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Some Conclusions on the Direct Product of Uninorms on Bounded Lattices

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Abstract

Recently, the notation of the order induced by uninorms (t-norms, nullnorms) has been studied widely. In this paper, we study on the direct product of uninorms on bounded lattices. Also, we define an order induced by uninorms which are a direct product of two uninorms on bounded lattices. Also, we investigate properties of introduced order.

Keywords: Uninorm; direct product; partial order

1 Introduction

Aggregation operators [21] play an important role in theories of fuzzy sets, fuzzy logic, fuzzy system modeling, expert systems, neural networks and approximate reasoning [16, 17, 21, 22, 29, 33]. Two basic types of aggregation functions, namely t-norms and t-conorms, were introduced by Schweizer and Sklar in [31].

Uninorms with the neutral element e on the unit interval $[0,1]$, as an important generalization of t-norms and t-conorms. In contrast to the definitions of t-norms and t-conorms, the difference is that uninorms allow the neutral element e lying anywhere in the unit interval $[0,1]$. In particular, a uninorm U is a t-norm T and t-conorm S when the case $e=1$ and $e=0$, respectively.

As a generalization of t-norms and t-conorms, Yager and Rybalov [32] introduced the concepts of uninorms, then Fodor et al. [19, 20] systematically studied them which are special aggregation functions with the neutral element $e\in[0,1]$. For uninorms on bounded lattices, there also arises much work. First, uninorms on bounded lattices were introduced by Karaçal and Mesiar [25] in 2015. They showed the existence of uninorms with neutral element e for an arbitrary element $e\in L\setminus\{0,1\}$ with underlying t-norms and t-conorms on an arbitrary bounded lattice. Also, they introduced the smallest and the greatest uninorm with the neutral element $e\in L\setminus\{0,1\}$.

In [14], direct product of triangular norms on product lattices was introduced and some of the algebraic properties were investigated.

In [27], T- partial order, denoted $\preceq_T$, defined by means of t-norms on a bounded lattice was introduced. Based on this study, in [18, 7] U-partial order, denoted $\preceq_U$ and F-partial order, denoted by $\preceq_F$ defined by means of uninorms and nullnorms, respectively.

In this paper, we study on the direct product of uninorms on bounded lattices. The present paper consists of four main parts. Firstly, we give in preliminaries some necessary definitions we will work with. In Section 3, we define an order induced by uninorms which are a direct product of two uninorms on bounded lattices. In Section 4, we define the set of comparable elements with respect to the U-partial order and we obtain some interesting results related to direct product of uninorms on $[0,1]^2$. In Section 5, some concluding remarks are added.

2 Preliminaries

A lattice is a partially ordered set $(L,\leq)$ in which each two element subset $\{x,y\}$ has an infimum, denoted as $x\wedge y$, and a supremum, denoted as $x\vee y$.
A bounded lattice $(L,\leq,0,1)$ is a lattice that has the bottom and top elements written as 0 and 1, respectively.
Given a bounded lattice $(L,\leq,0,1)$ and $a,b\in L$, if a and b are incomparable, in this case, we use the notation $a\parallel b$ (see [4, 5, 6, 9, 12, 23, 26]).

Definition 1 *[14]Let* $(L_1,\leq_1,0_1,1_1)$ *and* $(L_2,\leq_2,0_2,1_2)$ *be bounded lattices. Then,*

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$L_1 \times L_2 = (L_1 \times L_2, \leq, (0_1, 0_2), (1_1, 1_2))$ is a bounded lattice with partial order relation $\leq$, $\wedge$ and $\vee$ defined by

$$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 \leq_1 x_2 \quad \text{and} \quad y_1 \leq_2 y_2.$$

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge_1 x_2, y_1 \wedge_2 y_2).$$

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee_1 x_2, y_1 \vee_2 y_2).$$

In this study, we use the L_1 instead of $(L_1, \leq_1, 0_1, 1_1)$, L_2 instead of $(L_2, \leq_2, 0_2, 1_2)$ and $L_1 \times L_2$ instead of $(L_1 \times L_2, \leq, \wedge, \vee, (0_1, 0_2), (1_1, 1_2))$.

Definition 2 *[8, 30] Let L be a bounded lattice. A triangular norm T (briefly t-norm) is a binary operation on L that is commutative, associative, monotone and has neutral element* 1.

Example 1 *[28] The four basic t-norms T_M, T_P, T_L and T_D on $[0,1]$ are given by:*
$T_M(x,y) = \min(x,y)$,
$T_P(x,y) = x \times y$,
$T_L(x,y) = \max(x+y-1, 0)$,

$$T_D(x,y) = \begin{cases} 0 & (x,y) \in [0,1)^2, \\ \min(x,y) & \text{otherwise}. \end{cases}$$

Definition 3 *[2, 30] Let L be a bounded lattice. A triangular conorm S (briefly t-conorm) is a binary operation on L that is commutative, associative, monotone and has neutral element* 0.

Example 2 *[28] The four basic t-conorms S_M, S_P, S_L and S_D on $[0,1]$ are given by:*
$S_M(x,y) = \max(x,y)$,
$S_P(x,y) = x + y - x \times y$,
$S_L(x,y) = \min(x+y, 1)$,

$$S_D(x,y) = \begin{cases} 1 & (x,y) \in (0,1]^2, \\ \max(x,y) & \text{otherwise}. \end{cases}$$

Extremal t-norms $T_\wedge$ and T_W are defined on a bounded lattice as follows, respectively:
$T_\wedge(x,y) = x \wedge y$

$$T_W(x,y) = \begin{cases} x & if\ y = 1, \\ y & if\ x = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Similarly, the t-conorms $S_\vee$ and S_W can be defined.

Especially we have obtained $T_W = T_D$ and $T_\wedge = T_M$ for $L = [0,1] \subset R$.

Definition 4 *[14] Let L_1 and L_2 be bounded lattices and T_1 and T_2 be t-norms on L_1 and L_2, respectively. Then, the direct product $T_1 \times T_2$ of T_1 and T_2, defined by*

$$T_1 \times T_2((x_1, y_1), (x_2, y_2)) = (T_1(x_1, x_2), T_2(y_1, y_2))$$

is a t-norm on the product lattice $L_1 \times L_2$.

Definition 5 *[11] A t-norm T on L is divisible if the following condition holds:*

$$\forall x, y \in L \quad \text{with} \quad x \leq y \quad \text{there is a} \quad z \in L \quad \text{such that}$$

$$x = T(y, z).$$

Definition 6 *[1, 10, 13] Let L be a bounded lattice. An operation $U : L^2 \to L$ is called a uninorm on L, if it is commutative, associative, monotone and has a neutral element $e \in L$.*

We denote by $\mathscr{U}(e)$ the set of all uninorms on L with the neutral element $e \in L$. Also, we denote by $A(e) = L^2 \setminus ([0,e]^2 \cup [e,1]^2)$ and $I(U) = \{x \in L \mid U(x,x) = x\}$.

Theorem 1 *[19] Let $U : [0,1]^2 \to [0,1]$ be a uninorm with neutral element $e \in (0,1)$. Then the sections $x \mapsto U(x,1)$ and $x \mapsto U(x,0)$ are continuous in each point except perhaps for e if and only if U is given by one of the following formulas.*
(a) If $U(0,1) = 0$, then

$$U(x,y) = \begin{cases} eT(\frac{x}{e}, \frac{y}{e}) & ,(x,y) \in [0,e]^2 \\ e + (1-e)S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & ,(x,y) \in [e,1]^2 \\ \min(x,y) & ,(x,y) \in A(e), \end{cases} \tag{1}$$

where T is a t-norm and S is a t-conorm.
(b) If $U(0,1) = 1$, then the same structure holds, changing minimum by maximum in $A(e)$.

The class of uninorms as in case (a) will be denoted by $\mathscr{U}_{\min}$ and the class of uninorms as in case (b) by $\mathscr{U}_{\max}$. We will denote a uninorm U in $\mathscr{U}_{\min}$ with underlying t-norm T, underlying t-conorm S and neutral element e by $U \equiv \langle T, e, S \rangle_{\min}$ and in a similar way, a uninorm in $\mathscr{U}_{\max}$ by $U \equiv \langle T, e, S \rangle_{\max}$.

Proposition 1 *[24] Let L_1 and L_2 be bounded lattices and U_1 be a uninorm on L_1 with neutral element e_1 and U_2 be a uninorm on L_2 with neutral element e_2. Then the direct product $U_1 \times U_2$ of U_1 and U_2, defined by*

$$U_1 \times U_2((x_1, y_1), (x_2, y_2)) = (U_1(x_1, x_2), U_2(y_1, y_2))$$

is a uninorm on the product lattice $L_1 \times L_2$ with neutral element (e_1, e_2).

Definition 7 *[27] Let L be a bounded lattice, T be a t-norm on L. The order defined as follows is called a T− partial order (triangular order) for t-norm T:*

$$x \preceq_T y :\Leftrightarrow T(\ell, y) = x \text{ for some } \ell \in L.$$

Definition 8 *[18] Let L be a bounded lattice, S be a t-conorm on L. The order defined as follows is called an S− partial order for t-conorm S:*

$$x \preceq_S y :\Leftrightarrow S(\ell, x) = y \text{ for some } \ell \in L.$$

Definition 9 *[18] Let L be a bounded lattice and U be a uninorm with neutral element e on L. Define the following relation, for $x, y \in L$, as*

$$x \preceq_U y :\Leftrightarrow \begin{cases} \text{if } x, y \in [0,e] \text{ and there exist } k \in [0,e] \\ \qquad\qquad \text{such that } U(y,k) = x \text{ or,} \\ \text{if } x, y \in [e,1] \text{ and there exist } \ell \in [e,1] \\ \qquad\qquad \text{such that } U(x,\ell) = y \text{ or,} \\ \text{if } (x,y) \in L^* \text{ and } x \leq y. \end{cases} \quad (2)$$

where $I_e = \{x \in L \mid x \parallel e\}$ and $L^* = [0,e] \times [e,1] \cup [0,e] \times I_e \cup [e,1] \times I_e \cup [e,1] \times [0,e] \cup I_e \times [0,e] \cup I_e \times [e,1] \cup I_e \times I_e$.

Proposition 2 *[18] The relation $\preceq_U$ defined in (2) is a partial order on L.*

Note: The partial order $\preceq_U$ in (2) is called U-partial order on L.

Definition 10 *[3] Let L be a bounded lattice, U be a uninorm on L and let K_U be defined by*

$$K_U = \{x \in L \mid \text{for some } y \in L,\ [x < y \text{ and } x \npreceq_U y] \text{ or } [y < x \text{ and } y \npreceq_U x]\}.$$

3 $\preceq_{U_1 \times U_2}$-partial order

In this section, we define an order induced by uninorms which are a direct product of two uninorms on bounded lattices.

Definition 11 *Let L_1 and L_2 be bounded lattices, U_1 be a uninorm on L_1 with neutral element e_1 and U_2 be a uninorm on L_2 with neutral element e_2 and consider their direct product $U_1 \times U_2$ on $L_1 \times L_2$. Let $\preceq_{U_1}$ and $\preceq_{U_2}$ are partial orders induced by uninorms U_1 and U_2, respectively. Then, the relation $\preceq_{U_1 \times U_2}$ is defined by*

$$(x,y) \preceq_{U_1 \times U_2} (z,t) \Leftrightarrow x \preceq_{U_1} z \text{ and } y \preceq_{U_2} t$$

for all $(x,y), (z,t) \in L_1 \times L_2$.

Proposition 3 *Let U_1 be a uninorm on L_1 with neutral element e_1 and U_2 be a uninorm on L_2 with neutral element e_2 and consider their direct product $U_1 \times U_2$ on $L_1 \times L_2$. Then, the relation $\preceq_{U_1 \times U_2}$ defined in Definition 11 is a partial order on $L_1 \times L_2$.*

Proposition 4 *Let L_1 and L_2 be bounded lattices, U_1 be a uninorm on L_1 with neutral element e_1 and U_2 be a uninorm on L_2 with neutral element e_2 and consider their direct product $U_1 \times U_2$ on $L_1 \times L_2$. Then, $L_1 \times L_2$ is a bounded partially ordered set with respect to the $\preceq_{U_1 \times U_2}$ partial order.*

Example 3 *Consider the lattice $(L_1 = L_2 = \{0, m, n, e, p, s, k, t, 1\}, \leq, 0, 1)$ given in Fig. 1 and the uninorms U_1 and U_2 on $L_1 = L_2$ defined Table 1 and Table 2, respectively.*

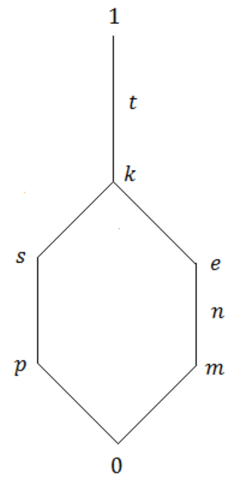

Figure 1: The lattice $L_1 = L_2$

Table 1: The uninorm U_1 on $L_1 = L_2$

U_1	0	m	n	e	p	s	k	t	1
0	0	0	0	0	p	s	k	t	1
m	0	m	m	m	p	s	k	t	1
n	0	m	n	n	p	s	k	t	1
e	0	m	n	e	p	s	k	t	1
p	p	p	p	p	1	1	1	1	1
s	s	s	s	s	1	1	1	1	1
k	k	k	k	k	1	1	1	1	1
t	t	t	t	t	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1

Since $U_1(m,n) = m$ and $U_2(m,n) = m$, then we obtain that $m \preceq_{U_1} n$ and $m \preceq_{U_2} n$. So, it is obtained $(m,m) \preceq_{U_1 \times U_2} (n,n)$ by Definition 11. Also, it can be show that $(k,k) \npreceq_{U_1 \times U_2} (t,t)$.

Remark 1 *Let L_1 and L_2 be bounded lattices, U_1 be a uninorm on L_1 with neutral element e_1 and U_2 be a uninorm on L_2 with neutral element e_2 and consider their direct product $U_1 \times U_2$ on $L_1 \times L_2$. Then,*

$$(x,y) \preceq_{U_1 \times U_2} (z,t) \Rightarrow x \leq_1 z \text{ and } y \leq_2 t$$

for all $(x,y), (z,t) \in L_1 \times L_2$.

Table 2: The uninorm U_2 on $L_1 = L_2$

U_2	0	m	n	e	p	s	k	t	1
0	0	0	0	0	p	s	k	t	1
m	0	m	m	m	p	s	k	t	1
n	0	m	n	n	p	s	k	t	1
e	0	m	n	e	p	s	k	t	1
p	p	p	p	p	p	s	k	t	1
s	s	s	s	s	s	s	k	t	1
k	k	k	k	k	k	k	k	t	1
t	t	t	t	t	t	t	t	t	1
1	1	1	1	1	1	1	1	1	1

Lemma 1 *Let L_1 and L_2 be bounded lattices, T_1 be a t-norm on L_1 and T_2 be a t-norm on L_2 and consider their direct product $T_1 \times T_2$ on $L_1 \times L_2$. $T_1 \times T_2$ is divisible if and only if T_1 and T_2 are divisible.*

Lemma 2 *Let L_1 and L_2 be bounded lattices, S_1 be a t-conorm on L_1 and S_2 be a t-conorm on L_2 and consider their direct product $S_1 \times S_2$ on $L_1 \times L_2$. $S_1 \times S_2$ is divisible if and only if S_1 and S_2 are divisible.*

Proposition 5 *Let L_1 and L_2 be bounded lattices, U_1 and U_2 be uninorms on L_1 and L_2 with neutral elements e_1 and e_2, respectively, T_1 and T_2 be t-norms on $[0,e_1]$ and $[0,e_2]$, respectively and S_1 and S_2 be t-conorms on $[e_1,1]$ and $[e_2,1]$, respectively. Consider direct products $U_1 \times U_2$ on $L_1 \times L_2$, $T_1 \times T_2$ on $[0,e_1] \times [0,e_2]$ and $S_1 \times S_2$ on $[e_1,1] \times [e_2,1]$. Then, $T_1 \times T_2$ and $S_1 \times S_2$ are divisible if and only if $\preceq_{U_1 \times U_2} = \leq$.*

Proposition 6 *[14] Let T_1 and T_2 be t-norms on $[0,1]$ and their direct product $T_1 \times T_2$ on $[0,1]^2$. $T_1 \times T_2$ is divisible if and only if $T_1 \times T_2$ is continuous.*

Proposition 7 *[14] Let S_1 and S_2 be t-conorms on $[0,1]$ and their direct product $S_1 \times S_2$ on $[0,1]^2$. $S_1 \times S_2$ is divisible if and only if $S_1 \times S_2$ is continuous.*

Corollary 1 *Let U_1 and U_2 be uninorms on $[0,1]$ with neutral elements e_1 and e_2, respectively, T_1 and T_2 be t-norms on $[0,e_1]$ and $[0,e_2]$, respectively and S_1 and S_2 be t-conorms on $[e_1,1]$ and $[e_2,1]$, respectively. Consider direct products $U_1 \times U_2$ on $L_1 \times L_2$, $T_1 \times T_2$ on $[0,e_1] \times [0,e_2]$ and $S_1 \times S_2$ on $[e_1,1] \times [e_2,1]$. Then, $T_1 \times T_2$ and $S_1 \times S_2$ are continuous if and only if $\preceq_{U_1 \times U_2} = \leq$.*

4 Some investigations on the set of comparable and incomparable elements with respect to the $\preceq_U$-partial order

In this section, we investigate some properties of direct product of uninorms on bounded lattice. We define comparable and incomparable elements with respect to the U partial order on bounded lattice. By using these definitions, we obtain some interesting results for direct product of uninorms on $[0,1]^2$.

Definition 12 *Let L be a bounded lattice and U be a uninorm on bounded lattice L. The set C_U is defined as follows:*

$$C_U = \{x \in L| \quad there \quad exist \quad y, y' \in L \setminus \{0, x, 1\}, \\ x \preceq_U y \quad and \quad y' \preceq_U x\}$$

It is clear that $\{0,1\} \notin C_U$.

Example 4 *Consider the uninorm $\overline{U_{\frac{1}{4}}} : [0,1]^2 \to [0,1]$ with neutral element $\frac{1}{4}$ defined by*

$$\overline{U_{\frac{1}{4}}}(x,y) = \begin{cases} \min(x,y) & (x,y) \in [0,\frac{1}{4}]^2, \\ 1 & (x,y) \in (\frac{1}{4},1]^2, \\ \max(x,y) & otherwise. \end{cases}$$

Then, $C_{\overline{U_{\frac{1}{4}}}} = (0, \frac{1}{4}]$.

Proposition 8 *Let L_1 and L_2 be bounded lattices, U_1 be a uninorm on L_1 with neutral element e_1 and U_2 be a uninorm on L_2 with neutral element e_2. If $\preceq_{U_1} \subseteq \preceq_{U_2}$, then $C_{U_1} \subseteq C_{U_2}$.*

Corollary 2 *Let L_1 and L_2 be bounded lattices, U_1 be a uninorm on L_1 with neutral element e_1 and U_2 be a uninorm on L_2 with neutral element e_2. If $\preceq_{U_1} = \preceq_{U_2}$, then $C_{U_1} = C_{U_2}$.*

Example 5 *Consider the uninorm $\underline{U_{\frac{1}{2}}} : [0,1]^2 \to [0,1]$ with neutral element $\frac{1}{2}$ defined by*

$$\underline{U_{\frac{1}{2}}}(x,y) = \begin{cases} 0 & (x,y) \in [0,\frac{1}{2})^2, \\ \max(x,y) & (x,y) \in [\frac{1}{2},1]^2, \\ \min(x,y) & otherwise. \end{cases}$$

and consider the uninorm $U := U_{\min(T^{nM}, S_M, \frac{1}{2})} : [0,1]^2 \to [0,1]$ with neutral element $\frac{1}{2}$ defined as follows:

$$U = U_{\min(T^{nM}, S_M, \frac{1}{2})}(x,y) = \begin{cases} 0 & (x,y) \in [0,\frac{1}{2}]^2, \\ & and\ x+y \leq \frac{1}{2}, \\ \max(x,y) & (x,y) \in [\frac{1}{2},1]^2, \\ \min(x,y) & otherwise. \end{cases}$$

It can be shown that $C_U = [\frac{1}{2}, 1)$ *and* $C_{\underline{U_{\frac{1}{2}}}} = [\frac{1}{2}, 1)$. *That is,* $C_U = C_{\underline{U_{\frac{1}{2}}}}$. *Since* $\frac{1}{5} \preceq_U \frac{1}{3}$ *and* $\frac{1}{5} \not\preceq_{\underline{U_{\frac{1}{2}}}} \frac{1}{3}$, *it does not need to be* $\preceq_U = \preceq_{\underline{U_{\frac{1}{2}}}}$.

The set C_U allows us to introduce the next equivalence relation on the class of all uninorms on bounded lattices.

Definition 13 *Define a relation* δ *on the class of all uninorms on bounded lattices by* $U_1 \delta U_2$

$$U_1 \delta U_2 :\Leftrightarrow C_{U_1} = C_{U_2}.$$

Lemma 3 *The relation* δ *given in Definition 13 is an equivalence relation.*

Definition 14 *For a given uninorm U on bounded lattice L, we denote by* $\overline{U}$ *the* δ *equivalence class linked to U, i.e,*

$$\overline{U} = \{U' | \quad U' \delta U\}.$$

If we take $L = [0, 1]$, then we obtain the following Proposition 9 and Proposition 10.

Proposition 9 *The set* $[0, 1]/\delta$ *of all equivalence classes of all uninorms on the unit interval* $[0, 1]$ *under* δ, *is uncountably infinite.*

Proposition 10 *Let* $e \in [0, 1]$. *If* $U \in \mathscr{U}(e)$, *then*

$$U(x,y) = \begin{cases} T_U(x,y) & (x,y) \in [0,e]^2, \\ S_U(x,y) & (x,y) \in [e,1]^2, \\ D(x,y) & (x,y) \in A(e), \end{cases}$$

where T_U *is a t-norm on* $[0, e]$, S_U *is a t-conorm on* $[e, 1]$ *and* $D : A(e) \to [0, 1]$ *is increasing and fulfills*

$\min(x, y) \le D(x, y) \le \max(x, y)$ *for* $(x, y) \in A(e)$ *by [15].*

If T_U *and* S_U *are continuous t-norm and t-conorm, respectively, then* $C_U = (0, 1)$.

Example 6 *Let* $e \in [0, 1]$. *Consider the uninorms* $U^{\min}$ *and* $U^{\max}$ *as unique idempotent uninorm* $U_e^{\min}$ *and* $U_e^{\max}$, *respectively:*

$$U^{\min}(x,y) = \begin{cases} \max(x,y) & (x,y) \in [e,1]^2, \\ \min(x,y) & \textit{otherwise}. \end{cases}$$

$$U^{\max}(x,y) = \begin{cases} \min(x,y) & (x,y) \in [0,e]^2, \\ \max(x,y) & \textit{otherwise}. \end{cases}$$

Then, it is obtained that $C_{U^{\min}} = (0, 1)$ *and* $C_{U^{\max}} = (0, 1)$.

The next example shows the importance of continuity in Proposition 10.

Example 7 *Consider the uninorm* $U := U_{\min(T^{nM}, S_M, \frac{1}{2})} : [0, 1]^2 \to [0, 1]$ *with neutral element* $\frac{1}{2}$ *defined in Example 5. Since* T^{nM} *is left continous t-norm, it need not be* $C_U = (0, 1)$. *Also, it is clear that* $C_U = [\frac{1}{2}, 1)$.

Remark 2 *We use the notation* K_U *denote the set of all elements from* $[0, 1]$ *admitting some incomparability with respect to* $\preceq_U$. *Note that any element* $x \in K_U$ *need not be incomparable to every element* $y \in [0, 1] \setminus \{0, 1\}$. *Considering the definition of* K_U, *it is easily seen that it is sufficient for an x to be incomparable to only one element y in order to be an element of* K_U. *So, we obtain different results for the sets* C_U *and* K_U *in Proposition 11 and Proposition 12, respectively.*

Proposition 11 *Let* L_1 *and* L_2 *be bounded lattices,* U_1 *be a uninorm on* L_1 *with neutral element* e_1 *and* U_2 *be a uninorm on* L_2 *with neutral element* e_2 *and consider their direct product* $U_1 \times U_2$ *on* $L_1 \times L_2$. *Then,*

$$C_{U_1 \times U_2} = C_{U_1} \times C_{U_2}.$$

Proposition 12 *Let* L_1 *and* L_2 *be bounded lattices,* U_1 *be a uninorm on* L_1 *with neutral element* e_1 *and* U_2 *be a uninorm on* L_2 *with neutral element* e_2 *and consider their direct product* $U_1 \times U_2$ *on* $L_1 \times L_2$. *Then,*

$$K_{U_1} \times K_{U_2} \subseteq K_{U_1 \times U_2}.$$

Remark 3 *The converse of the Proposition 12 may not be true. Here is an example illustrating such a case.*

Example 8 *Consider the greatest uninorm* $\overline{U_{\frac{1}{2}}}$: $[0, 1]^2 \to [0, 1]$ *with neutral element* $\frac{1}{2}$ *defined by*

$$\overline{U_{\frac{1}{2}}}(x,y) = \begin{cases} \min(x,y) & (x,y) \in [0,\frac{1}{2}]^2, \\ 1 & (x,y) \in (\frac{1}{2},1]^2, \\ \max(x,y) & \textit{otherwise}. \end{cases}$$

and the smallest uninorm $\underline{U_{\frac{1}{2}}} : [0, 1]^2 \to [0, 1]$ *with neutral element* $\frac{1}{2}$ *defined in Example 5.*

Now, we show that it need not to be $K_{\underline{U_{\frac{1}{2}}} \times \overline{U_{\frac{1}{2}}}} \subseteq K_{\underline{U_{\frac{1}{2}}}} \times K_{\overline{U_{\frac{1}{2}}}}$.

Remark 4 *If we take the uninorms* U_1 *and* U_2 *to be equal, then the converse of the Proposition 12 is true, i.e., equality is satisfied.*

Remark 5 *The converse of the Proposition 12 may be true for some special uninorms on the unit interval* $[0, 1]$. *Here is an example illustrating such a case.*

Example 9 *Consider the uninorm $U_1 : [0,1]^2 \to [0,1]$ with neutral element $\frac{1}{2}$ defined as follows:*

$$U_1(x,y) = \begin{cases} 0 & (x,y) \in [0,\frac{1}{2}]^2 \text{ and} \\ & x+y \leq \frac{1}{2} \text{ and } (x,y) \neq (\frac{1}{4},\frac{1}{4}), \\ \frac{1}{4} & (x,y) = (\frac{1}{4},\frac{1}{4}), \\ \max(x,y) & (x,y) \in [\frac{1}{2},1]^2, \\ \min(x,y) & \text{otherwise}. \end{cases}$$

and consider the uninorm $U := U_{\min(T^{nM},S_M,\frac{1}{2})} : [0,1]^2 \to [0,1]$ with neutral element $\frac{1}{2}$ defined in Example 5. We know that $K_U = (0,\frac{1}{2})$ by Aşıcı (see [3]). Similarly, it can be shown that $K_{U_1} = (0,\frac{1}{2})$. Also, it is clear that $K_{U\times U_1} = (0,\frac{1}{2}) \times (0,\frac{1}{2}) = K_U \times K_{U_1}$.

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5 Concluding remarks

Uninorms are generalizations of t-norms and t-conorms with neutral element to an arbitrary point from a bounded lattice. Uninorms are useful tool in many different fields, such as expert systems, neural networks, and fuzzy logic. Moreover, they have been used as aggregators in fuzzy logic in order to maintain as many logical properties as possible. Recently, the notation of the order induced by uninorms (t-norms, nullnorms) has been studied widely. First, the T - partial order obtained from a t-norm was defined by [27]. Based on these previous studies, the orders comprising the U-partial order and F-partial order obtained from the uninorm and nullnorm were defined by [7] and [18], respectively. The U-partial order is an extension of the T -partial order and S-partial order, so it is important to study the U-partial order to obtain more general conclusions. Thus, in this study, we have studied uninorms on bounded lattices. We have developed several new results in the domain of uninorms acting on bounded lattices, including the direct products of bounded lattices. Also, we have studied new partial orderings induced by uninorms on bounded lattices. processing, etc.

References

[1] E. Aşıcı, R. Mesiar, On generating uninorms on some special classes of bounded lattices, Fuzzy Sets and Systems, (2021) doi:10.1016/j.fss.2021.06.010.

[2] E. Aşıcı and R. Mesiar, Alternative approaches to obtain t-norms and t-conorms on bounded lattices, Iranian Journal of Fuzzy Systems 17 (2020), 121-138.

[3] E. Aşıcı, The equivalence of uninorms induced by the U-partial order, Hacettepe Journal of Mathematics and Statistics 48(2019) 439-450.

[4] E. Aşıcı, Equivalence classes of uninorms, Filomat 33:2 (2019) 571-582.

[5] E. Aşıcı, An extension of the ordering based on nullnorms, Kybernetika 55(2) (2019) 217-232.

[6] E. Aşıcı, On the properties of the F-partial order and the equivalence of nullnorms, Fuzzy Sets and Systems 346 (2018) 72-84.

[7] E. Aşıcı, An order induced by nullnorms and its properties, Fuzzy Sets and Systems 325 (2017), 35-46.

[8] E. Aşıcı, Some notes on the F-partial order, In: Kacprzyk J., Szmidt E., Zadrożny S., Atanassov K., Krawczak M. (eds) Advances in Fuzzy Logic and Technology 2017. IWIFSGN 2017, EUSFLAT 2017. Advances in Intelligent Systems and Computing, vol 641, pp 78-84 Springer, Cham, 2018.

[9] G. Birkhoff, Lattice Theory, 3 rd edition, Providence, 1967.

[10] T. Calvo, B. De Baets and J. Fodor, The functional equations of Frank and Alsina for uninorms and nullnorms, Fuzzy Sets and Systems 120 (2001), 385-394.

[11] J. Casasnovas and G. Mayor, Discrete t-norms and operations on extended multisets, Fuzzy Sets and Systems 159 (2008), 1165-1177.

[12] G.D. Çaylı, Uninorms on bounded lattices with the underlying t-norms and t-conorms, Fuzzy Sets and Systems 395 (2020), 107-129.

[13] G. D. Çaylı, Alternative approaches for generating uninorms on bounded lattices, Information Sciences 488 (2019) 111-139.

[14] B. De Baets and R. Mesiar, Triangular norms on product lattices, Fuzzy Sets and Systems 104 (1999), 61-75.

[15] J. Drewniak, P. Drygaś and E. Rak, Distributivity between uninorms and nullnorms, Fuzzy Sets and Systems 159 (2008), 1646-1657.

[16] D. Dubois, H. Prade, Fundamentals of Fuzzy Sets, Kluwer Acad. Publ., Boston 2000.

[17] D. Dubois, H. Prade, A review of fuzzy set aggregation connectives, Inf. Sci. 36(1985) 85-121.

[18] Ü. Ertuğrul, M.N. Kesicioğlu and F. Karaçal, Ordering based on uninorms, Information Sciences 330 (2016), 315-327.

[19] J.C. Fodor, R.R. Yager and A. Rybalov, Structure of uninorms, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 5 (1997), 411-427.

[20] J. Fodor and B. De Baets, A single-point characterization of representable uninorms, Fuzzy Sets and Systems 202 (2012), 89âĂŞ99.

[21] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, Aggregation Functions, Cambridge University Press, 2009.

[22] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, Aggregation functions: construction methods, conjunctive, disjunctive and mixed classes, Inf. Sci. 181(2011) 23-43.

[23] M. A. İnce, F. Karaçal, R. Mesiar, *Medians and nullnorms on bounded lattices*, Fuzzy Sets Syst. **289** (2016), 74-81.

[24] M. Kalina, On uninorms and nullnorms on direct product of bounded lattices, Open Physics 14 (2016), 321-327.

[25] F. Karaçal and R. Mesiar, Uninorms on bounded lattices, Fuzzy Sets and Systems 261 (2015), 33-43.

[26] F. Karaçal, M. A. İnce, R. Mesiar, *Nullnorms on bounded lattices*, Inf. Sci. **325** (2015), 227-235.

[27] F. Karaçal and M.N. Kesicioğlu, A T-partial order obtained from t-norms, Kybernetika 47 (2011), 300-314.

[28] E.P. Klement, R. Mesiar, E. Pap, *Triangular Norms*, Kluwer Academic Publishers, Dordrecht 2000.

[29] R. Mesiar, A. Kolesarova, A. Stupnanova Qua vadis aggregation, International Journal of General Systems 47(2) (2018) 97-117.

[30] S. Saminger, On ordinal sums of triangular norms on bounded lattices, Fuzzy Sets and Systems 157 (2006), 1403-1416.

[31] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific Journal of Mathematics 10 (1960) 313-334.

[32] R.R. Yager and A. Rybalov, Uninorm aggregation operators, Fuzzy Sets and Systems 80 (1996), 111-120.

[33] R.R. Yager, Aggregation operators and fuzzy systems modeling, Fuzzy Sets and Systems 67 (1994) 129-145.