cdot a)$, $\forall v \in E^*$ and $\forall a \in E_{uc}$. Then we have to demonstrate that $\mathscr{K}$ is a controllable fuzzy language, or that $(\mathscr{K}\mathscr{E}_{uc} \cap \mathscr{L}(\mathscr{P}))(v) \leqslant \mathscr{K}(v)$, $\forall v \in E^*$. For which, let $|v| = 0$. Then $(\mathscr{K}\mathscr{E}_{uc} \cap \mathscr{L}(\mathscr{P}))(v) = 0 \leqslant \mathscr{K}(v)$. Also, let $|v| > 0$. Then there exists $u \in E^*$ and $a \in E$ such that $v = u \cdot a$. If $a \in E_c$, then $(\mathscr{K}\mathscr{E}_{uc} \cap \mathscr{L}(\mathscr{P}))(v) = 0 \leqslant \mathscr{K}(v)$. Otherwise, we have $(\mathscr{K}\mathscr{E}_{uc} \cap \mathscr{L}(\mathscr{P}))(v) = (\mathscr{K}\mathscr{E}_{uc} \cap \mathscr{L}(\mathscr{P}))(u \cdot a) = \mathscr{K}(u) \wedge \mathscr{E}_{uc}(a) \wedge \mathscr{L}(\mathscr{P})(u \cdot a) = \mathscr{K}(u) \wedge \mathscr{K}(\varepsilon) \wedge \mathscr{L}(\mathscr{P})(u \cdot a) = \mathscr{K}(u \cdot a) = \mathscr{K}(v)$. Thus $(\mathscr{K}\mathscr{E}_{uc} \cap \mathscr{L}(\mathscr{P}))(v) \leqslant \mathscr{K}(v)$. Hence $\mathscr{K}$ is a controllable fuzzy language.

Now, we underlay the relationships between the fuzzy languages generated/marked by control system and controllable, $\mathscr{L}_m(\mathscr{P})$-closed fuzzy languages.

Proposition 4.2 *Let $\mathscr{P} = ((Q, \mathscr{Q}), (E^*, \cdot, \varepsilon), \delta_1, \delta_0, \delta_m)$ be the fuzzy partitioned automaton corresponding to a fuzzy automaton $\mathscr{A} = (Q, E, \delta, \delta_0, \delta_m)$. Then $\mathscr{K} \subseteq \mathscr{L}(\mathscr{P})$ is a controllable fuzzy language iff there exists a fuzzy supervisor S for $\mathscr{P}$ such that $\mathscr{L}^{\mathsf{S}} = \mathscr{K}$*

Proof: Let $\mathscr{K}$ be a controllable fuzzy language. Then we can construct a fuzzy supervisor S for $\mathscr{P}$ such that for all $v \in supp(\mathscr{L}(\mathscr{P}))$ and $a \in E$,

$$\mathsf{S}(v)(a) = \begin{cases} \mathscr{K}(\varepsilon) & \text{if } a \in E_{uc} \\ \mathscr{K}(v \cdot a) & \text{otherwise.} \end{cases}$$

Now, we have to demonstrate that $\mathscr{L}^{\mathsf{S}}(v) = \mathscr{K}(v)$, $\forall v \in E^*$. The proof of result is given by induction on length of v. Let $|v| = n$. Then for $n = 0$, the result is straightforward. We assume that the result is true for all $v \in E^*$ such that $|v| \leqslant n, n > 0$ and let $a \in E$. If $v \notin supp(\mathscr{L}(\mathscr{P}))$, then $\mathscr{L}(\mathscr{P})(v) = 0$, or that $\mathscr{L}(\mathscr{P})(v \cdot a) = 0$. Thus $\mathscr{L}^{\mathsf{S}}(v \cdot a) = \mathscr{K}(v \cdot a) = 0$. If $v \in supp(\mathscr{L}(\mathscr{P}))$ and $a \in E_{uc}$, then $\mathscr{L}^{\mathsf{S}}(v \cdot a) = \mathscr{L}^{\mathsf{S}}(v) \wedge \mathsf{S}(v)(a) \wedge \mathscr{L}(\mathscr{P})(v \cdot a) = \mathscr{K}(v) \wedge \mathscr{K}(\varepsilon) \wedge \mathscr{L}(\mathscr{P})(v \cdot a) = \mathscr{K}(v \cdot a)$. Otherwise, $\mathscr{L}^{\mathsf{S}}(v \cdot a) = \mathscr{L}^{\mathsf{S}}(v) \wedge \mathsf{S}(v)(a) \wedge \mathscr{L}(\mathscr{P})(v \cdot a) = \mathscr{K}(v) \wedge \mathscr{K}(v \cdot a) \wedge \mathscr{L}(\mathscr{P})(v \cdot a) = \mathscr{K}(v \cdot a)$. Thus $\mathscr{L}^{\mathsf{S}} = \mathscr{K}$.

Let S be a fuzzy supervisor for $\mathscr{P}$ such that $\mathscr{L}^{\mathsf{S}} = \mathscr{K}$. Then for all $v \in E^*$ and $a \in E_{uc}$, $\mathscr{K}(v \cdot a) = \mathscr{L}^{\mathsf{S}}(v \cdot a) = \mathscr{L}^{\mathsf{S}}(v) \wedge \mathsf{S}(v)(a) \wedge \mathscr{L}(\mathscr{P})(v \cdot a) = \mathscr{K}(v) \wedge \mathscr{L}^{\mathsf{S}}(\varepsilon) \wedge \mathscr{L}(\mathscr{P})(v \cdot a) = \mathscr{K}(v) \wedge \mathscr{K}(\varepsilon) \wedge \mathscr{L}(\mathscr{P})(v \cdot a)$. Hence $\mathscr{K}$ is a controllable fuzzy language.

Proposition 4.3 *Let $\mathscr{P} = ((Q, \mathscr{Q}), (E^*, \cdot, \varepsilon), \delta_1, \delta_0, \delta_m)$ be the fuzzy partitioned automaton corresponding to a fuzzy automaton $\mathscr{A} = (Q, E, \delta, \delta_0, \delta_m)$. Then $\mathscr{K} \subseteq \mathscr{L}(\mathscr{P})$ is a controllable and $\mathscr{L}_m(\mathscr{P})$-closed fuzzy language iff there exists a nonblocking fuzzy supervisor S for $\mathscr{P}$ such that $\mathscr{L}_m^{\mathsf{S}} = \mathscr{L}^{\mathsf{S}} = \mathscr{K}$.*

Proof: Let $\mathscr{K}$ be a controllable fuzzy language. Then from Proposition 4.2, there exists a fuzzy supervisor S for $\mathscr{P}$ such that $\mathscr{L}^{\mathsf{S}} = \mathscr{K}$. Now, we only demonstrate that S is a nonblocking fuzzy supervisor. For which, let $\mathscr{K}$ be an $\mathscr{L}_m(\mathscr{P})$-closed fuzzy language. Then $\mathscr{L}_m^{\mathsf{S}} = \mathscr{L}^{\mathsf{S}} \cap \mathscr{L}_m(\mathscr{P}) = \mathscr{K} \cap \mathscr{L}_m(\mathscr{P}) = \mathscr{K} = \mathscr{L}^{\mathsf{S}}$. Thus $\mathscr{L}_m^{\mathsf{S}} = \mathscr{L}^{\mathsf{S}}$. Thus S is a nonblocking fuzzy supervisor.

Conversely, let S be a nonblocking fuzzy supervisor for $\mathscr{P}$ such that $\mathscr{L}_m^{\mathsf{S}} = \mathscr{L}^{\mathsf{S}} = \mathscr{K}$. Then the results are follow from Proposition 4.2 and Definition 4.5.

Example 4.2 *In continuation to example 3.1, let $\mathscr{P}$ be the fuzzy partitioned automaton corresponding to a fuzzy automaton $\mathscr{A}$. In addition, let $E_c = \sigma_1$, $E_o = \sigma_2$, $\mathscr{K} : E^* \longrightarrow I$ be a fuzzy language such that for all $v \in E^*$, $\mathscr{K}(v) = 0.25$. It can be easily verified that $\mathscr{K}$ is a controllable (with respect to $\mathscr{L}(\mathscr{P})$ and E_{uc}) and $\mathscr{L}_m(\mathscr{P})$-closed fuzzy language. Then we can construct a non-blocking fuzzy supervisor S for $\mathscr{P}$ such that for all $v \in supp(\mathscr{L}(\mathscr{P})$ and $a \in E$, $\mathsf{S}(v)(a) = 0.25$. Now, for all $v \in E^*$, $\mathscr{L}^{\mathsf{S}}(v) = 0.25$. Also, $\mathscr{L}_m^{\mathsf{S}}(v) = \mathscr{L}^{\mathsf{S}}(v) \wedge \mathscr{L}_m(\mathscr{P})(v) = 0.25$. Thus $\mathscr{L}_m^{\mathsf{S}} = \mathscr{L}^{\mathsf{S}} = \mathscr{K}$.*

Proposition 4.4 *Let $\{\mathscr{K}_\omega \subseteq \mathscr{L}(\mathscr{P}) : \omega \in \Omega\}$ be the controllable fuzzy languages. Then $\cup\{\mathscr{K}_\omega : \omega \in \Omega\}$ and $\cap\{\mathscr{K}_\omega : \omega \in \Omega\}$ are also controllable fuzzy languages.*

Proof: The proof is straightforward.

Let $\mathscr{K} \subseteq \mathscr{L}(\mathscr{P})$ be a fuzzy language and define $\mathscr{C}_{in}(\mathscr{K}) = \{\mathscr{M} \in \mathscr{F}(E) : \mathscr{M} \subseteq \mathscr{K}$ is a controllable fuzzy language$\}$ and $\mathscr{C}_{out}(\mathscr{K}) = \{\mathscr{M} \in \mathscr{F}(E) : \mathscr{K} \subseteq \mathscr{M} \subseteq \mathscr{L}(\mathscr{P})$ is a controllable fuzzy language$\}$. Further, we define two fuzzy languages $\mathscr{K}^{\uparrow C}$ and $\mathscr{K}^{\downarrow C}$ corresponding to $\mathscr{C}_{in}(\mathscr{K})$ and $\mathscr{C}_{out}(\mathscr{K})$, respectively, are given as $\mathscr{K}^{\uparrow C} = \cup\{\mathscr{M} : \mathscr{M} \in \mathscr{C}_{in}(\mathscr{K})\}$ and $\mathscr{K}^{\downarrow C} = \cap\{\mathscr{M} : \mathscr{M} \in \mathscr{C}_{out}(\mathscr{K})\}$.

Then we have the following.

Proposition 4.5 *Let $\mathscr{K} \subseteq \mathscr{L}(\mathscr{P})$ be a fuzzy language. Then*

(i) $\mathscr{K}^{\uparrow C}$ is a supremal controllable fuzzy language such that $\mathscr{K}^{\uparrow C} \subseteq \mathscr{K}$;

(ii) $\mathscr{K}^{\downarrow C}$ is an infimal controllable fuzzy language such that $\mathscr{K} \subseteq \mathscr{K}^{\downarrow C}$; and

(iii) $\mathscr{K}$ *is an* $\mathscr{L}_m(\mathscr{P})$*-closed fuzzy language* $\Rightarrow \mathscr{K}^{\uparrow C}$ *is an* $\mathscr{L}_m(\mathscr{P})$*-closed fuzzy language.*

Proof: Follows from Proposition 4.4 and Definition 4.5.

Proposition 4.6 *Let* $\mathscr{P} = ((Q,\mathscr{Q}),(E^*,\cdot,\varepsilon),\delta_1,\delta_0,\delta_m)$ *be the fuzzy partitioned automaton corresponding to a fuzzy automaton* $\mathscr{A} = (Q,E,\delta,\delta_0,\delta_m)$ *and* $L_c \subseteq \mathscr{L}(\mathscr{P})$ *be a fuzzy language. Then there exists a fuzzy supervisor* S *for* $\mathscr{P}$ *such that* $\mathscr{L}^{\mathsf{S}} \subseteq L_c$ *with condition that if there exists a fuzzy supervisor* S' *such that* $\mathscr{L}^{\mathsf{S}'} \subseteq L_c$*. Then* $\mathscr{L}^{\mathsf{S}'} \subseteq \mathscr{L}^{\mathsf{S}}$*.*

Proof: Follows from Propositions 4.2 and 4.5.

Proposition 4.7 *Let* $\mathscr{P} = ((Q,\mathscr{Q}),(E^*,\cdot,\varepsilon),\delta_1,\delta_0,\delta_m)$ *be the fuzzy partitioned automaton corresponding to a fuzzy automaton* $\mathscr{A} = (Q,E,\delta,\delta_0,\delta_m)$ *and* $L_c \subseteq \mathscr{L}(\mathscr{P})$ *be* $\mathscr{L}_m(\mathscr{P})$*-closed. Then there exists a nonblocking fuzzy supervisor* S *for* $\mathscr{P}$ *such that* $\mathscr{L}^{\mathsf{S}} \subseteq L_c$ *with condition that if there exists a nonblocking fuzzy supervisor* S' *such that* $\mathscr{L}^{\mathsf{S}'} \subseteq L_c$*. Then* $\mathscr{L}^{\mathsf{S}'} \subseteq \mathscr{L}^{\mathsf{S}}$*.*

Proof: Follows from Propositions 4.3 and 4.5.

Proposition 4.8 *Let* $\mathscr{P} = ((Q,\mathscr{Q}),(E^*,\cdot,\varepsilon),\delta_1,\delta_0,\delta_m)$ *be the fuzzy partitioned automaton corresponding to a fuzzy automaton* $\mathscr{A} = (Q,E,\delta,\delta_0,\delta_m)$ *and* $L_c \subseteq \mathscr{L}(\mathscr{P})$ *be a fuzzy language. Then there exists a fuzzy supervisor* S *for* $\mathscr{P}$ *such that* $L_c \subseteq \mathscr{L}^{\mathsf{S}}$ *with condition that if there exists a fuzzy supervisor* S' *such that* $L_c \subseteq \mathscr{L}^{\mathsf{S}'}$*. Then* $\mathscr{L}^{\mathsf{S}} \subseteq \mathscr{L}^{\mathsf{S}'}$*.*

Proof: Follows from Propositions 4.2 and 4.5.

5 Conclusion

In this paper, we have introduced the concept of a fuzzy partitioned automaton corresponding to a given fuzzy automaton whose state-set is a space with a fuzzy partition of state-set of given automaton. Also, we have obtained the relationships between the fuzzy languages generated/marked by fuzzy partitioned automaton and fuzzy automaton. Interestingly, we have proposed the model of a fuzzy partitioned DES and its supervisory control theory under full observation, which is described by a fuzzy partitioned automaton. The behavior of fuzzy partitioned DES has been modelled by fuzzy language, which is generated by a fuzzy partitioned automata. Also, we have introduced a (fully observable) fuzzy supervisor. Further, we have introduced the concept of controlled system of a fuzzy partitioned DES under fuzzy supervisor. These can generate the strings that occur with minimum membership values in I under max-min composition. In this framework, we have focused on behavior of controlled system that can be achieved when controlling the fuzzy partitioned DES under fuzzy supervisor. Furthermore, we have introduced the concepts of a controllable and $\mathscr{L}_m(\mathscr{P})$-closed fuzzy languages. Finally, we have obtained the relationships between the fuzzy languages generated/marked by controlled system and controllable, $\mathscr{L}_m(\mathscr{P})$-closed fuzzy languages. Moreover, we have discussed the supremal and infimal controllable fuzzy languages.

In the future, we will provide an application of proposed model of fuzzy partitioned DES in medical diagnosis and treatment.

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