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Conditional Interval Valued Probability and Martingale Convergence Theorem

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Abstract

The aim of this contribution is to define a conditional probability for interval valued events. We show the connection between the conditional probability for interval valued events and the conditional probability for intuitionistic fuzzy events too. We formulate the properties of conditional probability for interval valued events. We prove a modification of martingale convergence theorem for conditional probability defined on a family of interval valued events too.

Keywords: Interval valued event, Interval valued state, Interval valued observable, Product, Conditional interval valued probability, Martingale convergence theorem.

1 Introductions

A notion of a conditional probability was studied on many multivalued structures, for example on MV-algebras (see [9, 18]), on σ-MV-algebras (see [7]) and on intuitionistic fuzzy sets (see [10, 16, 17]). Because there is a relation between the intuitionistic fuzzy events introduced by K. T. Atanassov in [1, 2, 3] and the interval valued events introduced by L. A. Zadeh in [19], it is interesting to study a conditional probability on interval valued events in relation to intuitionistic fuzzy events. In [12, 14, 13] the authors studied the martingale measures in connection with fuzzy approach in the financial area. They used the geometric Levy process, the Esscher transformed martingale measures and the minimal L^p equivalent martingale measure on the fuzzy numbers for an option pricing. A practical use of results is a good motivation for studying a theory of martingales. In this paper we try to define the notion of a conditional probability for interval valued events and we prove its properties. We show the connection to a conditional intuitionistic fuzzy probability too. We formulate a version of a martingale convergence theorem for interval valued events.

Note that in the whole text we use the notation "IF" for the short phrase "intuitionistic fuzzy" and the notation "IV" for the short phrase "interval valued".

First, we recall the basic notions (see [8, 11]).

Definition 1.1 *Let Ω be a nonempty set. An interval valued set $\mathbf{C}$ on Ω is a pair (π_C, ρ_C) of mappings $\pi_C, \rho_C : \Omega \to [0,1]$ such that $\pi_C \leq \rho_C$.*

Definition 1.2 *Start with a measurable space $(\Omega, \mathscr{S})$. Hence $\mathscr{S}$ is a σ-algebra of subsets of Ω. An interval valued set $\mathbf{C} = (\pi_C, \rho_C)$ is an IV-event, if $\pi_C, \rho_C : \Omega \to [0,1]$ are $\mathscr{S}$-measurable. The family of all IV-events on $(\Omega, \mathscr{S})$ will be denoted by $\mathscr{K}$.*

If $\mathbf{C} = (\pi_C, \rho_C) \in \mathscr{K}$, $\mathbf{D} = (\pi_D, \rho_D) \in \mathscr{K}$, then we define the Lukasiewicz binary operations $\widehat{\oplus}, \widehat{\odot}$ on $\mathscr{K}$ by

$$\mathbf{C}\widehat{\oplus}\mathbf{D} = \big((\pi_C + \pi_D) \wedge 1_\Omega, (\rho_C + \rho_D) \wedge 1_\Omega\big)$$

$$\mathbf{C}\widehat{\odot}\mathbf{D} = \big((\pi_C + \pi_D - 1_\Omega) \vee 0_\Omega, (\rho_C + \rho_D - 1_\Omega) \vee 0_\Omega\big)$$

and the partial ordering is given by

$$\mathbf{C} \preceq \mathbf{D} \Leftrightarrow \pi_C \leq \pi_D, \rho_C \leq \rho_D.$$

Moreover, it holds

$$\begin{aligned} \mathbf{C} \nearrow \mathbf{D} \quad &\Leftrightarrow \quad \pi_C \nearrow \pi_D, \rho_C \nearrow \rho_D, \\ \mathbf{C} \searrow \mathbf{D} \quad &\Leftrightarrow \quad \pi_C \searrow \pi_D, \rho_C \searrow \rho_D. \end{aligned}$$

In papers [8, 11] the authors studied the connection between the family of intuitionistic fuzzy events

$$\mathscr{F} = \{(\mu_A, \nu_A) \quad ; \quad \mu_A + \nu_A \leq 1_\Omega,\ \mu_A, \nu_A : \Omega \to [0,1] \text{ are } \mathscr{S}-\text{measurable functions}\}$$

with the operations, relation and continuity

$$\begin{aligned} \mathbf{A} \le \mathbf{B} &\Leftrightarrow \mu_A \le \mu_B, \nu_A \ge \nu_B, \\ \mathbf{A} \oplus \mathbf{B} &= \big((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega\big), \\ \mathbf{A} \odot \mathbf{B} &= ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega)), \\ \mathbf{A} \nearrow \mathbf{B} &\Leftrightarrow \mu_A \nearrow \mu_B, \nu_A \searrow \nu_B \end{aligned}$$

and the family $\mathscr{K}$ of interval valued events from *Definition 1.2*. They showed that these two systems are isomorphic by the mapping $\psi : \mathscr{F} \to \mathscr{K}$ given by

$$\psi\big((\mu_A, \nu_A)\big) = (\mu_A, 1_\Omega - \nu_A)$$

for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathscr{F}$. Therefore, the following relations hold

$$\psi(\mathbf{A} \oplus \mathbf{B}) = \psi(\mathbf{A}) \widehat{\oplus} \psi(\mathbf{B}), \quad (1)$$

$$\psi(\mathbf{A} \odot \mathbf{B}) = \psi(\mathbf{A}) \widehat{\odot} \psi(\mathbf{B}), \quad (2)$$

$$\mathbf{A} \le \mathbf{B} \Leftrightarrow \psi(\mathbf{A}) \preceq \psi(\mathbf{B}), \quad (3)$$

$$\mathbf{A}_n \nearrow \mathbf{A} \Leftrightarrow \psi(\mathbf{A}_n) \nearrow \psi(\mathbf{A}), \quad (4)$$

for each $\mathbf{A}_n, \mathbf{A}, \mathbf{B} \in \mathscr{F}$. They illustrated the connection between the intuitionistic fuzzy state $\mathbf{m} : \mathscr{F} \to [0,1]$ and the interval valued state $k : \mathscr{K} \to [0,1]$ and that was $\mathbf{m} = k \circ \psi$.

Definition 1.3 *Let $\mathscr{K}$ be the family of all IV-events in Ω. A mapping $k : \mathscr{K} \to [0,1]$ is called an interval valued state, if the following conditions are satisfied:*

(i) $k((1_\Omega, 1_\Omega)) = 1$, $k((0_\Omega, 0_\Omega)) = 0$*;*

(ii) if $\mathbf{C} \widehat{\odot} \mathbf{D} = (0_\Omega, 0_\Omega)$ *and* $\mathbf{C}, \mathbf{D} \in \mathscr{K}$*, then* $k(\mathbf{C} \widehat{\oplus} \mathbf{D}) = k(\mathbf{C}) + k(\mathbf{D})$*;*

(iii) if $\mathbf{C}_n \nearrow \mathbf{C}$ *(i.e.* $\pi_{C_n} \nearrow \pi_C, \rho_{C_n} \nearrow \rho_C$*), then* $k(\mathbf{C}_n) \nearrow k(\mathbf{C})$.

Recall that by an **intuitionistic fuzzy state m** we understand each mapping $\mathbf{m} : \mathscr{F} \to [0,1]$ which satisfies the following conditions (see [15]):

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$;

(ii) if $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ and $\mathbf{A}, \mathbf{B} \in \mathscr{F}$, then $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$;

(iii) if $\mathbf{A}_n \nearrow \mathbf{A}$ (i.e. $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$), then $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$.

Proposition 1.1 *If* $k : \mathscr{K} \to [0,1]$ *is an IV-state and* $\mathbf{m} = k \circ \psi : \mathscr{F} \to [0,1]$*, then* $\mathbf{m}$ *is an IF-state.*

Further in paper [4] we defined the notion of interval valued observable $z : \mathscr{B}(R) \to \mathscr{K}$ and we displayed the connection to the intuitionistic fuzzy observable $x : \mathscr{B}(R) \to \mathscr{F}$, which was $z = \psi \circ x$.

Let $\mathscr{J}$ be the family of all intervals in R of the form

$$[a,b) = \{x \in R : a \le x < b\}.$$

Then the σ-algebra $\sigma(\mathscr{J})$ is denoted $\mathscr{B}(R)$ and it is called the σ-algebra of Borel sets, its elements are called Borel sets.

Definition 1.4 *By an interval valued observable on $\mathscr{K}$ we understand each mapping $z : \mathscr{B}(R) \to \mathscr{K}$ satisfying the following conditions:*

(i) $z(R) = (1_\Omega, 1_\Omega)$, $z(\emptyset) = (0_\Omega, 0_\Omega)$*;*

(ii) if $A \cap B = \emptyset$*, then* $z(A) \widehat{\odot} z(B) = (0_\Omega, 0_\Omega)$ *and* $z(A \cup B) = z(A) \widehat{\oplus} z(B)$*;*

(iii) if $A_n \nearrow A$*, then* $z(A_n) \nearrow z(A)$.

Remark 1.5 *If we denote* $z(A) = \big(z^\flat(A), z^\sharp(A)\big)$ *for each* $A \in \mathscr{B}(R)$*, then* $z^\flat, z^\sharp : \mathscr{B}(R) \to \mathscr{T}$ *are observables, where* $\mathscr{T} = \{f : \Omega \to [0,1]; f \text{ is } \mathscr{S} - \text{measurable}\}$.

Remark 1.6 *Sometimes we need to work with n-dimensional IV-observable* $z : \mathscr{B}(R^n) \to \mathscr{K}$ *defined as a mapping with the following conditions:*

(i) $z(R^n) = (1_\Omega, 1_\Omega)$, $z(\emptyset) = (0_\Omega, 0_\Omega)$*;*

(ii) if $A \cap B = \emptyset$, $A, B \in \mathscr{B}(R^n)$*, then* $z(A) \widehat{\odot} z(B) = (0_\Omega, 0_\Omega)$ *and* $z(A \cup B) = z(A) \widehat{\oplus} z(B)$*;*

(iii) if $A_n \nearrow A$*, then* $z(A_n) \nearrow z(A)$ *for each* $A, A_n \in \mathscr{B}(R^n)$.

If $n = 1$ *we simply say that* z *is an IV-observable.*

Between IV-observable and IF-observable is the connection (see [4]).

Recall that by **intuitionistic fuzzy observable** on $\mathscr{F}$ we understand each mapping $x : \mathscr{B}(R) \to \mathscr{F}$ satisfying the following conditions (see [15]):

(i) $x(R) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) if $A \cap B = \emptyset$, then $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ and $x(A \cup B) = x(A) \oplus x(B)$;

(iii) if $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

Proposition 1.2 *Let* $\psi : \mathscr{F} \to \mathscr{K}$, $\psi((u,v)) = (u, 1_\Omega - v)$*. If* $x : \mathscr{B}(R) \to \mathscr{F}$ *is an IF-observable and* $z = \psi \circ x : \mathscr{B}(R) \to \mathscr{K}$*, then* z *is an IV-observable.*

If we denote $x(A) = \big(x^\flat(A), 1 - x^\sharp(A)\big)$ for each $A \in \mathscr{B}(R)$, then $x^\flat, x^\sharp : \mathscr{B}(R) \to \mathscr{T}$ are observables, where $\mathscr{T} = \{f : \Omega \to [0,1]; f \text{ is } \mathscr{S} - \text{measurable}\}$.

Theorem 1.7 *Let $z : \mathscr{B}(R) \to \mathscr{K}$ be an IV-observable, $k : \mathscr{K} \to [0,1]$ be an IV-state. Define the mapping $k_z : \mathscr{B}(R) \to [0,1]$ by the formula*

$$k_z(C) = k\big(z(C)\big),$$

for each $C \in \mathscr{B}(R)$. Then $k_z : \mathscr{B}(R) \to [0,1]$ is a probability measure. Moreover

$$k_z(C) = \mathbf{m}_x(C),$$

where $\mathbf{m}_x = \mathbf{m} \circ x$ is a probability measure induced by IF-state $\mathbf{m}$ and IF-observable x.

2 Conditional interval valued probability and martingale convergence theorem

In this section we formulate a conditional probability and a martingale convergence theorem for interval valued events. We are inspired by a notion of **conditional intuitionistic fuzzy probability** defined by B. Riečan in [16] as a Borel measurable function f (i.e. $B \in \mathscr{B}(R) \Longrightarrow f^{-1}(B) \in \mathscr{B}(R)$) such that

$$\int_B \mathbf{p}(\mathbf{A} \mid x)\, d\mathbf{m}_x = \mathbf{m}\big(\mathbf{A} \cdot x(B)\big)$$

for each $B \in \mathscr{B}(R)$, where $\mathbf{m} : \mathscr{F} \to [0,1]$ is the intuitionistic fuzzy state, $\mathbf{A} \in \mathscr{F}$ is an intuitionistic fuzzy event and $x : \mathscr{B}(R) \to \mathscr{F}$ is an intuitionistic fuzzy observable. Recall that a conditional intuitionistic fuzzy probability $\mathbf{p}(\mathbf{A} \mid x)$ has the following properties (see [5]):

Theorem 2.1 *Let $\mathscr{F}$ be a family of IF-events, $\mathbf{A} \in \mathscr{F}$, and $y : \mathscr{B}(R) \to \mathscr{F}$ be an IF-observable. Then $\mathbf{p}\big(\mathbf{A} \mid x\big)$ has the following properties:*

(i) $\mathbf{p}\big((0_\Omega, 1_\Omega) \mid x\big) = 0$, $\mathbf{p}\big((1_\Omega, 0_\Omega)|y\big) = 1$ *hold $\mathbf{m}_x$-almost everywhere;*

(ii) $0 \le \mathbf{p}\big(\mathbf{A} \mid x\big) \le 1$ *holds $\mathbf{m}_x$-almost everywhere;*

(iii) if $\bigodot_{i=1}^{\infty} \mathbf{A}_i = (0_\Omega, 1_\Omega)$, *then* $\mathbf{p}\Big(\bigoplus_{i=1}^{\infty} \mathbf{A}_i \mid x\Big) = \sum_{i=1}^{\infty} \mathbf{p}(\mathbf{A}_i \mid x)$ *holds $\mathbf{m}_x$-almost everywhere;*

(iv) if $\mathbf{A}_n \nearrow \mathbf{A}$, then the convergence $\mathbf{p}\big(\mathbf{A}_n \mid x\big) \nearrow \mathbf{p}\big(\mathbf{A} \mid x\big)$ holds $\mathbf{m}_x$-almost everywhere.

In paper [6] we defined the product operation for interval valued events.

Definition 2.2 *We say that a binary operation $\widehat{\cdot}$ on $\mathscr{K}$ is a product if it satisfies the following conditions:*

(i) $(1_\Omega, 1_\Omega)\widehat{\cdot}(\pi_C, \rho_C) = (\pi_C, \rho_C)$ *for each* $(\pi_C, \rho_C) \in \mathscr{K}$*;*

(ii) the operation $\widehat{\cdot}$ is commutative and associative;

(iii) if $(\pi_C, \rho_C)\widehat{\odot}(\pi_D, \rho_D) = (0_\Omega, 0_\Omega)$ *and* $(\pi_C, \rho_C), (\pi_D, \rho_D) \in \mathscr{K}$*, then*

$$(\pi_E, \rho_E)\widehat{\cdot}\big((\pi_C, \rho_C)\widehat{\oplus}(\pi_D, \rho_D)\big) = \\ = \big((\pi_E, \rho_E)\widehat{\cdot}(\pi_C, \rho_C)\big)\widehat{\oplus}\big((\pi_E, \rho_E)\widehat{\cdot}(\pi_D, \rho_D)\big)$$

and

$$\big((\pi_E, \rho_E)\widehat{\cdot}(\pi_C, \rho_C)\big)\widehat{\odot}\big((\pi_E, \rho_E)\widehat{\cdot}(\pi_D, \rho_D)\big) = \\ = (0_\Omega, 0_\Omega)$$

for each $(\pi_E, \rho_E) \in \mathscr{K}$*;*

(iv) if $(\pi_{C_n}, \rho_{C_n}) \searrow (0_\Omega, 0_\Omega)$, $(\pi_{D_n}, \rho_{D_n}) \searrow (0_\Omega, 0_\Omega)$ *and* $(\pi_{C_n}, \rho_{C_n}), (\pi_{D_n}, \rho_{D_n}) \in \mathscr{K}$*, then*

$$(\pi_{C_n}, \rho_{C_n})\widehat{\cdot}(\pi_{D_n}, \rho_{D_n}) \searrow (0_\Omega, 0_\Omega).$$

Now we are explaining the connection between the product operations on the family of interval valued events $\mathscr{K}$ and the family of intuitionistic fuzzy events $\mathscr{F}$ (see [6]).

Theorem 2.3 *If the operation $\cdot$ is a product on family of intuitionistic events $\mathscr{F}$ defined by*

$$(\mu_A, \nu_A)\cdot(\mu_B, \nu_B) = (\mu_A \cdot \mu_B, \nu_A + \nu_B - \nu_A \cdot \nu_B) = \\ = \big(\mu_A \cdot \mu_B, 1_\Omega - (1_\Omega - \nu_A) \cdot (1_\Omega - \nu_B)\big)$$

for each $\mathbf{A} = (\mu_A, \nu_A), \mathbf{B} = (\mu_B, \nu_B) \in \mathscr{F}$ and $\widehat{\cdot}$ is a product operation on a family of interval valued events $\mathscr{K}$ defined by

$$(\pi_C, \rho_C)\widehat{\cdot}(\pi_D, \rho_D) = (\pi_C \cdot \pi_D, \rho_C \cdot \rho_D)$$

for each $\mathbf{C} = (\pi_C, \rho_C), \mathbf{D} = (\pi_D, \rho_D) \in \mathscr{K}$ and $\psi : \mathscr{F} \to \mathscr{K}$ is a function given by $\psi\big((u,v)\big) = (u, 1-v)$, then

$$\psi(\mathbf{A} \cdot \mathbf{B}) = \psi(\mathbf{A})\,\widehat{\cdot}\,\psi(\mathbf{B})$$

for each $\mathbf{A}, \mathbf{B} \in \mathscr{F}$.

Now we are able to define a conditional interval valued probability.

Definition 2.4 *Let $z : \mathscr{B}(R) \to \mathscr{K}$ be an IV-observable, $\mathbf{C} \in \mathscr{K}$, $k : \mathscr{K} \to [0,1]$ be an IV-state. Then the conditional IV-probability $\widehat{\mathbf{p}}(\mathbf{C} \mid z) = \widehat{f}$ is a Borel measurable function (i.e. $B \in \mathscr{B}(R) \Longrightarrow \widehat{f}^{-1}(B) \in \mathscr{B}(R)$) such that*

$$\int_B \widehat{\mathbf{p}}(\mathbf{C} \mid z)\, dk_z = k\big(\mathbf{C}\,\widehat{\cdot}\,z(B)\big)$$

for each $B \in \mathscr{B}(R)$. There $\widehat{\cdot}$ is a product operation on a family $\mathscr{K}$.

In the following *Proposition* we show the connection to a conditional intuitionistic fuzzy probability.

Proposition 2.1 *Let* $\psi : \mathscr{F} \to \mathscr{K}$, $\psi((u,v)) = (u, 1_\Omega - v)$, $C \in \mathscr{K}$, $z : \mathscr{B}(R) \to \mathscr{K}$ *be an IV-observable. The function* $\widehat{\mathbf{p}}(\mathbf{C} \mid z)$ *from Definition 2.4 exists and moreover*

$$\widehat{\mathbf{p}}(\mathbf{C} \mid z) = \mathbf{p}(\mathbf{A} \mid x),$$

where $\mathbf{p}(\mathbf{A} \mid x)$ *is a conditional intuitionistic fuzzy probability,* $\mathbf{A} \in \mathscr{F}$, $x : \mathscr{B}(R) \to \mathscr{F}$ *is an IF-observable and* $\mathbf{C} = \psi(\mathbf{A})$, $z = \psi \circ x$.

Proof. Let $\psi : \mathscr{F} \to \mathscr{K}$, $\psi((u,v)) = (u, 1_\Omega - v)$, $C \in \mathscr{K}$, $z : \mathscr{B}(R) \to \mathscr{K}$ be an IV-observable. Since the function $\widehat{\mathbf{p}}(\mathbf{C} \mid z)$ is a conditional IV-probability, then by *Definition 2.4* we have

$$\int_B \widehat{\mathbf{p}}(\mathbf{C} \mid z)\, dk_z = k\big(\mathbf{C} \,\widehat{\cdot}\, z(B)\big)$$

for each $B \in \mathscr{B}(R)$. There $\widehat{\cdot}$ is a product operation on a family $\mathscr{K}$.

Using connections $\mathbf{C} = \psi(\mathbf{A})$, $z = \psi \circ x$, $\psi(\mathbf{A} \cdot \mathbf{B}) = \psi(\mathbf{A}) \,\widehat{\cdot}\, \psi(\mathbf{B})$, $\mathbf{m} = k \circ \psi$ from *Proposition 1.2*, *Theorem 2.3*, *Proposition 1.1* we obtain

$$\begin{aligned} k\big(\mathbf{C} \,\widehat{\cdot}\, z(B)\big) &= k\Big(\psi(\mathbf{A}) \,\widehat{\cdot}\, \psi\big(x(B)\big)\Big) = \\ &= k\Big(\psi\big(\mathbf{A} \cdot x(B)\big)\Big) = \mathbf{m}\big(\mathbf{A} \cdot x(B)\big). \end{aligned}$$

As $k_z = \mathbf{m}_x$ by *Theorem 1.7*, then

$$\int_B \widehat{\mathbf{p}}(\mathbf{C} \mid z)\, d\mathbf{m}_x = \mathbf{m}\big(\mathbf{A} \cdot x(B)\big) \tag{5}$$

for each $B \in \mathscr{B}(R)$. Therefore $\widehat{\mathbf{p}}(\mathbf{C} \mid z)$ is a conditional IF-probability and its existence is guaranteed.

On the other hand, if the Borel measurable function $\mathbf{p}(\mathbf{A} \mid x)$ is a conditional intuitionistic fuzzy probability, then

$$\int_B \mathbf{p}(\mathbf{A} \mid x)\, d\mathbf{m}_x = \mathbf{m}\big(\mathbf{A} \cdot x(B)\big) \tag{6}$$

and hence using (5) and (6) we obtain

$$\widehat{\mathbf{p}}(\mathbf{C} \mid z) = \mathbf{p}(\mathbf{A} \mid x).$$

□

Now we are looking at the properties of conditional IV-probability.

Theorem 2.5 *Let* $\mathscr{K}$ *be family of IV-events,* $\mathbf{C} \in \mathscr{K}$, *and* $z : \mathscr{B}(R) \to \mathscr{K}$ *be an IV-observable. Then a conditional IV-probability* $\widehat{\mathbf{p}}\big(\mathbf{C} \mid z\big)$ *has the following properties:*

(i) $\widehat{\mathbf{p}}\big((0_\Omega, 0_\Omega) \mid z\big) = 0$, $\widehat{\mathbf{p}}\big((1_\Omega, 1_\Omega) \mid z\big) = 1$ *hold* k_z*-almost everywhere;*

(ii) $0 \le \widehat{\mathbf{p}}\big(\mathbf{C} \mid z\big) \le 1$ *holds* k_z*-almost everywhere;*

(iii) if $\widehat{\bigodot}_{i=1}^{\infty} \mathbf{C}_i = (0_\Omega, 0_\Omega)$, *then* $\widehat{\mathbf{p}}\Big(\widehat{\bigoplus}_{i=1}^{\infty} \mathbf{C}_i \mid z\Big) = \sum_{i=1}^{\infty} \widehat{\mathbf{p}}(\mathbf{C}_i \mid z)$ *holds* k_z*-almost everywhere;*

(iv) if $\mathbf{C}_n \nearrow \mathbf{C}$, *then the convergence* $\widehat{\mathbf{p}}\big(\mathbf{C}_n \mid z\big) \nearrow \widehat{\mathbf{p}}\big(\mathbf{C} \mid z\big)$ *holds* k_z*-almost everywhere.*

Proof. Let $\psi : \mathscr{F} \to \mathscr{K}$, $\psi((u,v)) = (u, 1_\Omega - v)$, $C \in \mathscr{K}$, $z : \mathscr{B}(R) \to \mathscr{K}$ be an IV-observable. By *Proposition 2.1* we have

$$\widehat{\mathbf{p}}(\mathbf{C} \mid z) = \mathbf{p}(\mathbf{A} \mid x),$$

where $\mathbf{p}(\mathbf{A} \mid x)$ is a conditional IF-probability, $\mathbf{A} \in \mathscr{F}$, $x : \mathscr{B}(R) \to \mathscr{F}$ is an IF-observable and $\mathbf{C} = \psi(\mathbf{A})$, $z = \psi \circ x$. Moreover by *Theorem 1.7* we have $k_z = \mathbf{m}_x$. Then using properties a conditional IF-probability $\mathbf{p}(\mathbf{A} \mid x)$ from *Theorem 2.1* we have

(i) $\psi\big((0_\Omega, 1_\Omega)\big) = (0_\Omega, 0_\Omega)$, $\psi\big((1_\Omega, 0_\Omega)\big) = (1_\Omega, 1_\Omega)$ and therefore

$$\begin{aligned} \widehat{\mathbf{p}}\big((0_\Omega, 0_\Omega) \mid z\big) = \mathbf{p}\big((0_\Omega, 1_\Omega) \mid x\big) = 0, \\ \widehat{\mathbf{p}}\big((1_\Omega, 1_\Omega) \mid z\big) = \mathbf{p}\big((1_\Omega, 0_\Omega) \mid x\big) = 1 \end{aligned}$$

hold k_z-almost everywhere;

(ii) $0 \le \mathbf{p}(\mathbf{A} \mid x) = \widehat{\mathbf{p}}\big(\mathbf{C} \mid z\big) = \mathbf{p}(\mathbf{A} \mid x) \le 1$ holds k_z-almost everywhere;

(iii) if $\widehat{\bigodot}_{i=1}^{\infty} \mathbf{C}_i = (0_\Omega, 0_\Omega)$, then $\psi\big((0_\Omega, 1_\Omega)\big) = (0_\Omega, 0_\Omega) = \widehat{\bigodot}_{i=1}^{\infty} \mathbf{C}_i = \widehat{\bigodot}_{i=1}^{\infty} \psi(\mathbf{A}_i) = \psi\Big(\bigodot_{i=1}^{\infty} \mathbf{A}_i\Big)$. Hence $\bigodot_{i=1}^{\infty} \mathbf{A}_i = (0_\Omega, 1_\Omega)$. Therefore

$$\begin{aligned} \widehat{\mathbf{p}}\Big(\widehat{\bigoplus}_{i=1}^{\infty} \mathbf{C}_i \mid z\Big) &= \widehat{\mathbf{p}}\Big(\widehat{\bigoplus}_{i=1}^{\infty} \psi(\mathbf{A}_i) \mid z\Big) = \\ &= \widehat{\mathbf{p}}\Big(\psi\Big(\bigoplus_{i=1}^{\infty} \mathbf{A}_i\Big) \mid z\Big) = \\ &= \mathbf{p}\Big(\bigoplus_{i=1}^{\infty} \mathbf{A}_i \mid x\Big) = \\ &= \sum_{i=1}^{\infty} \mathbf{p}(\mathbf{A}_i \mid x) = \sum_{i=1}^{\infty} \widehat{\mathbf{p}}(\mathbf{C}_i \mid z) \end{aligned}$$

holds k_z-almost everywhere;

(iv) if $\mathbf{C}_n \nearrow \mathbf{C}$, i.e. by (4)

$$\psi(\mathbf{A}_n) \nearrow \psi(\mathbf{A}) \Leftrightarrow \mathbf{A}_n \nearrow \mathbf{A}.$$

Hence the convergence

$$\widehat{\mathbf{p}}\big(\mathbf{C}_n \mid z\big) = \mathbf{p}\big(\mathbf{A}_n \mid x\big) \nearrow \mathbf{p}\big(\mathbf{A} \mid x\big) = \widehat{\mathbf{p}}\big(\mathbf{C} \mid z\big)$$

holds k_z-almost everywhere. □

In paper [5] we formulate a modification of martingale convergence theorem for conditional IF-probability, see below.

Theorem 2.6 *Let $\mathscr{F}$ be a family of IF-events with product $\cdot$, $\mathbf{A} \in \mathscr{F}$, $y : \mathscr{B}(R) \to \mathscr{F}$ be an IF-observable, $\mathbf{m} : \mathscr{F} \to [0,1]$ be an IF-state, and $g, g_n : R \to R$ $(n = 1,2,\dots)$ be the Borel measurable functions such that $g_n^{-1}\big(\mathscr{B}(R)\big) \nearrow g^{-1}\big(\mathscr{B}(R)\big)$. Then the convergence*

$$\mathbf{p}(\mathbf{A} \mid y \circ g_n^{-1}) \to \mathbf{p}(\mathbf{A} \mid y \circ g^{-1})$$

holds $\mathbf{m}_{y \circ g^{-1}}$-almost everywhere.

Since there exists a connection between a conditional IF-probability and a conditional IV-probability, we try to prove the modification of this theorem for a conditional IV-probability.

Theorem 2.7 (Martingale Convergence Theorem) *Let $\mathscr{K}$ be a family of IV-events with product $\widehat{\cdot}$, $\mathbf{C} \in \mathscr{K}$, $z : \mathscr{B}(R) \to \mathscr{K}$ be an IV-observable, $k : \mathscr{K} \to [0,1]$ be an IV-state, and $g, g_n : R \to R$ $(n = 1,2,\dots)$ be the Borel measurable functions such that $g_n^{-1}\big(\mathscr{B}(R)\big) \nearrow g^{-1}\big(\mathscr{B}(R)\big)$. Then the convergence*

$$\widehat{\mathbf{p}}(\mathbf{C} \mid z \circ g_n^{-1}) \to \widehat{\mathbf{p}}(\mathbf{C} \mid z \circ g^{-1})$$

holds $k_{z \circ g^{-1}}$-almost everywhere.

Proof. Let $\psi : \mathscr{F} \to \mathscr{K}$, $\psi((u,v)) = (u, 1_\Omega - v)$. Using connections between states and observables in family of IV-events and IF-events given by $\mathbf{m} = k \circ \psi$, $z = \psi \circ x$ from *Propositions 1.1 and 1.2* we obtain

$$\begin{aligned} z \circ g_n^{-1} &= \psi \circ x \circ g_n^{-1} = \psi\big(x \circ g_n^{-1}\big), \\ z \circ g^{-1} &= \psi \circ x \circ g^{-1} = \psi\big(x \circ g^{-1}\big), \\ \mathbf{m}\big(x \circ g^{-1}\big) &= k \circ \psi \circ x \circ g^{-1} = k\big(z \circ g^{-1}\big). \quad (7) \end{aligned}$$

Hence by *Proposition 2.1* we have

$$\begin{aligned} \widehat{\mathbf{p}}(\mathbf{C} \mid z \circ g_n^{-1}) &= \mathbf{p}(\mathbf{A} \mid x \circ g_n^{-1}), \\ \widehat{\mathbf{p}}(\mathbf{C} \mid z \circ g^{-1}) &= \mathbf{p}(\mathbf{A} \mid x \circ g_n^{-1}), \end{aligned}$$

where $\mathbf{p}(\mathbf{A} \mid x \circ g_n^{-1})$ and $\mathbf{p}(\mathbf{A} \mid x \circ g^{-1})$ are the conditional intuitionistic fuzzy probabilities, $\mathbf{A} \in \mathscr{F}$, $x : \mathscr{B}(R) \to \mathscr{F}$ is an IF-observable and $\mathbf{C} = \psi(\mathbf{A})$.

Finally using (7) and *Theorem 2.6* we obtain that the convergence

$$\widehat{\mathbf{p}}(\mathbf{C} \mid z \circ g_n^{-1}) \to \widehat{\mathbf{p}}(\mathbf{C} \mid z \circ g^{-1})$$

holds $k_{z \circ g^{-1}}$-almost everywhere. □

3 Conclusion

In this paper we introduced the notion of conditional probability for interval valued events and we showed the connection with a conditional intuitionistic fuzzy probability. We formulated the martingale convergence theorem for the conditional interval valued probability. As fuzzy sets are special cases of the intuitionistic fuzzy sets and there exists the connection between the interval valued sets and the intuitionistic fuzzy sets, so we obtained the results for a more general situation.

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