Triangular Fuzzy Relational Products of Level Fuzzy Relations

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Abstract

This paper firstly aims at investigating distinct properties of the Bandler-Kohout subproduct and superproduct of level fuzzy relations or level relations. We show that these triangular products preserve several desirable properties similar to those valid for the compositions of standard fuzzy relations. Moreover, we provide the relationship between the fuzzy cut of the Bandler-Kohout products of fuzzy relations and the same products of the fuzzy cut of the same arguments. Secondly, we discuss the appropriateness of the use of the suggested products for the classification task. The positive impact of such products is demonstrated on a numerical example and the real application of the Dragonfly classification problem.

Keywords: Fuzzy relational compositions, Fuzzy relational products, Bandler-Kohout triangular products, Cutability, α-level fuzzy relation, Classification, Dragonflies.

1 Introduction

The topic of fuzzy relational compositions has been widely and intensively explored by numerous authors, since its initial investigation in the late 70's and the early 80's by Bandler and Kohout, see e.g., [12, 6, 2, 1]. Its application areas are diverse, covering medical diagnosis, fuzzy inference systems, fuzzy relational equations, data mining, or fuzzy control (cf. [19, 18, 8, 13]. Let us mention that the topic has been recently extended in various directions. For instance, the compositions incorporating excluding features and unavoidable features, the compositions employing fuzzy quantifiers, and the BK-superproduct and subproduct based on the grouping features, see e.g., [5, 17, 3]. It is worth mentioning that these approaches have been successfully applied to the practical classification of the Dragonflies in biology [17].

The cutting of fuzzy relational compositions by a threshold level α was intensively approached in [7]. In particular, the authors have shown the relationship between the cutability of a fuzzy relational composition and the composition of the cutability of fuzzy relations of the same arguments. A similar relationship applying the notion of fuzzy cutability into the basic composition of fuzzy relations was discussed in [7] as well. Several contributions have taken into account the use of the cutability and fuzzy cutability of fuzzy relations. For instance, the studies involving the rough approximation based on level fuzzy sets, and the combination of rough and fuzzy sets based on level sets were concentrated in [14]. In [10], the authors applied the cutting fuzzy relations to explain the experimental result of the land evaluation. In [20], a new method to information retrieval system using the fuzzy cut of fuzzy relations was addressed. The use of fuzzy cut of fuzzy expressions was also applied to the solvability of fuzzy relational equations [9]. In [9], the cutability threshold level α is chosen based on the assumption that truth degrees smaller than α are less reliable than those greater than α. For more valuable results, we may refer to e.g., [15, 11].

In this paper, we focus on the Bandler-Kohout triangular products of cutting fuzzy relations. In particular, apart from studying various properties, we show how the suggested products are appropriate for the classification problems in biology. We discuss a way of approaching the cutting threshold and present the experiment on practical examples.

2 Fuzzy relational compositions

We recall some basic definitions of fuzzy relational compositions and their recent extensions on the concept of excluding features [4] and unavoidable features [17]. In the sequel, we use the operations from a residuated lattice $\mathscr{L} = \langle [0,1], \wedge, \vee, \otimes, \to 0, 1\rangle$ and additional negation $\neg$ defined in the standard way that $\neg a = a \to 0$. Moreover, by $\mathscr{F}(U)$, we denote the set of all fuzzy sets on a given universe U. Let X,Y,Z be nonempty finite universes of samples/objects, features, and classes, respectively. Fuzzy relation $R \in \mathscr{F}(X\times Y)$ encodes the information for the relationship between samples and features and fuzzy relation $S \in \mathscr{F}(Y\times Z)$ expresses the dependence of the features on the classes.

2.1 Compositions of fuzzy relations

Definition 1. Let $R \in \mathscr{F}(X\times Y), S \in \mathscr{F}(Y\times Z)$. Then, for all $x \in X$ and $z \in Z$, the *basic composition* $\circ$, *BK-subproduct* $\lhd$, *BK-superproduct* $\rhd$, and *BK-square product* $\square$ of R and S are fuzzy relations on $X\times Z$ defined by:

$$(R\circ S)(x,z) = \bigvee_{y\in Y} (R(x,y)\otimes S(y,z)),$$
$$(R\lhd S)(x,z) = \bigwedge_{y\in Y} (R(x,y)\to S(y,z)),$$
$$(R\rhd S)(x,z) = \bigwedge_{y\in Y} (S(y,z)\to R(x,y)),$$
$$(R\square S)(x,z) = \bigwedge_{y\in Y} (R(x,y)\leftrightarrow S(y,z)),$$

In order to illustrate the meaning of the compositions, we may consider the context of classification of animals in biology. Then, the value $(R\circ S)(x,z)$ shows how much it is true, that "*animal x carries at least one feature belonging to class z*". The value $(R\lhd S)(x,z)$ stands for the truth degree of the predicate, that "*all features of animal x belong to class z*". Similarly, $(R\rhd S)(x,z)$ means how much the truth degree should be, that "*animal x carries all the features that belong to class z* ". The semantics of $(R\square S)(x,z)$ can be explained as the truth degree of the predicate "*animal x has all features of class z and at the same time, all features of x belong to z*". Note, that the triangular products and square product provide a sort of strengthening the initial suspicion given by the basic composition.

2.2 The concept of excluding features

The incorporation of excluding features in the fuzzy relational compositions was intensively studied in [4]. The influence of this approach was demonstrated on the real Dragonfly classification problem. Let $E \in \mathscr{F}(Y\times Z)$ with the semantics that $E(y,z)$ expresses the truth degree of the predicate, that "*feature y is excluding for class z*".

Definition 2. [4] Let $R \in \mathscr{F}(X\times Y)$, $S,E \in \mathscr{F}(Y\times Z)$. Then, the composition of R and S incorporating E is a fuzzy relation $R\circ S^{\backprime}E \in \mathscr{F}(X\times Z)$ defined by:

$$(R\circ S^{\backprime}E)(x,z) = (R\circ S)(x,z)\otimes\neg(R\circ E)(x,z)\ .$$

The composition $R\circ S^{\backprime}E$ can be interpreted as follows. The value $(R\circ S^{\backprime}E)(x,z)$ stands for the truth degree of predicate, that "*animal x has at least one feature belonging to the class z, and at the same time, it does not have any feature that would be excluding the class z*".

2.3 The concept of unavoidable features

The concept of unavoidable features (or typical features) in fuzzy relational compositions was concentrated in [17]. This approach has a similar construction to the approach of the composition incorporating excluding features, and it has been shown to be an appropriate model for the classification task. In addition to fuzzy relations R,S,E, fuzzy relation $U \in \mathscr{F}(Y\times Z)$ is considered with the semantics described as follows. The value $U(y,z)$ shows the truth degree up to which that "*y is an unavoidable feature of class z*".

Definition 3. [17] Let $R \in \mathscr{F}(X\times Y)$, and let $S,U \in \mathscr{F}(Y\times Z)$. Then, $(R@S)^{\rhd U}$ for $@ \in \{\circ,\lhd,\rhd,\square\}$ are fuzzy relations on $X\times Z$ defined by:

$$(R@S)^{\rhd U}(x,z) = (R@S)(x,z)\otimes(R\rhd U)(x,z).$$

The value $(R\circ S)^{\rhd U}(x,z)$ expresses the truth degree of the sentence, that "*animal x has at least one feature belonging to class z, and at the same time, it has all features that are unavoidable for z*". Note, this approach may incorporate the concept of excluding features.

Definition 4. [17] Let $R \in \mathscr{F}(X\times Y)$, and let $S,E,U \in \mathscr{F}(Y\times Z)$. Then, $(R@S^{\backprime}E)^{\rhd U}$ for $@ \in \{\circ,\lhd,\rhd,\square\}$ is a fuzzy relation on $X\times Z$ defined by:

$$(R@S^{\backprime}E)^{\rhd U}(x,z) = (R@S)(x,z)\otimes\neg(R\circ E)(x,z)\otimes (R\rhd U)(x,z)\ .$$

The value $(R\circ S^{\backprime}E)^{\rhd U}(x,z)$ means that how much it is true, that "*animal x has at least one feature belonging to class z and it has no features that would be excluding the class z, and moreover, it carries all features that are typical for class z*".

3 Triangular products of level relations

3.1 Definitions and basic properties

We review some basic facts about level sets and level fuzzy sets.

Definition 5. (cf. [7, 14]) Let $R \in \mathscr{F}(X \times Y)$ and let $\alpha \in [0,1]$. The *α-cut* or *α-level* relation of R is the relation R_α on $X \times Y$ defined by

$$R_\alpha(x,y) = \begin{cases} 1 & \text{if } R(x,y) \geq \alpha; \\ 0 & \text{otherwise.} \end{cases}$$

Definition 6. (cf. [7, 14]) Let $R \in \mathscr{F}(X \times Y)$ and let $\alpha \in [0,1]$. The *fuzzy α-cut* or *α-level fuzzy* relation of R is the fuzzy relation $\tilde{R}_\alpha$ on $X \times Y$ defined by

$$\tilde{R}_\alpha(x,y) = \begin{cases} R(x,y) & \text{if } R(x,y) \geq \alpha; \\ 0 & \text{otherwise.} \end{cases}$$

It holds that $\tilde{R}_\alpha \subseteq R$ for all $\alpha \in [0,1]$. Moreover, for $\alpha = 0$, it is obvious that $R_\alpha = X \times Y$ and $\tilde{R}_\alpha = R$. For the simplicity, we call α-level relation and α-level fuzzy relation by level relation and level fuzzy relation, respectively. Let $R, R_1, R_2 \in \mathscr{F}(X \times Y)$ and $\alpha, \beta \in [0,1]$. The following properties are known for level relations and level fuzzy relations (cf. [16, 14]):

$$R_1 \subseteq R_2 \;\Rightarrow\; (R_1)_\alpha \subseteq (R_2)_\alpha\ ; \tag{1}$$
$$R_1 \subseteq R_2 \;\Rightarrow\; (\tilde{R}_1)_\alpha \subseteq (\tilde{R}_2)_\alpha\ ; \tag{2}$$
$$(R_1 \cup R_2)_\alpha = (R_1)_\alpha \cup (R_2)_\alpha\ ; \tag{3}$$
$$(\widetilde{R_1 \cup R_2})_\alpha = (\tilde{R}_1)_\alpha \cup (\tilde{R}_2)_\alpha\ ; \tag{4}$$
$$(R_1 \cap R_2)_\alpha = (R_1)_\alpha \cap (R_2)_\alpha\ ; \tag{5}$$
$$(\widetilde{R_1 \cap R_2})_\alpha = (\tilde{R}_1)_\alpha \cap (\tilde{R}_2)_\alpha \tag{6}$$
$$\alpha \leq \beta \;\Rightarrow\; R_\alpha \supseteq R_\beta\ ; \tag{7}$$
$$\alpha \leq \beta \;\Rightarrow\; \tilde{R}_\alpha \supseteq \tilde{R}_\beta\ . \tag{8}$$

3.2 Additional properties

This subsection investigates distinct properties related to the triangular products of level fuzzy relations or level relations. We show that several properties that are similar to those preserved in the standard fuzzy relational compositions are valid for the considered products as well. Let X, Y, Z, W be finite non-empty universes and let symbols $\bigcap$ and $\bigcup$ stand for Gödel intersection and union, respectively. Let $R, R_1, R_2 \in \mathscr{F}(X \times Y)$ and let $S, S_1, S_2 \in \mathscr{F}(Y \times Z)$, and $T \in \mathscr{F}(Z \times W)$.

Proposition 1. (Monotonicity) *For all* $\alpha \in [0,1]$*:*

$$R_1 \subseteq R_2 \;\Rightarrow\; (\tilde{R}_1)_\alpha \lhd S \supseteq (\tilde{R}_2)_\alpha \lhd S\ ;$$
$$R_1 \subseteq R_2 \;\Rightarrow\; (\tilde{R}_1)_\alpha \rhd S \subseteq (\tilde{R}_2)_\alpha \rhd S\ .$$

Sketch of the proof: The proof uses property (2) and the isotonocity and antitonicity of implication $\rightarrow$. □

Corollary 1. (Containment) *For all* $\alpha \in [0,1]$*:*

$$\tilde{R}_\alpha \lhd S \supseteq R \lhd S\ , \qquad R \lhd \tilde{S}_\alpha \subseteq R \lhd S\ ;$$
$$\tilde{R}_\alpha \rhd S \subseteq R \rhd S\ , \qquad R \rhd \tilde{S}_\alpha \supseteq R \rhd S\ .$$

Sketch of the proof: The proof is directly derived from Proposition 1 using the fact that $\tilde{R}_\alpha \subseteq R, \forall \alpha \in [0,1]$. □

Proposition 2. (Antitonicity w.r.t. ordering of levels) *For all* $\alpha, \beta \in [0,1]$*:*

$$\alpha \leq \beta \;\Rightarrow\; \tilde{R}_\alpha \lhd S \subseteq \tilde{R}_\beta \lhd S\ ;$$
$$\alpha \leq \beta \;\Rightarrow\; \tilde{R}_\alpha \rhd S \supseteq \tilde{R}_\beta \rhd S\ .$$

Sketch of the proof: It is sufficient to apply (8) and the isotonocity and antitonicity of $\rightarrow$. □

Proposition 3. (Union) *For all* $\alpha \in [0,1]$*:*

$$R \lhd (S_1 \cup S_2)_\alpha \supseteq (R \lhd (S_1)_\alpha) \cup (R \lhd (S_2)_\alpha)\ ; \tag{9}$$
$$R \rhd (S_1 \cup S_2)_\alpha = (R \rhd (S_1)_\alpha) \cap (R \rhd (S_2)_\alpha)\ ; \tag{10}$$
$$R \lhd (\widetilde{S_1 \cup S_2})_\alpha \supseteq (R \lhd (\tilde{S}_1)_\alpha) \cup (R \lhd (\tilde{S}_2)_\alpha)\ ; \tag{11}$$
$$R \rhd (\widetilde{S_1 \cup S_2})_\alpha = (R \rhd (\tilde{S}_1)_\alpha) \cap (R \rhd (\tilde{S}_2)_\alpha)\ . \tag{12}$$

Sketch of the proof: The proof directly uses properties (3) and (4). □

Proposition 4. (Intersection) *For all* $\alpha \in [0,1]$*:*

$$R \lhd (S_1 \cap S_2)_\alpha = (R \lhd (S_1)_\alpha) \cap (R \lhd (S_2)_\alpha)\ ; \tag{13}$$
$$R \rhd (S_1 \cap S_2)_\alpha \supseteq (R \rhd (S_1)_\alpha) \cup (R \rhd (S_2)_\alpha)\ ; \tag{14}$$
$$R \lhd (\widetilde{S_1 \cap S_2})_\alpha = (R \lhd (\tilde{S}_1)_\alpha) \cap (R \lhd (\tilde{S}_2)_\alpha)\ ; \tag{15}$$
$$R \rhd (\widetilde{S_1 \cap S_2})_\alpha \supseteq (R \rhd (\tilde{S}_1)_\alpha) \cup (R \rhd (\tilde{S}_2)_\alpha)\ . \tag{16}$$

Sketch of the proof: The proof applies properties (5) and (6). □

The following properties concentrate on the relationship between the level fuzzy relation of the BK-subproduct of fuzzy relations and the BK-subproduct of the level fuzzy relations of the same arguments. We provide the results for the use of the Gödel implication.

Proposition 5. *Let $\rightarrow$ from $\mathscr{L}$ be a Gödel implication. Then, for all $\alpha \in [0,1]$, it holds that*

$$(\widetilde{R \lhd S})_\alpha \subseteq \tilde{R}_\alpha \lhd \tilde{S}_\alpha\ . \tag{17}$$

Sketch of the proof: The case $\alpha = 0$ is trivial, so let us assume that $\alpha \neq 0$. Assume to the contrary that there exist such $x \in X$ and $z \in Z$ that

$$(\widetilde{R \lhd S})_\alpha(x,z) > (\tilde{R}_\alpha \lhd \tilde{S}_\alpha)(x,z)\ .$$

Clearly $(\widetilde{R \lhd S})_\alpha(x,z) > 0$, i.e.,

$$0 < \alpha \leq (R \lhd S)(x,z) = (\widetilde{R \lhd S})_\alpha(x,z)\ .$$

Now, let $Y' \subseteq Y$ such that $\forall y' \in Y'$: $R(x,y') < S(y',z)$. Evidently,

$$\bigwedge_{y' \in Y'} (R(x,y') \rightarrow S(y',z)) = 1$$
$$\bigwedge_{y' \in Y'} (\tilde{R}_\alpha(x,y') \rightarrow \tilde{S}_\alpha(y',z)) = 1\ .$$

However, for all $y'' \in Y \setminus Y'$ we get $R(x,y'') \geq S(y'',z)$ and also $S(y'',z) \geq \alpha$ because $(R \lhd S)(x,z) \geq \alpha$. This leads to $\tilde{R}_\alpha(x,y'') = R(x,y'')$ and $\tilde{S}_\alpha(y'',z) = S(y'',z)$, $\forall y'' \in Y \setminus Y'$. Thus,

$$(\widetilde{R \lhd S})_\alpha(x,z) = (R \lhd S)(x,z) = (\tilde{R}_\alpha \lhd \tilde{S}_\alpha)(x,z)$$

which is in contradiction with the original assumption. □

Proposition 6. *Let $\rightarrow$ from $\mathscr{L}$ be a Gödel implication. Then, for all $\alpha \in [0,1]$, it holds that*

$$(\widetilde{R \rhd S})_\alpha \subseteq \tilde{R}_\alpha \rhd \tilde{S}_\alpha \ . \tag{18}$$

Sketch of the proof: Using the same technique of contradiction as given in the proof of Proposition 5, let us focus on the case $\alpha > 0$. Indeed, let there exist $x \in X$ and $z \in Z$ such that

$$(\widetilde{R \rhd S})_\alpha(x,z) > (\tilde{R}_\alpha \rhd \tilde{S}_\alpha)(x,z) \ .$$

Then, it is clear that $0 < \alpha \leq (R \rhd S)(x,z) = (\widetilde{R \rhd S})_\alpha(x,z)$. Let $Y' \subseteq Y$ such that $\forall y' \in Y'$: $S(y',z) < R(x,y')$. Then, it holds that

$$\bigwedge_{y' \in Y'} (S(y',z) \rightarrow R(x,y')) = 1$$

$$\bigwedge_{y' \in Y'} (\tilde{S}_\alpha(y',z) \rightarrow \tilde{R}_\alpha(x,y')) = 1 \ .$$

Since $(R \rhd S)(x,z) \geq \alpha$, it must hold that $S(y'',z) \geq R(x,y'')$ and $R(x,y'') \geq \alpha$ for $y'' \in Y \setminus Y'$. Hence, $\tilde{R}_\alpha(x,y'') = R(x,y'')$, $\tilde{S}_\alpha(y'',z) = S(y'',z)$, $\forall y'' \in Y \setminus Y'$, and thus,

$$(\widetilde{R \rhd S})_\alpha(x,z) = (R \rhd S)(x,z) = (\tilde{R}_\alpha \rhd \tilde{S}_\alpha)(x,z)$$

which contradicts with the original assumption. □

Proposition 7 recalls the properties provided in [7].

Proposition 7. *[7] Let $\rightarrow$ from $\mathscr{L}$ be a Gödel implication. Then, for all $\alpha \in [0,1]$, it holds that*

$$(R \lhd S)_\alpha \subseteq R_\alpha \lhd S_\alpha \ ; \quad (R \rhd S)_\alpha \subseteq R_\alpha \rhd S_\alpha \ . \tag{19}$$

Inclusions (17), (18) and (19) lead to the following propositions.

Proposition 8. *Let $\rightarrow$ in $\mathscr{L}$ be a Gödel implication. Then, for all $\alpha \in [0,1]$, it holds that*

$$R_\alpha \lhd (S_1 \cup S_2)_\alpha \supseteq (R \lhd S_1)_\alpha \cup (R \lhd S_2)_\alpha \ ; \tag{20}$$
$$R_\alpha \rhd (S_1 \cup S_2)_\alpha \supseteq (R \rhd S_1)_\alpha \cap (R \rhd S_2)_\alpha \ ; \tag{21}$$
$$\tilde{R}_\alpha \lhd (\widetilde{S_1 \cup S_2})_\alpha \supseteq (\widetilde{R \lhd S_1})_\alpha \cup (\widetilde{R \lhd S_2})_\alpha \ ; \tag{22}$$
$$\tilde{R}_\alpha \rhd (\widetilde{S_1 \cup S_2})_\alpha \supseteq (\widetilde{R \rhd S_1})_\alpha \cap (\widetilde{R \rhd S_2})_\alpha \ ; \tag{23}$$
$$R_\alpha \lhd (S_1 \cap S_2)_\alpha \supseteq (R \lhd S_1)_\alpha \cap (R \lhd S_2)_\alpha \ ; \tag{24}$$
$$R_\alpha \rhd (S_1 \cap S_2)_\alpha \supseteq (R \rhd S_1)_\alpha \cup (R \rhd S_2)_\alpha \ ; \tag{25}$$
$$\tilde{R}_\alpha \lhd (\widetilde{S_1 \cap S_2})_\alpha \supseteq (\widetilde{R \lhd S_1})_\alpha \cap (\widetilde{R \lhd S_2})_\alpha \ ; \tag{26}$$
$$\tilde{R}_\alpha \rhd (\widetilde{S_1 \cap S_2})_\alpha \supseteq (\widetilde{R \rhd S_1})_\alpha \cup (\widetilde{R \rhd S_2})_\alpha \ . \tag{27}$$

Sketch of the proof: Property (22) is directly derived from (11) and (17). Similarly, property (24) is implied from (13) and (19). The other properties can be derived analogously. □

Proposition 9. (Residuation and Exchange properties) *Let $\rightarrow$ from $\mathscr{L}$ be a Gödel implication. Then, for all $\alpha \in [0,1]$, it holds that*

$$((R \circ S) \lhd T)_\alpha \subseteq R_\alpha \lhd (S \lhd T)_\alpha \subseteq (R_\alpha \circ S_\alpha) \lhd T_\alpha \ ;$$
$$\left(\widetilde{(R \circ S) \lhd T}\right)_\alpha \subseteq \tilde{R}_\alpha \lhd (\widetilde{S \lhd T})_\alpha \subseteq (\tilde{R}_\alpha \circ \tilde{S}_\alpha) \lhd \tilde{T}_\alpha \ ;$$
$$(R \lhd (S \rhd T))_\alpha \subseteq (R_\alpha \lhd S_\alpha) \rhd T_\alpha \ ;$$
$$\left(R \lhd \widetilde{(S \rhd T)}\right)_\alpha \subseteq \left(\tilde{R}_\alpha \lhd \tilde{S}_\alpha\right) \rhd \tilde{T}_\alpha \ .$$

Sketch of the proof: It can be observed that

$$\begin{aligned}((R \circ S) \lhd T)_\alpha = (R \lhd (S \lhd T))_\alpha &\subseteq R_\alpha \lhd (S \lhd T)_\alpha \\ &\subseteq R_\alpha \lhd (S_\alpha \lhd T_\alpha) = (R_\alpha \circ S_\alpha) \lhd T_\alpha \ .\end{aligned}$$

$$\begin{aligned}(R \lhd (S \rhd T))_\alpha \subseteq R_\alpha \lhd (S \rhd T)_\alpha &\subseteq R_\alpha \lhd (S_\alpha \rhd T_\alpha) \\ &= (R_\alpha \lhd S_\alpha) \rhd T_\alpha \ .\end{aligned}$$

The other properties are derived similarly. □

4 Triangular products of level relations and classification problem

4.1 Reasoning part

Although the BK-subproduct has an impressive impact on narrowing the classes serving the classification task, it usually bring a rather low accuracy. For this point, we may refer to the real experiment on the Dragonfly classification studied in the previous work [17]. One of the reasons is that the BK-subproduct uses the universal quantifier and some of the given features do not belong to a given class fully while a particular sample/animal carries those features fully or carries them in rather high truth degrees. To improve this drawback regarding the accuracy, we may use the BK-subproduct that applies to a level fuzzy relation and a level relation. In fact, we approach to employ $\tilde{R}_\beta \lhd S_\alpha$ so that the value $(\tilde{R}_\beta \lhd S_\alpha)(x,z)$ would give a desirable semantics, that "*all features carried by animal x in high degrees are essential features of class z*". This semantics resembles the human thinking when associating a class in nature to a particular animal. For example, given an animal which "has wings", "flies well", "be completely black", and "eats fruits sometimes". Then, one may immediately connect it with a kind of bird e.g., Raven, or Crow. Why it is so? Why it does not belong to the class of Bats as Bats have these features as well? In this case, he/she may give the reason, that it is due to "the three first mentioned features are essential/representative for the Birds class. Especially, black color is well-known for Crow or Raven". Note, if the BK-subproduct of standard fuzzy relations is applied then the obtained membership degree will be rather low as birds are colorful and thus, feature "be complete black" belongs to this class at a low truth degree. Consequently, it may lead to a misclassification to the Bats class, if the given animal is a Crow.

To simulate the proposed semantics, we opt a high value β for level fuzzy relation $\tilde{R}_\beta$. In this framework, we approach $\beta \geq 0.7$. Level relation S_α, which encodes the information for the features being essential for the classes, can be approached in the way that α is a minimum membership degree among $S(y,z)$ where y is considered as an undoubtedly essential feature of class z. In practical examples, α can be determined with help of an expert. Now, let z be a given class. Then, by undoubtedly essential features, we mean that those are *major features* which are carried by any animal in z, or *well-known features* of z (or a subclass of z). Note, the well-known features are not necessary to be major features due to the existence of some objects in a given class that do not carry them. For instance, major features of the Birds class are known as having feather, beak, wings, laying eggs, etc. while characteristic of "well flying", or "singing well" can be considered as a well-known feature.

Remark 1. *It is important to note that every feature y that belongs to a given class z in a non-zero degree, i.e., $S(y,z) > 0$, is possibly essential for class z. This is because if a particular animal carries such a feature then it can be associated to z as its correct class. Due to this fact, it should not be a conflict when elaborating a relation S_α based on determining features that are undoubtedly essential for class z. Indeed, if $S(y,z) \geq \alpha$ then y may be viewed as an essential one of z, i.e., $(y,z) \in S_\alpha$, even this feature is not visible in advance to be undoubtedly essential for z.*

The concept of unavoidable/typical features can be incorporated into the proposed BK-subproduct to increase the expressive power and the applicable potential. Then, the semantics of $(\tilde{R}_\beta \lhd S_\alpha)^{\rhd U}(x,z)$ can be formulated as "*all features that x carries in high degrees are essential for class z, and at the same time, all typical features of z are carried by x in high degrees*".

4.2 Illustrative example

In order to observe the behavior of the use of the proposed triangular product $(\tilde{R}_\beta \lhd S_\alpha)^{\rhd U}$, we consider a simple yet illustrative example regarding the classification problem of animals. The example is very similar yet slightly modified from the one provided in [4]. Let $X = \{x_1, x_2, \ldots, x_6\}$, $Y = \{y_1, y_2, \ldots, y_9\}$, $Z = \{z_1, z_2, \ldots, z_7\}$ be sets of particular animals, features of animals and classes of animals, respectively. The elements in X, Y, Z are given as

- x_1 – Platypus, x_2 – Emu, x_3 – Hairless dog, x_4 – Aligator, x_5 – Goldfish, x_6 – Puffin,
- y_1 – animal flies, y_2 – animal has feathers, y_3 – animal has fins, y_4 – animal has claws, y_5 – animal has hair, y_6 – animal has teeth, y_7 – animal has a beak, y_8 – animal has scales, y_9 – animal swims,
- z_1 – Bird, z_2 – Fish, z_3 – Dog, z_4 – Equidae, z_5 – Mosquito, z_6 – Monotreme, and z_7 – Reptile.

The given task is to classify the animals to their classes. Fuzzy relations $S \in \mathscr{F}(Y \times Z)$ and $R \in \mathscr{F}(X \times Y)$ are given in Table 1, Table 3. Furthermore, we may consider fuzzy relation $U \in \mathscr{F}(Y \times Z)$ regarding the unavoidable features as in Table 2. Value $U(y_2, z_1) = 1$ means that feature y_1 is truly carried by any animal in class z_1. $U(y_1, z_1) = 0$ stands for that y_1 is not an unavoidable feature of z_1 as it is known that some birds e.g., Emu or Penguins does not carry this feature at all. If it is an unavoidable feature, the use of the BK-superproduct does not classify these kinds of birds to z_1. Similarly, $U(y_8, z_2)$ should be equal to 0 as it is clear that e.g., Pangasius fish does not have scale.

S	z_1	z_2	z_3	z_4	z_5	z_6	z_7
y_1	0.8	0	0	0	1	0	0
y_2	1	0	0	0	0	0	0
y_3	0	1	0	0	0	0.5	0
y_4	0.9	0	1	0	0	0.8	0.3
y_5	0	0	0.8	1	0	0.9	0
y_6	0	0.6	1	1	0	0	0.7
y_7	1	0.1	0	0	0	0.5	0
y_8	0.7	0.9	0	0	0	0	1
y_9	0.5	1	0.8	0.6	0.1	0.7	0.8

Table 1: Relationship of features and classes.

For the illustrative purpose, we may approach the fuzzy cut of R at $\beta = 0.7$ i.e., consider $\tilde{R}_{0.7}$. Relation S_α is approached based on the determination of some well-known or major features of the given classes. This can be done by learning from the literature for the relationship between features and classes of animals. In details, features y_1, y_2, y_4, y_7 are undoubtedly essential for class z_1. Similarly, y_3, y_8, y_9 are indeed essential for z_2 as they play a very important role in identifying an object belonging to the class of Fish. Features y_4, y_5, y_6 are known for z_3, and y_5, y_6 are crucial for z_4. Similarly, y_5, y_7, y_9 can be seen as the obviously essential ones of z_6. Finally, it is most known that several

U	z_1	z_2	z_3	z_4	z_5	z_6	z_7
y_1	0	0	0	0	1	0	0
y_2	1	0	0	0	0	0	0
y_3	0	1	0	0	0	0	0
y_4	1	0	1	0	0	1	0
y_5	0	0	0	1	0	0.5	0
y_6	0	0	1	1	0	0	0.5
y_7	1	0	0	0	0	0.5	0
y_8	0	0	0	0	0	0	1
y_9	0	1	0	0	0	0.5	0

Table 2: Relationship regarding unavoidable features.

R	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
x_1	0	0	0	1	1	0	1	0	0.9
x_2	0	1	0	1	0	0	1	0.5	0.4
x_3	0	0	0	1	0.2	1	0	0	0.7
x_4	0	0	0	1	0	1	0	1	0.9
x_5	0	0	1	0	0	0.9	0	1	1
x_6	1	1	0	1	0	0	1	0.4	0.9

Table 3: Relationship of animals and features.

kinds of Reptile have claws such as Lizards, or Alligators, so apart from y_6, y_9, feature y_4 is taken into account to be important as well. Hence, we may approach $\alpha = S(y_4, z_7) = 0.3$ (and thus consider $S_{0.3}$) as a minimum truth degree among the ones assigning to the visibly essential features for the classes.

If we use the Gödel algebra as the underlying algebraic structure then we obtain the result given in Tables 4-6.

$R \lhd S$	z_1	z_2	z_3	z_4	z_5	z_6	z_7
x_1	0	0	0	0	0	0.5	0
x_2	0.9	0	0	0	0	0	0
x_3	0	0	1	0	0	0	0
x_4	0	0	0	0	0	0	0.3
x_5	0	0.6	0	0	0	0	0
x_6	0.5	0	0	0	0	0	0

Table 4: Subproduct of standard fuzzy relations.

The BK-subproduct $R \lhd S$ classifies each animal to its correct class, however, it lower too much the membership degrees. The triangular products $\tilde{R}_{0.7} \lhd S_{0.3}$ and $(\tilde{R}_{0.7} \lhd S_{0.3})^{\rhd U}$ bring a more promising result. In fact, apart from animal x_3 (Hairless dog), which has the suspicion belonging to class z_7 (Reptile) besides its correct one z_3, the composition $\tilde{R}_{0.7} \lhd S_{0.3}$ classifies the other animals into their unique correct classes with highest membership degrees 1. When the unavoidable features are applied, the product $(\tilde{R}_{0.7} \lhd S_{0.3})^{\rhd U}$ brings a significant improvement by eliminating false suspicions brought by $\tilde{R}_{0.7} \lhd S_{0.3}$. In particular, the membership degree expressing the belonging of x_3 into class z_7 is completely eliminated.

$\tilde{R}_{0.7} \lhd S_{0.3}$	z_1	z_2	z_3	z_4	z_5	z_6	z_7
x_1	0	0	0	0	0	1	0
x_2	1	0	0	0	0	0	0
x_3	0	0	1	0	0	0	1
x_4	0	0	0	0	0	0	1
x_5	0	1	0	0	0	0	0
x_6	1	0	0	0	0	0	0

Table 5: Subproduct of cutting relations.

$(\tilde{R}_{0.7} \lhd S_{0.3})^{\rhd U}$	z_1	z_2	z_3	z_4	z_5	z_6	z_7
x_1	0	0	0	0	0	1	0
x_2	1	0	0	0	0	0	0
x_3	0	0	1	0	0	0	0
x_4	0	0	0	0	0	0	1
x_5	0	1	0	0	0	0	0
x_6	1	0	0	0	0	0	0

Table 6: Subproduct with unavoidable features of cutting relations.

5 Dragonfly classification problem

This section demonstrates the impact of suggested triangular products i.e., $\tilde{R}_\beta \lhd S_\alpha$ and $(\tilde{R}_\beta \lhd S_\alpha)^{\rhd U}$ on the real classification of Odonata (Dragonflies). For the purpose of comparing with the former approaches of the compositions incorporating excluding features, unavoidable features, we consider the experiment that has been studied in [17]. The data-set of the Dragonfly classification contains 52940 testing samples of Dragonflies (X), 60 features of the particular samples (Y), and 140 classes (including males and females of 70 species) (Z). The used features cover 6 distinct colors, 14 intervals of altitudes, 36 decades in the year, and 4 morphological categories combining two kinds of Anisoptera and Zygoptera and their size (big/small). Fuzzy relations $S, E, U \in \mathscr{F}(Y \times Z)$ are constructed by an odonatologist which express the relationship between features and classes, excluding features and classes, and unavoidable features and classes, respectively. The dependence of considered features and testing samples are expressed in relation $R \in \mathscr{F}(X \times Y)$. The choice of the level fuzzy relation $\tilde{R}_\beta$ and level relation S_α is advisable to approach such that β is higher than or equal to 0.7 and α should not be greater than 0.5. To illustrate the results, let us approach $\beta = 0.7$, $\alpha = 0.5$ and $\alpha = 0.4$.

The aim is to classify each dragonfly into one of 140 classes. The following measures are used to evaluate the performance of the applied compositions [17]:

- rankM – the arithmetic mean (over X) of the numbers of classes with assigned membership degrees that are greater than or equal to the membership degree assigned to the correct class;
- #correctMax – the number of samples with assigned membership degrees to the correct classes are maximal comparing with the assigned membership degrees to the other classes.

The *rankM* represents the narrowness of the classes serving the task of the classification. Especially, it shows how many classes in average are in the suspicion to become the correct class of a given sample. Let us call the set of such potential classes as the *guessed set*. The measure #correctMax represents the "accuracy" of the classification that shows how many samples are truly classified to their correct classes in the highest truth degrees. Although there is a trade-off between the narrowness of the guessed set and the robustness of the accuracy when evaluating a method and compare it with others, it was discussed [17] that the narrower the guessed set is, the better for the system.

For the illustrative purpose, the Łukasiewicz MV-algebra has been used. The result is provided in Table 7. We note that the results obtained from the compositions differing from $\tilde{R}_\beta \lhd S_\alpha$ and $(\tilde{R}_\beta \lhd S_\alpha)^{\rhd U}$ have been discussed in detail in [17]. Among them, the use of the composition incorporating excluding features and unavoidable features $(R \circ S^{\backslash} E)^{\rhd U}$ brings the most promising result. Indeed, it shows the robustness on narrowing the guessed set from the initial suspicion 132.83 (given by $R \circ S$) down to 12.27 classes on average. Moreover, the accuracy of the classification is very high as it reaches 99.10%. Let us focus on the suggested triangular products. As the use of $\alpha = 0.4$ in the products provides the same results as that of $\alpha = 0.5$, it is sufficient to focus on only one case e.g., $\alpha = 0.5$. Subproduct $\tilde{R}_{0.7} \lhd S_{0.5}$ lowers the size of initial guessed set from 132.83 down to 22.68 and thus, it has a better performance than approaches $R \rhd S, R \square S, R \rhd U$, and $(R \circ S)^{\rhd U}$. Although it does not bring the maximal accuracy degree as in the case of using those approaches, the obtained degree is notably high (95.48%). Achieving such a high degree of accuracy accompanying with the noteworthy reduction on the size of the guessed set, demonstrates the effectiveness of the proposed BK-subproduct. The BK-subproduct of the standard fuzzy relations $R \lhd S$ provides a much weaker result in terms of accuracy (69.19%), though it yields an incredible result in narrowing the initial suspicions (from 132.83 to 9.88).

The composition $(\tilde{R}_{0.7} \lhd S_{0.5})^{\rhd U}$ employing unavoidable features brings a further improvement. In fact, comparing to $\tilde{R}_{0.7} \lhd S_{0.5}$, it remarkably decreases the size of the guessed set (from 22.68 to 17.63) while still preserving the accuracy in a high degree that even higher than 95.48% provided by $\tilde{R}_{0.7} \lhd S_{0.5}$. It is obvious that this approach does not outperform the model $(R \circ S^{\backslash} E)^{\rhd U}$. However, the difference in the number of the guessed classes is not too big (12.27 compared to 17.63), and at the same time, the drop in accuracy is negligible (99.10% compared to 96.09%). In the comparison with model $R \circ S^{\backslash} E$, the loss of the accuracy provided by the considered approach is also non-significantly (98.97% compared to 96.09%) while it may bring a better improvement on decreasing the average size of the guessed set.

6 Conclusion

We have studied various valid properties of the Bandler-Kohout products applying to level relations and level fuzzy relations. Furthermore, we have shown that those approaches of the triangular products may bring efficiency in serving the classification task. In other words, they construct the intuitive semantics suitable for the classification and their application brings a positive impact. Let us emphasize that it is not the aim to show that the suggested products outperform the other classification approaches, but to reveal their rather strong potential in practical applications. Also, their combination with the existing methods may form more effective and powerful tools. For instance, the combination with generalized quantifiers that use natural quantifiers such as "most", "majority" instead of universal quantifier "for all" is a trivial direction. Furthermore, future effort will be addressed to extending the study into different types of fuzzy relational compositions. Moreover, it should be a discussion regarding the change of the results when distinct threshold values are considered.

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References

[1] R. Belohlavek, Sup-t-norm and inf-residuum are one type of relational product: Unifying framework and consequences, Fuzzy Sets and Systems 197 (2012) 45–58.

Compositions	rankM	#correctMax
$R \circ S$	132.83	52940 (100.00%)
$R \lhd S$	9.88	36628 (69.19%)
$R \rhd S$ $(R \square S)$	140.00	52940 (100.00%)
$R \circ S^{\backslash} E$	18.37	52396 (98.97%)
$R \rhd U$	26.48	52940 (100.00%)
$(R \circ S)^{\rhd U}$	26.29	52940 (100.00%)
$(R \lhd S)^{\rhd U}$	7.67	38022 (71.82%)
$(R \circ S^{\backslash} E)^{\rhd U}$	12.27	52465 (99.10%)
$\tilde{R}_{0.7} \lhd S_{0.5}$	22.68	50545 (95.48%)
$\tilde{R}_{0.7} \lhd S_{0.4}$	22.68	50545 (95.48%)
$(\tilde{R}_{0.7} \lhd S_{0.5})^{\rhd U}$	17.63	50870 (96.09%)
$(\tilde{R}_{0.7} \lhd S_{0.4})^{\rhd U}$	17.63	50870 (96.09%)

Table 7: Dragonfly classification experimental results.

[2] L. Běhounek, M. Daňková, Relational compositions in fuzzy class theory, Fuzzy Sets and Systems 160 (8) (2009) 1005–1036.

[3] N. Cao, M. Štěpnička, M. Burda, A. Dolnỳ, How to enhance, use and understand fuzzy relational compositions, in: Beyond Traditional Probabilistic Data Processing Techniques: Interval, Fuzzy etc. Methods and Their Applications, Springer, 2020, pp. 121–136.

[4] N. Cao, M. Štěpnička, M. Burda, A. Dolný, Excluding features in fuzzy relational compositions, Expert Systems with Applications 81 (2017) 1–11.

[5] N. Cao, M. Štěpnička, M. Holčapek, Extensions of fuzzy relational compositions based on generalized quantifer, Fuzzy Sets and Systems 339 (2018) 73 – 98.

[6] B. De Baets, E. Kerre, Fuzzy relational compositions, Fuzzy Sets and Systems 60 (1993) 109–120.

[7] B. De Baets, E. Kerre, The cutting of compositions, Fuzzy Sets and Systems 62 (3) (1994) 295–309.

[8] A. Di Nola, S. Sessa, W. Pedrycz, E. Sanchez, Fuzzy relation equations and their applications to knowledge engineering, Vol. 3, Springer Science & Business Media, 2013.

[9] S. Gottwald, W. Pedrycz, Solvability of fuzzy relational equations and manipulation of fuzzy data, Fuzzy Sets and Systems 18 (1) (1986) 45–65.

[10] R. Groenemans, E. Van Ranst, E. Kerre, Fuzzy relational calculus in land evaluation, Geoderma 77 (2-4) (1997) 283–298.

[11] S. Kannan, R. K. Mohapatra, New notions for fuzzy equivalence using α-cut relation, Journal of Physics: Conference Series 1344 (1) (2019) 012040.

[12] L. J. Kohout, E. Kim, The role of bk-products of relations in soft computing, Soft Computing 6 (2) (2002) 92–115.

[13] C. K. Lim, C. S. Chan, A weighted inference engine based on interval-valued fuzzy relational theory, Expert Systems with Applications 42 (7) (2015) 3410–3419.

[14] W.-N. Liu, J. Yao, Y. Yao, Rough approximations under level fuzzy sets, in: International Conference on Rough Sets and Current Trends in Computing, Springer, 2004, pp. 78–83.

[15] A. Pourabdollah, J. M. Mendel, R. I. John, Alpha-cut representation used for defuzzification in rule-based systems, Fuzzy Sets and Systems 399 (2020) 110–132.

[16] T. Radecki, Fuzzy set theoretical approach to document retrieval, Information Processing & Management 15 (5) (1979) 247–259.

[17] M. Štěpnička, N. Cao, M. Burda, A. Dolnỳ, S. Ožana, The concept of unavoidable features in fuzzy relational compositions, Knowledge-Based Systems 196 (2020) 105785.

[18] M. Stepnicka, B. Jayaram, On the suitability of the bandler–kohout subproduct as an inference mechanism, IEEE Transactions on Fuzzy Systems 18 (2) (2010) 285–298.

[19] H. P. Vigier, A. Terceño, A model for the prediction of “diseases” of firms by means of fuzzy relations, Fuzzy Sets and Systems 159 (17) (2008) 2299–2316.

[20] R. B. Zenner, R. M. De Caluwe, E. E. Kerre, A new approach to information retrieval systems using fuzzy expressions, Fuzzy Sets and Systems 17 (1) (1985) 9–22.