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An Overview of Graded Structures of Opposition with Intermediate Quantifiers

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Abstract

The paper is focused on an overview of graded structures of opposition with generalized intermediate quantifiers. This article focuses mainly on achieving two goals. Firstly, classical structures of opposition (square, hexagon, cube) are introduced. Then, graded extensions of mentioned structures are constructed using linguistic expressions of natural language.

Keywords: Fuzzy natural logic, Generalized intermediate quantifiers, Graded Peterson's square of opposition, Graded Peterson's cube of opposition

1 Introduction

The efforts to understand a natural language were already made by Aristotle in the times of Ancient Greece. Using the Aristotle's square of opposition, he tried to understand classical quantifiers. It was not until modern times when new generalized quantifiers appeared, which was a contribution of a Polish mathematician Andrzej Mostowski [20] in the 50's of the last century.

Numerous mathematicians followed his work. A few years later, this topic began to be interesting for linguists and philosophers, who laid a question of how to formalize expressions such as "for several", etc. Mostowski's exact approach to generalized quantifiers was further elaborated in works of Barwise and Copper ([16]).

These works inspired many authors to deal with the studies of various special types of generalized quantifiers. A special group of quantifiers, which are common in a natural language, was worked out by P.L. Peterson in his book *Intermediate quantifiers*. In [32], the author continued in the work of Thompson, who was the first to introduce quantifiers "A few", "Many", and "Most" in 1982. Later in his book, Peterson came up with one more quantifier "Almost all" and studied intermediate quantifiers from the point of view of their position in the Peterson's square of opposition. At the beginning of his book, he presented the basic idea characterizing the group of *intermediate quantifiers*:

Intermediate quantifiers express logical quantities which fall between Aristotle's two quantities of categorical propositions - universal and particular.

Fuzzy quantifiers, as it proved later that they characterize many objects very well, were proposed by L.A. Zadeh in [38] in 1983. Fuzzy quantifiers were represented using fuzzy numbers.

2 Classical Structures of Opposition

In this section, we review classical structures of opposition, which are called square, hexagon and cube. We will start with the Aristotle's square of opposition. Before considering a generalization of the mentioned structures, we will also refresh the hexagon of opposition and the cube of opposition.

2.1 Aristotle's square of opposition

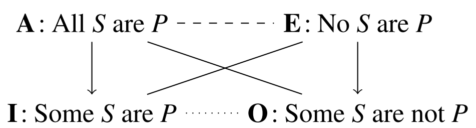
A natural language used in a common human life carries an incredibly strong expressive power, which helps us determine the validity of any statement. Special expressions of a natural language used to characterize the number/amount of objects having a certain property are called quantifiers. It concerns expressions, such as "All", "Almost all", "Roughly half", "Many", "More than a third", etc.

Classical quantifiers "All" and "Some" were analyzed by the Aristotle's square of opposition, which was first studied and proposed by a philosopher Aristotle in [33] in Ancient Greece for the purposes of analyzing logical relationships between various quantified statements. Aristotle introduced logic which currently holds his name and where the crucial role is played by the Aristotle's square of opposition with categorical syllogisms.

From the point of view of predicate logic, the Aristotle's square of opposition is formed by the properties of quantifiers *contrary, sub-contrary, sub-alterns, and contradictory*, which are defined as follows:

- We say that two formulas are *contradictory* if in any model they cannot be both true, and they cannot be both false.
- We say that two formulas are *contrary* if in any model they cannot be both true, but both can be false.
- We say that two formulas are *sub-contrary* if in any model they cannot be both false, but both can be true.
- A formula is *sub-altern* of another one, called a *super-altern* formula, if, in any model, it must be true if its super-altern is true. At the same time, the super-altern must be false if the sub-altern is false.

Below is presented the Aristotle's square of opposition with individual properties. The full diagonal line between formulas **A**, **O** and formulas **E**, **I** denotes them to be contradictory. The dashed line between individual formulas **A** and **E** denotes contrary. Similarly, the dotted line between **I** and **O** characterizes sub-contrary and, finally, the arrow describes the sub-altern property.



It proved later that Aristotle's logic is full of imperfections, which then slowed down its subsequent development. One of the most famous examples is the impossibility to work with individual subjects of type "Socrates" as the statement "Socrates is a man" cannot be interpreted in Aristotle's logic. More information can be found in [7].

The first one to find a way to the currently known first-order predicate logic was a German logician Gottlob Frege (see [15]). Using the first-order predicate logic, the basic Aristotle's statement were transformed into the following formulas:

$$\textbf{A}: \text{All } S \text{ are } P \qquad (\forall x)(Sx \Rightarrow Px) \wedge (\exists x)Sx, \tag{1}$$

$$\textbf{E}: \text{No } S \text{ are } P \qquad (\forall x)(Sx \Rightarrow \neg Px), \tag{2}$$

$$\textbf{I}: \text{Some } S \text{ are } P \qquad (\exists x)(Sx \wedge Px), \tag{3}$$

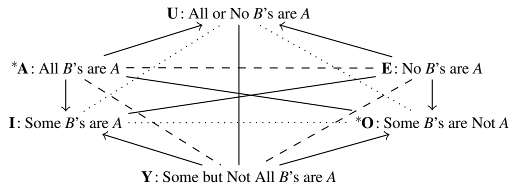
$$\textbf{O}: \text{Some } S \text{ are not } P \qquad (\exists x)(Sx \wedge \neg Px) \vee \neg(\exists x)Sx. \tag{4}$$

A very important role is played by the "existential import" which was discussed in [23]. At this point we will just mention other authors who have dealt with this issue [1, 3]. From the propositional logic point of view, the Aristotle's square was later developed in [18, 30]. In addition, it was even more deeply elaborated from the first-order predicate logic point of view by other authors in [2, 9, 32].

2.2 The Hexagon of opposition

An extension into the hexagon was discovered by the French logicians Jacoby, Sesmat and Blanché [34, 6], and it was also studied by Bézeiau [4]. A different version of hexagon is the so-called *Sherwood-Czezowski hexagon* [11, 35]. The arrows of subalternation are the same, but crucial difference = horizontal diagonal is contrary instead of contradictory, so there are fewer subcontraries as well.

In this subsection, we will be interested in Béziau approach, who in [4] suggested to extend a square of opposition by adding two new formulas **U** and **Y** that are defined as the disjunction of the two top corners of the square and the conjunction of the two bottom corners.



The diagonal lines represent contradictories, which means that the formulas **A** and **E** are contraries, **A** and **E** entail **U**, while **Y** entails both formulas **I** and **O**. The formulas **I** and **O** are sub-contraries. It is interesting to see that the logical hexagon obtains three Aristotle's squares of opposition, namely, **AEIO**, **AYOU** and **EYUI**.

In [36], we can find differences between the Aristotle hexagon and Duality hexagon. A logical hexagon with many examples was described in [13].

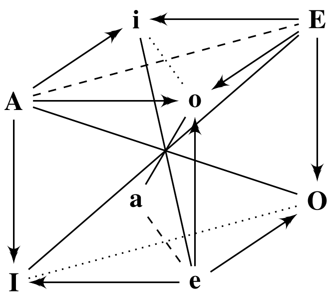


Figure 1: Moretti's cube of opposition

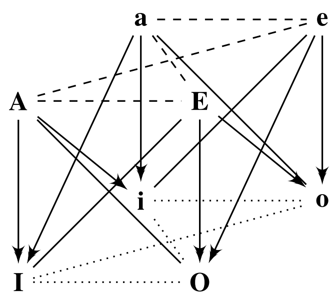


Figure 2: JK-cube of opposition

2.3 The cube of opposition

The objective of this subchapter is to familiarize the reader with the structures of opposition which are called cubes of opposition although Béziau states in his publication [5] that cubes of opposition do not exist, but there are only combinations of squares of opposition. Nevertheless, there is a large group of authors (Pellissier, Moretti, Dubois, Prade, Rico, Ciucci, Moyse, Lesot) who deal with these interesting structures of opposition.

In this chapter, we will study the Moretti's cube of opposition [19] and the Johnson-Keynes' cube (JK-cube) [17].

Let's recall that the Aristotle's square of opposition is formed by two positive and two negative quantifiers. Changing S and P into their negation, $\neg S$ and $\neg P$ respectively, leads to another similar square of opposition **aeio**. Then the 8 formulas, **A, E, I, O, a, e, i, o** create a *cube of opposition*, which requires that $S, P, \neg P, \neg S$ are non-empty. These assumptions ensure the consistency of both squares of opposition forming a cube of opposition.

To construct a cube of opposition as an extension of the Aristotle's square, we have to define new formulas as follows:

$$\textbf{a} : \text{All } \neg S\text{'s are not } P \qquad (\forall x)(\neg Sx \Rightarrow \neg Px), \tag{5}$$

$$\textbf{e} : \text{All } \neg S\text{'s are } P \qquad (\forall x)(\neg Sx \Rightarrow Px), \tag{6}$$

$$\textbf{i} : \text{Some } \neg S\text{'s are not } P \qquad (\exists x)(\neg Sx \wedge \neg Px), \tag{7}$$

$$\textbf{o} : \text{Some } \neg S\text{'s are } P \qquad (\exists x)(\neg Sx \wedge Px). \tag{8}$$

It is interesting to observe that the Moretti's cube of opposition is based on four vertices **A**, **E**, **a**, **e** from which arrows leave, while the four vertices where the arrow arrive are **I**, **O**, **i**, **o**. As we can see above, the Moretti's cube of opposition is formed by six squares of opposition **AEIO**, **AaOo**, **AeOi**, **aEoI**, **eEiI** and **aeoi**.

Another interesting structure of opposition is the JK-cube of opposition **AEIOaeio**, which was initially presented as an octagon by Johnson and Keynes consisting of 4 squares **AEIO**, **aeio**, **AeOi** and **aEoI**.

2.4 Peterson's square of opposition

In this section, we continue with an extension of the Aristotle's square of opposition into the Peterson's square of opposition by *intermediate quantifiers* "Almost all, Most and Many".

The first version of the Peterson's (complete) square of opposition was introduced as a generalization of the classical Aristotelian one introduced by Peterson in (1979) in [31]. He, at first, considered the intermediate quantifiers "Almost-all" and "Many" with an assumption that he took "Most" in the sense of "Almost all". Thompson, in [37], followed up Peterson's work and proposed the intermediate quantifier "Most" as synonymous with "more than half". Peterson in his book started with an explanation of how to integrate new quantifiers in the Aristotle's square. Later, in [32], he showed how "A few" comes in. He explained relationships between "A few", "Many" and "Most" by introducing Aristotle's square replacing main formulas $\mathbf{A}, \mathbf{E}, \mathbf{I}, \mathbf{O}$ by quantifiers "A few" and "Most", "Most" and "Many", "Few" and "Many", respectively.

We conclude this section with a note that in a similar way to the Aristotelian hexagon construction by adding new formulas **U** and **Y**, we can construct the Peterson's hexagon of opposition.

3 Graded structures of opposition

The second part of this paper is devoted to graded structures of opposition with generalized intermediate quantifiers (graded Peterson's square cube and hexagon of opposition). We will start with special expressions of a natural language which are called intermediate quantifiers because they lie between classical

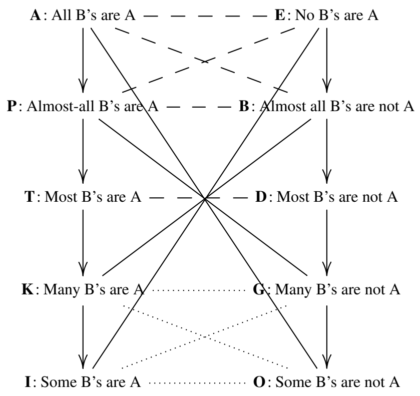


Figure 3: Peterson's square of opposition

ones (universal and existential). Intermediate quantifiers are expressions of a natural language, for example, *A few, Most, Several, Almost all*, etc. We can find them in several areas from common life :

- *Most people during the COVID pandemic work at home*.
- *Many people after undergoing COVID have worsened chronic diseases*.
- *Almost all shares grow with growing economy.*

3.1 Fuzzy type theory

In this section, we will remind the main concepts and properties of the fuzzy type theory (higher-order fuzzy logic) and the theory of evaluative linguistic expressions. We will not go into the detail. The reader can find details in several papers [22, 24, 25].

Recall at this point, that the formal theory of intermediate quantifiers is developed within Łukasiewicz fuzzy type theory (Ł-FTT). The algebra of truth values is a linearly ordered $\mathrm{MV}_{\boldsymbol{\Delta}}$-algebra extended with the delta operation (see [10, 29]). A special case is the standard Łukasiewicz MV_{Δ}-algebra.

$$\mathscr{L} = \langle [0,1], \vee, \wedge, \otimes, \rightarrow, 0, 1, \Delta \rangle. \quad (9)$$

The basic syntactical objects of Ł-FTT are classical, namely the concepts of type and formula. The atomic types are ε (elements) and o (truth values). General types are denoted by Greek letters $\alpha, \beta, \ldots$. We will omit the type whenever it is clear from the context. A set of all types is denoted by *Types*.

The *language* consists of variables $x_\alpha, \ldots$, special constants $c_\alpha, \ldots$ ($\alpha \in$ *Types*), symbol λ, and parentheses. The connectives (which are special constants) are *fuzzy equality/equivalence* $\equiv$, *conjunction* $\boldsymbol{\wedge}$, *implication* $\Rightarrow$, *negation* $\neg$, *strong conjunction* $\&$, *strong disjunction* ∇, *disjunction* $\vee$, and *delta* $\boldsymbol{\Delta}$. The fuzzy type theory is *complete*, i.e., a theory T is consistent iff it has a (Henkin) model ($\mathscr{M} \models T$). We sometimes apply its equivalent version: $T \vdash A_o$ iff $T \models A_o$.

3.2 Evaluative linguistic expressions

An important role in our model of intermediate quantifiers is played by *evaluative linguistic expressions*.

Evaluation language expressions play a very important role, because we use it to define generalized intermediate quantifiers. We are speaking about expressions of a natural language, such as *small, medium, big, very short, more or less deep, quite roughly strong, extremely high*, etc. Their theory is the basic constituent of the fuzzy natural logic.

The semantics of evaluative linguistic expressions is formulated in a special formal theory T^{Ev} of Ł-FTT which was introduced in [25] and less formally explained in [28] where also formulas for direct computation are provided.

Another motivation, fundamental assumptions, and formalization of our theory are in detail described in [25].

3.3 Definition of Intermediate Quantifiers

The theory of intermediate quantifiers is a special formal theory T^{IQ} of Ł-FTT extending T^{Ev}. A detailed structure of T^{IQ} and precise definitions can be found in [22, 26].

In [26], the first version of the definition of generalized quantifier was introduced. Later, in [23], we explained the motivation why it was necessary to modify this definition. Further comparison of the two definitions and a detailed explanation of the examples were given in [21].

Below we introduce modified definitions of positive intermediate quantifiers as follows:

Definition 3.1 *Let Ev be a formula representing an evaluative expression, x be variables and A, B, z be formulas. Then either of the formulas represents the quantifier "⟨Quantifier⟩ B's are A".*

$$(Q^{\forall}_{Ev} x)(B,A) \equiv (\exists z)[(\forall x)((B|z)x \Rightarrow Ax) \wedge Ev((\mu B)(B|z))], \quad (10)$$

$$(Q^{\exists}_{Ev} x)(B,A) \equiv (\exists z)[(\exists x)((B|z)x \wedge Ax) \wedge Ev((\mu B)(B|z))]. \quad (11)$$

By putting of the concrete evaluative expressions we obtain the list of generalized intermediate quantifiers as follows:

Definition 3.2 [23]

$$\begin{aligned}
\textbf{\textit{A:}}\ \textit{All B are A} &:= (\forall x)(Bx \Rightarrow Ax),\\
\textbf{\textit{E:}}\ \textit{No B are A} &:= (\forall x)(Bx \Rightarrow \neg Ax),\\
\textbf{\textit{P:}}\ \textit{Almost all B are A} &:= (Q^{\forall}_{BiEx}x)(B,A),\\
\textbf{\textit{B:}}\ \textit{Almost all B are not A} &:= (Q^{\forall}_{BiEx}x)(B,\neg A),\\
\textbf{\textit{T:}}\ \textit{Most B are A} &:= (Q^{\forall}_{BiVe}x)(B,A),\\
\textbf{\textit{D:}}\ \textit{Most B are not A} &:= (Q^{\forall}_{BiVe}x)(B,\neg A),\\
\textbf{\textit{K:}}\ \textit{Many B are A} &:= (Q^{\forall}_{\neg Sm\ \bar{\nu}}x)(B,A),\\
\textbf{\textit{G:}}\ \textit{Many B are not A} &:= (Q^{\forall}_{\neg Sm\ \bar{\nu}}x)(B,\neg A),\\
\textbf{\textit{F:}}\ \textit{A few B are A} &:= (Q^{\forall}_{SmSi}x)(B,A),\\
\textbf{\textit{V:}}\ \textit{A few B are not A} &:= (Q^{\forall}_{SmSi}x)(B,\neg A),\\
\textbf{\textit{S:}}\ \textit{Several B are A} &:= (Q^{\forall}_{SmVe}x)(B,A),\\
\textbf{\textit{Z:}}\ \textit{Several B are not A} &:= (Q^{\forall}_{SmVe}x)(B,\neg A),\\
\textbf{\textit{I:}}\ \textit{Some B are A} &:= (\exists x)(Bx \wedge Ax),\\
\textbf{\textit{O:}}\ \textit{Some B are not A} &:= (\exists x)(Bx \wedge \neg Ax).
\end{aligned}$$

By ****A, *E, *P, *B, *T, *D, *K, *G, *F, *V, *S, *Z, *I, *O*** *we denote intermediate quantifiers with presupposition. For the precise definition see [23].*

Proposition 3.3 ([23]) *Let $\mathcal{M}$ be a model such that $B = \mathcal{M}(y) \subsetneq M_\alpha$, $Z = \mathcal{M}(z) \subsetneq M_\alpha$. Then for any $m \in M_\alpha$*

$$\mathcal{M}(y|z)(m) = (B|Z)(m) = \begin{cases} B(m), & \textit{if } B(m) = Z(m),\\ 0 & \textit{otherwise}. \end{cases}$$

One can see that the operation $B|Z$ takes only those elements $m \in M_\alpha$ from the fuzzy set B whose membership $B(m)$ is equal to $Z(m)$, otherwise $(B|Z)(m) = 0$. If there is no such an element, then $B|Z = \emptyset$. We can thus use various fuzzy sets Z to "pick proper elements" from B.

3.4 Graded Peterson's square

Needless to say, the graded Peterson square of opposites is constructed as an extension of the graded Aristotle's square. Let us recall the work of other authors who in their publications studied the graded Aristotle's square from the point of view of other mathematical theories. A generalization of the square of opposition to many-valued logic was introduced by Dubois and Prade in [12]. The authors proposed a graded Aristotle's square of opposition and a cube of opposition including its graded version that associates the traditional square of opposition with a dual square of opposition. Another graded extensions of Aristotle's square, named polygons of opposition, are constructed by using quantifier-based operators in fuzzy formal concept analysis (see [8]).

In Subsection 2.4, we explained how Peterson approached the construction of a square of opposition with intermediate quantifiers. The main idea is based on the position of intermediate quantifiers, all of which lie among the classical ones. This idea has been followed up in the approach of introducing mathematical definitions using evaluative language expressions that behave monotonically. The next step was to formally define the properties that form the classical Peterson square in fuzzy logic. The complete graded Peterson's square with generalized intermediate quantifiers was formally proved in [22]. Below we recall main *generalized* definitions (contrary, contradictory, sub-contrary and sub-alterns), which generalized the classical ones.

Definition 3.4 *Let T be a consistent theory of Ł-FTT, $\mathcal{M} \models T$ be a model, and P_1, P_2 be closed formulas.*

- *P_1 and P_2 are* contraries *if $\mathcal{M}(P_1) \otimes \mathcal{M}(P_2) = 0$.*
- *P_1 and P_2 are* weak-contraries *if $0 < \mathcal{M}(P_1) \otimes \mathcal{M}(P_2) < 1$.*
- *P_1 and P_2 are* subcontraries *if $\mathcal{M}(P_1) \oplus \mathcal{M}(P_2) = 1$.*
- *P_1 and P_2 are* contradictories *if both $\mathcal{M}(\mathbf{\Delta} P_1) \otimes \mathcal{M}(\mathbf{\Delta} P_2) = 0$ and $\mathcal{M}(\mathbf{\Delta} P_1) \oplus \mathcal{M}(\mathbf{\Delta} P_2) = 1$.*
- *P_2 is a* subaltern *of P_1 if $\mathcal{M}(P_1) \leq \mathcal{M}(P_2)$.*

In Figure 4, we introduce graded Peterson's square **AEPBTDKGFVSZIO** which will be from now called graded **7**-square of opposition. This square was constructed as an extension of the basic graded Peterson's square [22] by adding new quantifiers "A few" and "Several", which were designed and studied in detail in [27].

Recall that all the above properties between the individual intermediate quantifiers have been syntactically proven in our previous papers (see [22, 23]) and therefore they are fulfilled in every model of T^{IQ}. The quantifier Many, which behaves ambiguously, has a special position in our theory. In [23], we have shown that there are two models of how to interpret the quantifier Many.

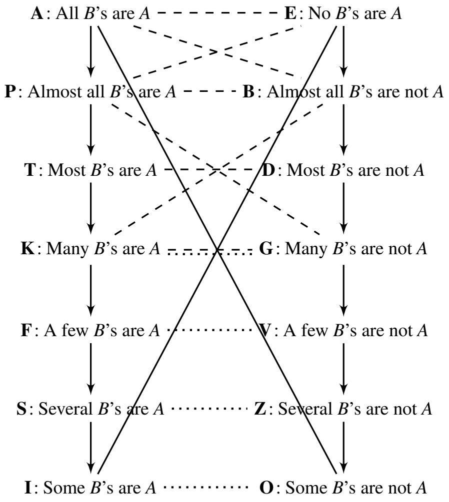


Figure 4: **7**-graded Peterson's square of opposition

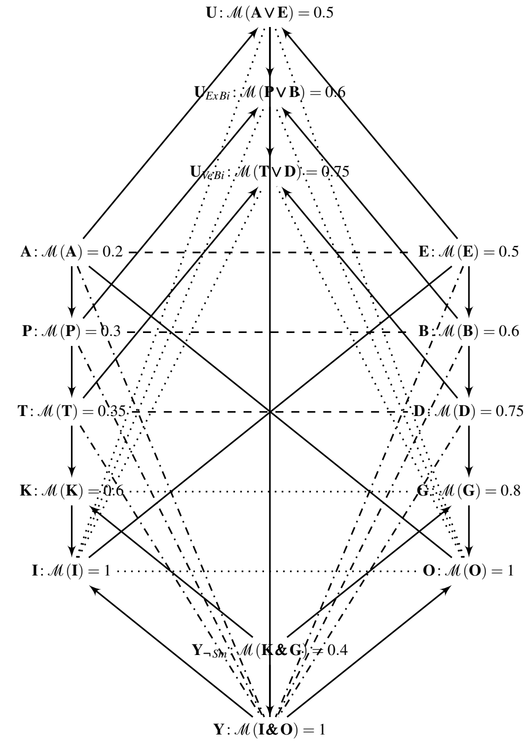


Figure 5: Graded Peterson's hexagon of opposition

3.5 Graded Peterson's hexagon

We start with definitions of new generalized intermediate quantifiers as follows:

- $\mathbf{U}_{ExBi} :=$ Almost all B are A or Almost all B are not A
- $\mathbf{U}_{VeBi} :=$ Most B are A or Most B are not A
- $\mathbf{Y}_{\neg Sm} :=$ Many B are AandMany B are not A.

3.6 Example of graded logical hexagon with intermediate quantifiers

Let us consider a model $\mathcal{M} \models T[B]$ such that $T^{\mathrm{IQ}} \vdash (\exists x)Bx$ and let $\mathcal{M}(\mathbf{A}) = a > 0$ (e.g., $a = 0.2$). The degrees inside in generalized Peterson's square follow from the definitions of contrary, weak-contrary, contradictories, sub-contraries and subalterns. The formula **I** is sub-contrary with $\mathbf{U}, \mathbf{U}_{ExBi}, \mathbf{U}_{VeBi}$ as well as **O** is sub-contrary with $\mathbf{U}, \mathbf{U}_{ExBi}, \mathbf{U}_{VeBi}$. The formula **U** is superaltern of all of them and the formula **Y** is sub-altern of all of them.

By an extension of Peterson's square adding quantifiers "A few" and "Several" we can define new quantifiers that can represent new lower and upper forms of quantifiers.

3.7 Graded Peterson's cube

This subsection is devoted to graded Peterson's cube of opposition which is constructed as an extension of graded Aristotle's cube of opposition. Another cube of opposition, a structure that generalizes the square of opposition invented in ancient logic which can be generated from the composition of a binary relation with a subset, by the effect of set complementation on the subset, on the relation, or on the result of the composition was introduced in [14].

At this point, let's just remind that the construction of a graded Aristotle's cube of opposites from a classical Aristotle's square is based on the same idea as in the classical case replacing B and A into their negation, $\neg B$ and $\neg A$. We can introduce mathematical definitions of generalized quantifiers[1] which form the second graded Peterson's square.

We continue with an extension of graded **7**-square of opposition **AEPBTDKGFVSZIO**, which was introduced as a generalization of Peterson's square, to the graded Peterson's cube of opposition **aepbtdkgfvszio** with intermediate quantifiers. Let us recall that **aepbtdkgfvszio** form the second square of opposition and so all mathematical properties can be proved analogously.

Below we present examples of natural language linguistic expressions which represent new forms of intermediate quantifiers forming a graded cube of opposition as follows:

[1] $(Q^{\forall}_{Ev}x)(\neg B, \neg A) \equiv (\exists z)[(\forall x)((\neg B|z)x \Rightarrow \neg Ax) \wedge Ev((\mu(\neg B))(\neg B|z))]$

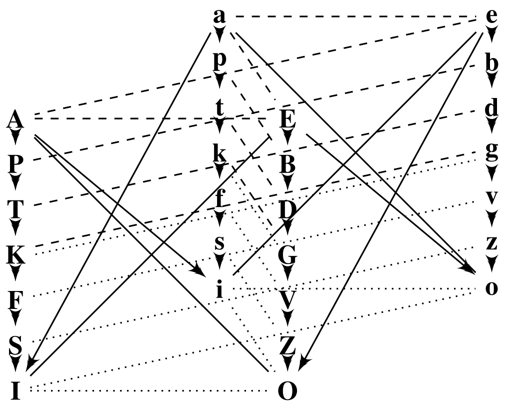


Figure 6: **7**-graded Peterson's cube of opposition

Let us consider a model $\mathscr{M} \models T[B]$. Let us, for example, assume that

- **d**:*Most people who do not smoke have higher lung capacity.*

is true in the degree 0.8. The from the property of contrary with **T** we observe that

- **T**:*Most people who smoke have higher lung capacity*

may be true at most in the degree 0.2. Because, **T** is superaltern of **S**, we can conclude, that

- **S**:*Several people who smoke have higher lung capacity*

could be true, for example, in the degree 0.4 as well as

- **Z**:*Several people who smoke do not have higher lung capacity*

may be true in the degree 0.9 because **S** and **Z** are subcontrary.

In Figure 4, we introduce a **7**-graded cube of opposition with seven generalized intermediate quantifiers as a generalization of the graded Peterson's square. All the properties were syntactically proven in [21].

4 Conclusion

This article was devoted to an overview of the structures of opposition. In the beginning we reminded the readers the classical structures of opposition (Aristotle's square, Blanché hexagon and). Furthermore, by introducing additional squares we presented J-K and Moretti's cubes. In the second part, we generalized all the mentioned structures and introduced graded structures of opposition with intermediate quantifiers. The idea for future is to compare graded Aristotle's square proposed by Dubois et. all with our approach. The main difference between the two approaches is in the different definitions of the quantifier Some. Another idea is to continue studying generalized logical syllogisms which relate to proposed graded structures of opposition.

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References

[1] D. Abusch, R. M., Empty-domain efeects for presuppositional and nonpresuppositional determiners, in: H. Kamp, B. Partee (eds.),Context Dependency in the Analysis of Linguistic Meaning, Elsevier, Amsterdam (2004) 7.

[2] M. Afshar, C. Dartnell, D. Luzeaux, J. Sallantin, Y. Tognetti, Aristotle's square revisited to frame discovery science, Journal of Computers 2 (2007) 54–66.

[3] D. Beaver, Presupposition, in: J. van Benthem, A. ter Meulen (Eds.), The Handbook of Logic and Language, Elsevier, Amsterdam, 2004, p. 939–1008.

[4] J. Béziau, The power of the hexagon, Logica Universalis 6(1-2) (2012) 1–43.

[5] J. Béziau, There is no cube of opposition, In: Beziau, J.-Y., Basti, G. (eds.) The Square of Oppositions: A Cornerstone of Thought, Studies in Universal Logic (2017) 179–193.

[6] R. Blanché, Structures intellectuelles essai sur l'organisation des concepts.

[7] I. Bocheński, Ancient Formal Logic, North-Holland Publishing Company, Amsterdam, 1951.

[8] S. Boffa, P. Murinová, V. Novák, Graded polygons of opposition in fuzzy formal concept analysis, International Journal of Approximate Reasoning 132 (2021) 128–153.

[9] M. Brown, Generalized quantifiers and the square of opposition, Notre Dame Journal of Formal Logic 25 (1984) 303–322.

[10] R. L. O. Cignoli, I. M. L. D'Ottaviano, D. Mundici, Algebraic Foundations of Many-valued Reasoning, Kluwer, Dordrecht, 2000.

[11] T. Czezowski, On certain peculiarities of singular propositions, Mind 64 (255) (1955) 392–395.

[12] D. Dubois, H. Prade, A. Rico, Graded cubes of opposition and possibility theory with fuzzy events, International Journal of Approximate Reasoning 84 (2017) 168–185.

[13] P. H. Dubois, D., From blanch?e?s hexagonal organization of concepts to formal con- cept analysis and possibility theory, Logica Universalis 6 (2012) 149–?169.

[14] P. H. Dubois, D., A. Rico, Structures of opposition and comparisons: Boolean and gradual cases, Logica Universalis 14 (2020) 115–149.

[15] G. Frege, Begriffsschrift (Concept Script), eine der arithmetischen nachgebildete Formelsprache des reinen Denkens, Halle a. S., 1879.

[16] R. C. J. Barwise, Generalized quantifiers and natural language, Linguistics and Philosophy 4 (1981) 159–219.

[17] J. N. Keynes, Studies and Exercises in Formal Logic, Part II, 3rd edn, p.144, MacMilan, New York, 1894.

[18] L. Miclet, H. Prade, Analogical proportions and square of oppositions, In: A. Laurent et al. (ed.) Proc. 15th Int. Conf. on Information Processing and Management of Uncer- tainty in Knowledge-Based Systems, July 15-19, Montpellier, CCIS 443 (2014) 324–334.

[19] A. Moretti, The geometry of opposition, Ph.D. thesis, University of Neuchâtel, Neuchâtel (2009).

[20] A. Mostowski, On a generalization of quantifiers, Fundamenta Mathematicae 44 (1957) 12–36.

[21] P. Murinová, Graded structures of opposition in fuzzy natural logic, Logica Universalis 265 (2020) 495–522.

[22] P. Murinová, V. Novák, Analysis of generalized square of opposition with intermediate quantifiers, Fuzzy Sets and Systems 242 (2014) 89–113.

[23] P. Murinová, V. Novák, The theory of intermediate quantifiers in fuzzy natural logic revisited and the model of "many", Fuzzy sets and systems 388 (2020) 56–89.

[24] V. Novák, On fuzzy type theory, Fuzzy Sets and Systems 149 (2005) 235–273.

[25] V. Novák, A comprehensive theory of trichotomous evaluative linguistic expressions, Fuzzy Sets and Systems 159 (22) (2008) 2939–2969.

[26] V. Novák, A formal theory of intermediate quantifiers, Fuzzy Sets and Systems 159 (10) (2008) 1229–1246.

[27] V. Novák, P. Murinová, A formal model of the intermediate quantifiers "a few", "several" and "a little", in: Proc. IFSA-NAFIPS 2019, Lafayette, USA, 2019.

[28] V. Novák, I. Perfilieva, A. Dvořák, Insight into Fuzzy Modeling, Wiley & Sons, Hoboken, New Jersey, 2016.

[29] V. Novák, I. Perfilieva, J. Močkoř, Mathematical Principles of Fuzzy Logic, Kluwer, Boston, 1999.

[30] R. Pellissier, "setting" n-opposition, Logica Universalis 2 (2008) 235–263.

[31] P. Peterson, On the logic of "few","many" and "most", Notre Dame Journal of Formal Logic 20 (1979) 155–179.

[32] P. Peterson, Intermediate Quantifiers. Logic, linguistics, and Aristotelian semantics, Ashgate, Aldershot, 2000.

[33] W. D. Ross, Aristotle's Prior and Posterior Analytics, Clarendom Press, Oxford, 1949.

[34] A. Sesmat, Logique II: Les raisonnements, la logistique, Hermann, 1951.

[35] W. Shirwood, N. Kretzmann, William of Sherwood's Introduction to logic, University of Minnesota Press, 1966.

[36] H. Smesaert, The classical aristotelian hexagon versus the modern duality hexagon, Logica Universalis 6 (2012) 171–199.

[37] B. E. Thompson, Syllogisms using "few","many" and "most", Notre Dame Journal of Formal Logic 23 (1982) 75–84.

[38] L. A. Zadeh, A computational approach to fuzzy quantifiers in natural languages, Computers and Mathematics 9 (1983) 149–184.