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Fuzzy Relations and Monometrics: Some Correspondences

Megha Gupta [a] and *Balasubramaniam Jayaram*[a]
[a]Department of Mathematics, Indian Institute of Technology Hyderabad, INDIA.
ma16m18p100001@iith.ac.in,jbala@math.iith.ac.in

Abstract

Fuzzy relations that are not T-transitive have not merited as much scrutiny as their T-transitive counterparts. Recently monometrics on a given betweenness set, or a B-set, has garnered a lot of attention, especially for their role in decision making and penalty based data aggregation. In this work, we show that from non strict T-transitive relations one can obtain both betweenness relations, satisfying various types of transitivities, and monometrics on them. In fact, it can be shown that the natural or canonical betweenness obtained from a given distance function can be seen as being obtained from an appropriate non strict T-transitive similarity relation. Further, a typical practical problem in this context is that of obtaining a monometric on a given B-set. Our investigations show that there exists a one-one correspondence between non strict T-transitive similarity relations and monometrics defined on the given B-set. This study, thus, shows the usefulness of non strict T-transitive fuzzy relations as much as their T-transitive counterparts.

Keywords: Betweenness relation, Monometric, Fuzzy relations, Non Strict T-transitivity.

1 Introduction

Fuzzy relations on an $\mathscr{X} \neq \emptyset$ that are T-transitive, i.e., satisfy the following inequality

$$T(E(x,y),E(y,z)) \leq E(x,z)\ ,\ x,y,z \in \mathscr{X}\ . \qquad \text{(T-T)}$$

w.r.t. a t-norm T are very well studied.

It is well-known that a pseudo-metric d on $\mathscr{X}$ induces a corresponding equivalence relation (see Section 2.1) E on $\mathscr{X}$ - which is T-transitive w.r.t. an Archimedean generated t-norm T - and vice-versa [1]. Further, De Baets and Mesiar [2] have carried out a complete study on the relationship between T-Equalities and metrics. It is clear that if an E satisfies (T-T) w.r.t. a t-norm T, then it is T-transitive also for any $T' < T$. However, there may exist t-norms $T^* > T$ w.r.t. which E may not satisfy (T-T). For instance, let $\hat{T}$ denote the largest t-norm w.r.t. which an E satisfies (T-T). Clearly, then for every $T^* > \hat{T}$ there will exist $x,y,z \in \mathscr{X}$ such that the inequality in (T-T) will fail. In fact, the minimum t-norm is one such that can be considered for the T^*.

Example 1 ([1], Example 2)**.** *Consider a finite partition $\{A_1,\cdots,A_n\}$, $n \in \mathbb{N}_o$, of a universe $\mathscr{X}$ and define the mapping $k : \mathscr{X} \to \{1,\cdots,n\}$ by $k(x) = i \iff x \in A_i$. Then the following similarity relation (see Section 2.1)*

$$E(x,y) = \frac{1}{|k(x)-k(y)|+1}$$

*is a non-*min *transitive relation on the set $\mathrm{B}_{\preceq} = \{(x,y,z) \in \mathscr{X}^3 \mid k(x) < k(y) < k(z)\}$. Clearly, the triple (x,y,z) such that $x \in A_1$, $y \in A_2$ and $z \in A_3$ is one such for which the T-transitivity does not hold with $T = \min$, since $E(x,y) = E(y,z) = \frac{1}{2}$ while $E(x,z) = \frac{1}{3}$.*

Example 2 ([1], Example 3)**.** *Consider $\mathscr{X} = \{x,y,z\}$ and define the similarity relation E as follows:*

$$E(x,y) = 1$$
$$E(y,z) = 1$$
$$E(x,z) = 0.5$$

Then, the triple (x,y,z) is one such for which the T-transitivity does not hold with any t-norm T, since $T(E(x,y),E(y,z)) = 1$ while $E(x,z) = 0.5$.

In short, unless an E is min-transitive, there will always exist a t-norm T and $x,y,z \in \mathscr{X}$ such that the strict T-transitivty (see Section 2.1) is invalid. Given a fuzzy relation E and a t-norm T, we say that E is not strictly T-transitive if there exist $x,y,z \in \mathscr{X}$ s.t.

$$T(E(x,y),E(y,z)) \geq E(x,z) \ . \tag{1}$$

Similarly, E is considered non T-transitive if for some $x,y,z \in \mathscr{X}$ the strict inequality in (1) is satisfied.

1.1 Motivation for this work:

Non strict T-transitive relations have not merited any attention so far. In this submission, we investigate these relations, especially those that are also similarity relations, i.e., reflexive and symmetric, and investigate the correspondences, if any, between such fuzzy relations and some known types of metrics or distance functions.

1.2 Contributions of the work:

Recently, the role of monometrics m on a betweenness set $(\mathscr{X},\mathrm{B})$, i.e., a set $\mathscr{X} \neq \emptyset$ with a special ternary relation B, has started attracting attention [6] [5]. Pérez-Fernández and De Baets associated a betweenness relation with a metric space such that the metric becomes a monometric on this betweenness relation [5].

Our investigations on non strict T-transitive relations have led us to the following twin contributions.

Firstly, we show that from similarity relations E that are not strictly T-transitive we can obtain such betweenness relations - denoted $\mathrm{B}_{E,T}, \mathrm{B}^*_{E,T}$ - on the set and in some special cases show that these are related to the natural or canonical betweenness obtained from a given distance function.

A typical practical problem, in this context, is that of obtaining a monometric on a given B-set. Our second contribution is in showing the existence of a correspondence between non strict T-transitive similarity relations and monometrics defined on the given B-set. The betweenness relations $\mathrm{B}_{E,T}, \mathrm{B}^*_{E,T}$ defined using the pair (E,T) as above are seen to play an important role in obtaining these monometrics.

2 Non Strict T-Transitivity and Betweenness Relations

In this section, we discuss two ways of obtaining betweenness relations, viz., (BET) and (BET*) , from fuzzy similarity relations that are not strictly T-transitive and show that, in fact, one of them is in one-one correspondence to the natural or canonical betweenness relation obtained from distance functions [5].

2.1 Fuzzy Relations and their Transitivities

Fuzzy binary relations on $\mathscr{X} \neq \emptyset$ have been studied along different approaches - from being part of clustering ensembles and inference schemes in practical applications to obtaining relaxed partitions and, specifically for their correspondences to special distance functions, in theoretical settings. In each of these contexts these relations are expected to satisfy further desirable properties, for instance to determine whether they capture the underlying similarity or dissimilarity.

In the sequel, a symmetric fuzzy binary relation R on an $\mathscr{X} \neq \emptyset$ is said to be

- a similarity relation, if it is reflexive, i.e., $R(x,x) = 1$ for all $x \in \mathscr{X}$.
- a strong similarity relation, if it is strongly reflexive, i.e., $R(x,y) = 1 \iff x = y$.
- T-equivalence (T-equality) relation on $\mathscr{X}$ if it is a similarity (strong similarity) relation that satisfies T-transitivity (T-T) w.r.t. a t-norm T.
- Strict T-equivalence (T-equality) relation on $\mathscr{X}$ if it is a similarity (strong similarity) relation that satisfies the strict inequality of T-transitivity (T-T) w.r.t. a t-norm T.

2.2 Betweenness Relations on an $\mathscr{X}$

There are many definitions of betweenness since the work of [3], see for instance, [7], [4, 5] all of which seem to differ in the type of transitivities considered.

Definition 1. *Let* B *be a ternary relation on an* $\mathscr{X} \neq \emptyset$ *and consider the following properties: For any* $o,x,y,z \in \mathscr{X}$,

$$(x,y,z) \in \mathrm{B} \iff (z,y,x) \in \mathrm{B} \ , \tag{BS}$$

$$(x,y,z) \wedge (x,z,y) \in \mathrm{B} \iff y = z \ , \tag{BU}$$

$$(x,y,z) \in \mathrm{B} \Longrightarrow x \neq y \neq z \ . \tag{BD}$$

and the following set of transitivities:

$$\left.\begin{array}{r}(o,x,y) \in \mathrm{B}\\ \wedge\ (o,y,z) \in \mathrm{B}\\ \wedge\ (x,y,z) \in \mathrm{B}\end{array}\right\} \Longrightarrow (o,x,z) \in \mathrm{B} \ , \tag{BT}$$

$$\left.\begin{array}{r}(o,x,y) \in \mathrm{B}\\ \wedge\ (o,y,z) \in \mathrm{B}\end{array}\right\} \iff \left\{\begin{array}{ll} & (o,x,z) \in \mathrm{B}\\ \wedge & (x,y,z) \in \mathrm{B}\end{array}\right. , \tag{BST}$$

$$\left.\begin{array}{r}(o,x,y) \in \mathrm{B}\\ \wedge\ (o,y,z) \in \mathrm{B}\\ \wedge\ (x,y,z) \notin \mathrm{B}\end{array}\right\} \Longrightarrow (o,x,z) \in \mathrm{B} \ . \tag{BWT}$$

- B *is said to be a **betweenness** relation if* B *satisfies* (BS), (BU) *and any one of the transitivities, viz.,* (BST), (BT) *or* (BWT).
- B *is said to be a **strict betweenness** relation if* B *satisfies* (BS), (BD) *and any one of the transitivities, viz.,* (BST), (BT) *or* (BWT).

Remark 1. • $(\mathscr{X}, \mathrm{B})$ *is known as a (strict) Betweenness set or a (strict)* B*-set. Also,* $(x,y,z) \in \mathrm{B}$ *is read as 'y is in between x and z'.*

- *Clearly, only one of* (BU) *or* (BD) *can be true for any ternary relation* B.
- *However, while* (BST) *implies* (BT)*, neither does* (BST) *or* (BT) *imply* (BWT) *nor does* (BWT) *imply either of* (BST) *or* (BT).

2.3 Betweenness Relations from Similarity Relations

In this section, we derive two types of betweenness relations on a given set $\mathscr{X}$ based on the similarity relations defined on it.

Theorem 1. *Let* $E : \mathscr{X} \times \mathscr{X} \to [0,1]$ *be a strong similarity relation and let* T *be a t-norm. The ternary relation* $\mathrm{B}_{E,T}$ *defined as in* (BET) *is a betweenness relation and satisfies the* (BT) *transitivity:*

$$\mathrm{B}_{E,T} = \{(x,y,z) \in \mathscr{X}^3 \mid E(x,z) = T[E(x,y),E(y,z)]\} . \tag{BET}$$

Proof. Firstly, note that $\mathrm{B}_{E,T} \neq \emptyset$ since $(x,x,x) \in \mathrm{B}_{E,T}$ for every $x \in \mathscr{X}$.

(a) Let $(x,y,z) \in \mathrm{B}_{E,T}$. By the commutativity of T and E, we have that $(z,y,x) \in \mathrm{B}_{E,T}$ and hence $\mathrm{B}_{E,T}$ satisfies (BS).

(b) Let $(x,y,z) \in \mathrm{B}_{E,T}$ and $(x,z,y) \in \mathrm{B}_{E,T}$. Then from the definition of E and the associativity of T we obtain $E(x,y) = T[E(x,y),T[E(y,z),E(z,y)]]$. From this, it follows that

$$T[E(y,z),E(y,z)] = 1 \implies E(y,z) = 1 \implies y = z .$$

(c) Let $(o,x,y),(o,y,z),(x,y,z) \in \mathrm{B}_{E,T}$. Then the following chain of equalities show that $(o,x,z) \in \mathrm{B}_{E,T}$:

$$\begin{aligned} E(o,z) &= T[E(o,y),E(y,z)] , \\ &= T[T[E(o,x),E(x,y)],E(y,z)] , \\ &= T[E(o,x),T[E(x,y),E(y,z)]] , \\ &= T[E(o,x),E(x,z)] . \end{aligned}$$

□

The following result can be proven similarly.

Theorem 2. *Let* $E : \mathscr{X} \times \mathscr{X} \to [0,1]$ *be a similarity relation, not necessarily strong, and let* T *be a t-norm s.t.* E *is not* T*-transitive. The ternary relation* $\mathrm{B}^*_{E,T}$ *defined as in* (BET*) *is a strict betweenness relation and satisfies* (BWT)*:*

$$\mathrm{B}^*_{E,T} = \{(x,y,z) \in \mathscr{X}^3 \mid E(x,z) < T[E(x,y),E(y,z)]\} . \tag{BET*}$$

Remark 2. *(i) If* $(x,y,z) \in \mathrm{B}^*_{E,T}$*, then* $E(x,z) < 1$.

(ii) If $(x,y,z) \in \mathrm{B}^*_{E,T}$*, then* $E(x,y) > 0$ *and* $E(y,z) > 0$. *Equivalently, if* $E(x,y) = 0$ *then* $(x,y,z) \notin \mathrm{B}^*_{E,T}$ *for any* $z \in \mathscr{X}$.

(iii) If $(x,y,z) \in \mathrm{B}^*_{E,T}$*, then* $E(x,z) < E(x,y)$ *and* $E(x,z) < E(y,z)$.

Example 3. *(i) Let* $\mathscr{X} = \mathbb{R}$. *Define the similarity relation as follows:*

$$E(x,y) = \begin{cases} 1, & x = y , \\ \dfrac{1+|x-y|}{2+3|x-y|}, & x \neq y . \end{cases}$$

Then E *is not* $T_{\mathbf{M}}$*-transitive. For instance, consider the triple* $(x,y,z) = (0,1,2)$. *Then*

$$T_{\mathbf{M}}(E(x,y),E(y,z)) = 0.4 \nleq 0.375 = E(x,z) .$$

The betweenness relations $\mathrm{B}_{E,T_{\mathbf{M}}}, \mathrm{B}^*_{E,T_{\mathbf{M}}}$ *are given as follows:*

$$\mathrm{B}_{E,T_{\mathbf{M}}} = \big\{(x,y,z) \in \mathscr{X}^3 \mid [x=y] \vee [y=z]\big\} .$$

$$\mathrm{B}^*_{E,T_{\mathbf{M}}} = \left\{(x,y,z) \in \mathscr{X}^3 \middle| \begin{array}{c} [|x-y| < |x-z|] \\ \wedge \; [|y-z| < |x-z|] \end{array}\right\} .$$

(ii) Let $\mathscr{X} = [0,1]$. *Define the similarity relation as follows:*

$$E(x,y) = 1-(x-y)^2 .$$

Then E *is not* $T_{\mathbf{LK}}$*-transitive. For instance, consider the triple* $(x,y,z) = (0,0.5,1)$. *Then*

$$T_{\mathbf{LK}}(E(x,y),E(y,z)) = 0.5 \nleq 0 = E(x,z) .$$

The betweenness relations $\mathrm{B}_{E,T_{\mathbf{LK}}}, \mathrm{B}^*_{E,T_{\mathbf{LK}}}$ *are given as follows:*

$$\mathrm{B}_{E,T_{\mathbf{LK}}} = \left\{(x,y,z) \in \mathscr{X}^3 \middle| \begin{array}{rc} & [y=x] \\ \vee & [y=z] \\ \vee & [(x-z)^2 = 1] \end{array}\right\} .$$

$$\mathrm{B}^*_{E,T_{\mathbf{LK}}} = \left\{(x,y,z) \in \mathscr{X}^3 \middle| \begin{array}{rc} & [x<y<z] \\ \vee & [z<y<x] \\ \vee & [(x-z)^2 > 1] \end{array}\right\} .$$

(iii) *Let $\mathscr{X} = \mathbb{R}$. Define the similarity relation as follows:*

$$E(x,y) = \frac{1}{1+(x-y)^2}.$$

Then E is not-$T_{\mathbf{M}}$-transitive. For instance, consider the triple $(x,y,z) = (0,1,2)$. Then

$$T_{\mathbf{M}}(E(x,y),E(y,z)) = 0.5 \not\leq 0.2 = E(x,z)\ .$$

The betweenness relations $\mathrm{B}_{E,T_{\mathbf{M}}}, \mathrm{B}^*_{E,T_{\mathbf{M}}}$ are given as follows:

$$\mathrm{B}_{E,T_{\mathbf{M}}} = \left\{ (x,y,z) \in \mathscr{X}^3 \,\middle|\, \begin{matrix} [y=z] \vee [y+z=2x] \\ \wedge \\ [y=x] \vee [x+y=2z] \end{matrix} \right\}.$$

$$\mathrm{B}^*_{E,T_{\mathbf{M}}} = \left\{ (x,y,z) \in \mathscr{X}^3 \,\middle|\, \begin{matrix} [z<y] \vee [2x<y+z] \\ \wedge \\ [y<x] \vee [x+y<2z] \end{matrix} \right\}.$$

2.4 Betweenness from distance functions

Betweenness relations have also been generated from distance functions. In this section we begin by presenting the natural betweenness relation B_d that can be obtained from a metric d on $\mathscr{X}$ [5]. However, as can be seen, this relation can be generalised for arbitrary distance functions too.

Definition 2. *Let $d : \mathscr{X} \times \mathscr{X} \to [0,\infty)$ and consider the following properties for any $x,y,z \in \mathscr{X}$:*

$$d(x,x) = 0\ , \qquad \text{(P1*)}$$

$$d(x,y) = 0 \iff x = y\ , \qquad \text{(P1)}$$

$$d(x,y) = d(y,x)\ , \qquad \text{(P2)}$$

$$d(x,z) \leq d(x,y) + d(y,z)\ . \qquad \text{(P3)}$$

*d is called a **distance** function if it satisfies* (P1*) *and* (P2). *Further, it is called a pseudo-metric if it also satisfies* (P3). *A pseudo-metric is a metric if it also satisfies* (P1).

Definition 3. *Consider a distance d on a set $\mathscr{X}$. Let us define ternary relations $\mathrm{B}_d, \mathrm{B}^*_d$ as follows:*

$$\mathrm{B}_d = \{(x,y,z) \in \mathscr{X}^3 \mid d(x,z) = d(x,y) + d(y,z)\}\ , \qquad \text{(NBD)}$$

$$\mathrm{B}^*_d = \{(x,y,z) \in \mathscr{X}^3 \mid d(x,z) > d(x,y) + d(y,z)\}\ . \qquad \text{(NBD*)}$$

Theorem 3 (cf. [5])**.** B_d *is a betweenness relation. Further,*

1. B_d *satisfies the* (BT) *transitivity but not the* (BWT).
2. *If d satisfies the triangle inequality then* B_d *enjoys the* (BST) *transitivity.*

Theorem 4. B^*_d *is a strict betweennness relation and satisfies the* (BWT) *transitivity.*

Example 4. *Let $x,y \in \mathbb{R}$. Let us consider a distance function $d(x,y) = (x-y)^2$.*

(i) *While $\mathrm{B}_d = \{(x,y,z) \mid y = x \text{ or } y = z\}$ we have $\mathrm{B}^*_d = \{(x,y,z) \mid x<y<z \vee z<y<x\}$.*

(ii) *Interestingly, both B_d and B^*_d satisfy* (BST) *and* (BT) *transitivities, while* (BWT) *is vacuously true.*

Example 5. *Let $x,y \in \mathbb{R}^m$. Let us consider the fractional distance*

$$d(x,y) = \left(\sum_{i=1}^{m} |x_i - y_i|^p \right)^{\frac{1}{p}}\ ,$$

where $0 < p < 1$. It can be shown that B^*_d does not satisfy either (BST) *or* (BT).

Let us consider the case of $p = \frac{1}{2}$ and $m = 2$, i.e., $x,y,z \in \mathbb{R}^2$. Then the following shows that while some set of triples may satisfy (BST)*, in general it is not true.*

(i) *Consider the points $O = (1.05, 0.78)$, $x = (0.21, 0.9)$, $y = (0.25, 0.25)$ and $z = (0.28, -1.13) \in \mathbb{R}^2$. Then it can be easily verified that $(O,x,y), (O,y,z), (x,y,z)$ and $(O,x,z) \in \mathrm{B}^*_d$, clearly showing that B^*_d satisfies* (BST).

(ii) *On the other hand, if we consider the points $O = (0.28, -1.14)$, $x = (0.4, -1)$, $y = (-1.69, -0.97)$ and $z = (-1.59, -0.12) \in \mathbb{R}^2$, it can be easily verified that $(O,x,y), (O,y,z), (x,y,z) \in \mathrm{B}^*_d$ but $(O,x,z) \notin \mathrm{B}^*_d$, clearly showing that B^*_d does not satisfy either* (BST) *or* (BT).

2.5 Inclusions among $\mathrm{B}_{E,T}, \mathrm{B}^*_{E,T}, \mathrm{B}_d$ and B^*_d

Intrestingly, as the following result shows the natural or canonical betweenness relation B_d obtained from a distance function through (NBD) is related to, and in some cases in one-one correspondence with, the ternary relation $\mathrm{B}_{E,T}$ from (BET).

Lemma 1. *(i) Let d be a distance function on $\mathscr{X}$ and B_d be the betweenness relation obtained through* (NBD). *Then there exists a pair (E,T) such that E is a similarity relation that is not strictly T-transitive and $\mathrm{B}_d \subseteq \mathrm{B}_{E,T}$.*

(ii) *Let T be a strict Archimedean t-norm with an additive generator f and E be a non strict T-transitive relation on $\mathscr{X}$ and $\mathrm{B}_{E,T}$ be the betweenness relation obtained through* (BET). *Then there exists a distance function d on $\mathscr{X}$ such that $\mathrm{B}_{E,T} \subseteq \mathrm{B}_d$.*

Proof. (i) Let us assume that $(x,y,z) \in \mathrm{B}_d$. Then,

$$d(x,z) = d(x,y) + d(y,z).$$

Let T be a strict Archimedean t-norm with additive generator f (i.e., $f(0) = \infty$) and define $E : \mathscr{X} \times \mathscr{X} \to [0,1]$ by

$$E(x,y) = f^{-1}\left(d(x,y)\right) .$$

Clearly, E is a similarity relation, i.e., reflexive and symmetric. Now, for any $(x,y,z) \in \mathrm{B}_d$, we have

$$\begin{aligned}
&E(x,z) = f^{-1}\left(d(x,z)\right) ,\\
\Longrightarrow\ &E(x,z) = f^{-1}\left(d(x,y) + d(y,z)\right) ,\\
\Longrightarrow\ &E(x,z) = f^{-1}\left(f(E(x,y)) + f(E(y,z))\right) ,\\
\Longrightarrow\ &E(x,z) = T(E(x,y),E(y,z)) ,\\
\Longrightarrow\ &(x,y,z) \in \mathrm{B}_{E,T} \text{ i.e., } \mathrm{B}_d \subseteq \mathrm{B}_{E,T} .
\end{aligned}$$

(ii) Let us define $d : \mathscr{X} \times \mathscr{X} \to [0,\infty)$ as

$$d(x,y) = f(E(x,y)).$$

Clearly, d satisfies both (P1*) and (P2) and hence is a distance function. For any $(x,y,z) \in \mathrm{B}_{E,T}$,

$$\begin{aligned}
&T(E(x,y),E(y,z)) = E(x,z)\\
\Longrightarrow\ &f^{-1}\left(f(E(x,y)) + f(E(y,z))\right) = E(x,z)\\
\Longrightarrow\ &f(E(x,y)) + f(E(y,z)) = f(E(x,z))\\
\Longrightarrow\ &d(x,y) + d(y,z) = d(x,z) ,
\end{aligned}$$

and thus $\mathrm{B}_{E,T} \subseteq \mathrm{B}_d$.

□

Lemma 2. *(i) Let d be a distance function on $\mathscr{X}$ and B^*_d be the betweenness relation obtained through* (NBD*)*. Then there exists a pair (E,T) such that E is a similarity relation that is not strictly T-transitive and $\mathrm{B}^*_d \subseteq \mathrm{B}^*_{E,T}$.*

(ii) Let T be a strict Archimedean t-norm with an additive generator f and E be a non strict T-transitive relation on $\mathscr{X}$ and $\mathrm{B}^*_{E,T}$ be the strict betweenness relation obtained through (BET*)*. Then there exists a distance function d on $\mathscr{X}$ such that $\mathrm{B}^*_{E,T} \subseteq \mathrm{B}^*_d$.*

Example 6. *Let $\mathscr{X} = [0,1]$. Consider the distance functions d from Examples 4 and 5, which do not satisfy* (P3)*. Let us define $E(x,y) = e^{-d(x,y)}$ and take T as the product t-norm $T_{\mathbf{P}}$. Then, $\mathrm{B}_d = \mathrm{B}_{E,T_{\mathbf{P}}}$ and $\mathrm{B}^*_d = \mathrm{B}^*_{E,T_{\mathbf{P}}}$.*

3 Monometrics and non strict T-Transitive relations

So far we have studied betweenness relations that were obtainable from both non strict T-transitive fuzzy relations and distance functions. A typical practical problem is that of obtaining a monometric on a given B-set.

In this section, firstly, we show that non strict T-transitive similarity relations E not only give us betweenness relations but also derivable monometrics on the generated B-set. Secondly, we show that, given a B-set $(\mathscr{X},\mathrm{B})$, there exists a one-one correspondence between non strict T-transitive similarity relations and monometrics on it, thus highlighting the important role played by non strict T-transitive fuzzy relations in obtaining monometrics.

3.1 Monometrics on a B-set $(\mathscr{X},\mathrm{B})$

We begin with a slightly more generalised definition of a monometric than what is presented in [4].

Definition 4 (cf. [4], Definition 6)**.** *Consider a betweenness set $(\mathscr{X},\mathrm{B})$. A distance function $m : \mathscr{X} \times \mathscr{X} \to [0,\infty)$ is called a monometric (w.r.t. B) if for every $(x,y,z) \in \mathrm{B}$, it holds that:*

$$\max(m(x,y),m(y,z)) \le m(x,z) . \qquad \text{(MC)}$$

Example 7. *Let $(\mathscr{X},\preceq)$ be a partially ordered set and let us define a betweenness relation B on $\mathscr{X}$ by*

$$\mathrm{B}_{\preceq} = \{(x,y,z) \in \mathscr{X}^3 \mid x \preceq y \preceq z \ \vee\ z \preceq y \preceq x\}.$$

(i) Consider $\mathscr{X} = [0,1]$ and define $x \preceq y \iff x \le y$. The distance $m(x,y) = (x-y)^2$ is a monometric w.r.t. $\mathrm{B}_{\preceq}$.

(ii) Consider $\mathscr{X} = \mathbb{R}^2$ and define $x \preceq y \iff x_1 \le y_1$ and $x_2 \le y_2$. Clearly, $m(x,y) = \|x-y\|_2$ is a monometric w.r.t. $\mathrm{B}_{\preceq}$.

(iii) Consider $\mathscr{X} = [-1,1]^2$ and define

$$\bar{x} \preceq \bar{y} \iff \begin{cases} & \left[\bar{x} = \bar{y}\right] ,\\ \text{or if} & \left[|x_1| \ge |y_1|\right] \text{and} \left[|x_2| > |y_2|\right] ,\\ \text{or if} & \left[|x_1| > |y_1|\right] \text{and} \left[|x_2| \ge |y_2|\right] . \end{cases}$$

Clearly, $m(x,y) = \sum_{i=1}^{2} \big||x_i| - |y_i|\big|$ is a monometric w.r.t. $\mathrm{B}_{\preceq}$.

It is well known that given a distance function d, on a set equipped with the corresponding betweenness relation B_d, acts as a monometric, see [5]. In fact, it can be easily seen that d is a monometric w.r.t. the obtained strict betweenness relation B^*_d. The following results show that not only do non strict T-transitive similarity relations E give us betweenness relations but also derivable monometrics on the generated B-set.

Theorem 5. *Let $E : \mathscr{X} \times \mathscr{X} \to [0,1]$ be a similarity relation and let T be a t-norm s.t. E is non T-transitive. Then there exists a monometric on the strict* B*-set* $(\mathscr{X}, \mathrm{B}^*_{E,T})$*.*

Proof. Firstly, note that since E is a similarity relation on $\mathscr{X}$, we have from Theorem 2 that $(\mathscr{X}, \mathrm{B}^*_{E,T})$ is a strict B-set. Since $T \leq \min$ we have that for every $(x,y,z) \in \mathrm{B}^*_{E,T}$, we have

$$\min(E(x,y), E(y,z)) \geq T(E(x,y), E(y,z)) > E(x,z) ,$$

and hence $\mathrm{B}^*_{E,T} \subseteq \mathrm{B}^*_{E,T_\mathbf{M}}$ and $(\mathscr{X}, \mathrm{B}^*_{E,T_\mathbf{M}})$ is also a strict B-set. Let $f : [0,1] \to [0,\infty]$ such that $f(1) = 0$ be any strictly decreasing continuous function, then the following inequalities show that $m_E = f \circ E$ is a monometric on $(\mathscr{X}, \mathrm{B}^*_{E,T})$: For any $(x,y,z) \in \mathrm{B}^*_{E,T}$, we have

$$\begin{aligned} & T(E(x,y), E(y,z)) > E(x,z) \\ \Longrightarrow\ & \min(E(x,y), E(y,z)) > E(x,z) \\ \Longrightarrow\ & f(\min(E(x,y), E(y,z)) < f(E(x,z)) \\ \Longrightarrow\ & \max(f \circ E(x,y), f \circ E(y,z)) \leq f \circ E(x,z) \\ \Longrightarrow\ & \max(m_E(x,y), m_E(y,z)) \leq m_E(x,z) . \end{aligned}$$

□

Theorem 6. *Let $E : \mathscr{X} \times \mathscr{X} \to [0,1]$ be a similarity relation which is non strict T-transitive for some t-norm T. Then there exists a monometric on the* B*-set* $(\mathscr{X}, \mathrm{B}_{E,T})$*.*

The final result in this section shows that given a B-set $(\mathscr{X}, \mathrm{B})$, there exists a one-one correspondence between non strict T-transitive similarity relations and monometrics defined on B.

Theorem 7. *Consider a betweenness set* $(\mathscr{X}, \mathrm{B})$*.*

(i) If $m : \mathscr{X} \times \mathscr{X} \to [0,\infty)$ is a monometric w.r.t. B*, then there exists a t-norm T and a similarity relation $E : \mathscr{X} \times \mathscr{X} \to [0,1]$ such that E is a non strict T-transitive similarity relation on $\mathscr{X}$.*

(ii) If $E : \mathscr{X} \times \mathscr{X} \to [0,1]$ is a non strict T-transitive similarity relation on $\mathscr{X}$ w.r.t. a t-norm T, then there exists a monometric m on B*-set.*

Example 8. *(i) For the Example 7 (i), if we define $E : \mathscr{X} \times \mathscr{X} \to [0,1]$ by*

$$E(x,y) = 1 - (x-y)^2 ,$$

then E is a non min*-transitive similarity relation on $\mathscr{X}$.*

(ii) Consider the Example 7 (ii). Define $E : \mathscr{X} \times \mathscr{X} \to [0,1]$ by

$$E(x,y) = \frac{1}{1 + \|x-y\|_2} .$$

Then E is a non min*-transitive similarity relation on $\mathscr{X}$.*

4 Some Concluding Remarks

In this submission, we have explored non strict T-transitive similarity relations and shown that one can construct both betweenness relations and monometrics on them. In fact, it was shown that the natural betweenness relation obtained from a distance can, in some cases, be obtained from non strict T-transitive relations. Given the impactful role of monometrics in many decision making and optimisation applications and context, we believe our work presents an interesting perspective of fuzzy relations themselves. Further, our work also shows that similarity relations, either T-transitive or not, continue to be influential in their reach and applications.

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