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Inf- and Sup-preserving Aggregation Functions

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Abstract

We introduce inf-lattices, sup-lattices and convexity algebras and we define closure or interior operators and congruences in these algebraic structures. We prove that any inf-lattice can be represented as a quotient of a power set lattice with respect to a congruence. Moreover we consider closure operators defined on a poset and we provide a characterization of inf-preserving maps between inf-lattices. Inf-preserving aggregation functions are described.

Keywords: Inf-lattice, convexity algebra, inf-preserving function, interior operator, congruence, aggregation operator.

1 Introduction

A closure space is an abstract generalization of the notion of closing a set with respect to some underlying operation. Closure spaces are extensively studied in topology, lattice theory and in several other branches of mathematics as well. In this contribution it is proposed a link between a class of inf-preserving or sup-preserving function and convex operators.

We first introduce a notion of infinitary preorder as in [1] considering inf-lattices and sup-lattice then we study convexity algebra as a pointfree convexity space as introduced in [7].

It is proved that any inf-lattice can be represented as a quotient inf-lattice of a power set lattice with respect to a congruence and that also sup-lattices and convexity algebras can be represented as a quotient of a power set sup-lattice or convexity algebra.

Then we focus on the closure and interior operators defined in a partially ordered spaces and we prove that every inf-preserving function can be represented by an interior operator. We can obtain a dual result for sup-preserving functions and also a representation of homomorphisms of convexity algebras.

We conclude the paper with a result on inf-preserving aggregation functions that considers the framework proposed in [6].

2 Inf-lattices and convexity algebras

We are going to consider structures with an infinitary operation as in [1]. In our notation a poset as an ordered set is denoted by $(P,\leq)$. If a poset is bounded then we denote the greatest element, the top, by 1, and the smallest, the bottom, by 0. An inf-lattice is a set L with a partial order $\leq$ such that if $\{x_i : i \in I\}$ is a subset of X then $\wedge x_i$ exists in L and a sup-lattice is a set X with a partial order $\leq$ such that if $\{x_i : i \in I\}$ is a subset of X then $\vee x_i$ exists in L. It can be proved that if the poset $(L,\leq)$ has the least element 0 and the greatest element 1 then a inf-lattice is a complete lattices.

If L_1, and L_2, are inf-lattices a function $f\colon L_1 \to L_2$ is a homomorphism from L_1, to L_2, if $f(\bigwedge_{i\in I} x_i) = \bigwedge_{i\in I} f(x_i)$ for every $\{x_i \in L_1 : i \in I\}$. we can also define an homomorphism between sup-lattices.

Congruences in algebraic structures are relevant from the point of view of aggregation theory as proved in [4] and in [5].

We consider a concept of congruence relation in a partially ordered sets introduced in [2]. An equivalence relation $\sim$ in a inf-lattice L is said to be a congruence if the following properties are satisfied

(i) if $x_1 \leq x_2$ and $x_1 \sim y_1$ then there is an element $y_2 \in L$ such that $x_2 \sim y_2$ and $y_1 \leq y_2$;

(ii) if $x_1 \leq x_2$ and $x_2 \sim y_2$ then there is an element $y_1 \in L$ such that $x_1 \sim y_1$ and $y_1 \leq y_2$;

(iii) if $x \leq z \leq y$ and $x \sim y$ then $x \sim z$;

(iv) if $x_i \sim y_i$ for every $i \in I$ then $\bigwedge_{i\in I} x_i \sim \bigwedge_{i\in I} y_i$.

Note that the first three properties (see Theorem 3.1. in [2]) define the quotient poset $L/\sim$, where the partial ordering $\leq$ on $L/\sim$ defined as $[x] \leq [y]$ if and only if for any $x \in [x]$ there is $y \in [y]$ such that $x \leq y$ and simultaneously for any $y \in [y]$ there is $x \in [x]$ such that $x \leq y$. The last axiom defines the quotient inf-lattice $L/\sim$ with respect to the meet defined by $\bigwedge[x_i] = [\bigwedge x_i]$.
A congruence on a sup-lattice can be defined in a dual mode.
Now we introduce the notion of convexity algebra as proposed and studied in [7]. A poset L is a convexity algebra if the following properties are satisfied:

(i) L has arbitrary meets;

(ii) if $\{x_i : i \in I\}$ is a totally ordered subset of L then $\{x_i : i \in I\}$ has a join in L;

(iii) for any double index family $\{x_{i,j} : i \in I, j \in J_i\}$ if $\{x_{i,j} : j \in J_i\}$ is totally ordered for every $i \in I$ and if $\{\bigwedge_{i\in I} x_{i,f(i)} : f \in F\}$ is totally ordered for every $i \in I$ then

$$\bigwedge_{i\in I}\bigvee_{j\in J_i} x_{i,j} = \bigvee_{f\in F}\bigwedge_{i\in I} x_{i,f(i)},$$

where $F = \prod J_i$ and $\prod J_i = \{f : I \mapsto \bigcup_{i\in I} : \forall i \in I, f(i) \in J_i\}$.

A convexity algebra has the least element 0 and the greatest element 1 by letting the index sets be empty in conditions i) and ii) above. A convexity algebra is an inf-lattice . We introduce also the notion of homomorphism between convexity algebras.
If L_1, and L_2, are convexity algebras a function $f\colon L_1 \to L_2$ is a homomorphism from L_1, to L_2, if the following conditions are satisfied:

(i) $f(\bigwedge_{i\in I} x_i) = \bigwedge_{i\in I} f(x_i)$ for every $\{x_i \in L_1 : i \in I\}$;

(ii) $f(\bigvee_{i\in I} x_i) = \bigvee_{i\in I} f(x_i)$ for every totally ordered set $\{x_i \in L_1 : i \in I\}$.

An equivalence relation $\sim$ in a convexity algebra L is congruence whenever it is a congruence with respect to the inf-lattice structure such that if $x_i \sim y_i$ for every $i \in I$ where $\{x_i : i \in I\}$ and $\{y_i : i \in I\}$ are totally ordered subset of L then $\bigvee_{i\in I} x_i \sim \bigvee_{i\in I} y_i \in C$.

3 Closure systems and closure operators

Let us recall some well known and general notions related to closure systems and operators, pointing to our notation. A closure system $\mathscr{C}$ on a nonempty set X is a collection of subsets of X, closed under arbitrary set intersections. The set $\emptyset$ belongs to $\mathscr{C}$ by letting the index set be empty in the intersection condition. Note that every inf-lattice is a closure system. A closure operator on a set X is a map $c\colon \mathcal{P}(X) \to \mathcal{P}(X)$ on the power set $\mathcal{P}(X)$, which for all $A,B \subseteq X$ fulfils the following properties:

(i) $A \subseteq c(A)$

(ii) if $A \subseteq B$ then $c(A) \subseteq c(B)$

(iii) $c(c(A)) = c(A)$

for every $A,B \subseteq X$. If $A = c(A)$, then A is called a closed set. The family $\mathscr{F}$ of closed sets is a closure system on the same set. On the other hand, if $\mathscr{C}$ is a closure system on X, then the map $c\colon \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$c(A) = \bigwedge\{C \in \mathscr{F}, C \subseteq A\}$$

is a closure operator on X. This correspondence among closure systems and closure operators is one-to-one.
A closure system is a complete lattice under inclusion, and as a converse,the collection of principal ideals of a complete lattice is a closure system, which is, under inclusion, order isomorphic with the lattice itself.
The notion of an abstract convexity structure (convex space, convex structure) studied in [8] is considered.
If $\mathscr{C}$ is a closure system on X, such that if $\{X_i : i \in I\}$ is totally ordered subset of $\mathscr{C}$ with respect to inclusion then $\bigcup_{i\in I} X_i \in \mathscr{C}$ then $(X,\mathscr{C})$ is a convexity space. We call $\mathscr{C}$ the convexity of X and an element of $\mathscr{C}$ is a convex set in X. The set X and $\emptyset$ belong to $\mathscr{C}$. Every convexity space is a convexity algebra with set-theoretical operations. Let X be a non empty set and $\mathcal{P}(X)_{fin}$ the subset of finite sets of X. Then it can be proved (see Proposition 2.1 and 2.2 of [8]) that a mapping $c\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is the closure operator of a convexity space if and only if it is the closure operator of the closure space and

$$c(A) = \bigcup_{B\in 2^A_{fin}} c(B)$$

for every $A \in \mathcal{P}(X)$. If also this property is satisfied the operator is called a finitary closure operator.

4 Representation of inf-lattices

The main example of a inf-lattice and of a convexity algebra is the power set $\mathcal{P}(X)$ of a non empty set X.
It can be proved that any inf-lattice can be represented as a quotient inf-lattice of a power set lattice with respect to a congruence(see [1]).

Proposition 1. *Let L an inf-lattice and $\mathcal{P}(L)$ the inf-lattice of subsets of L. The relation $\sim$ in $\mathcal{P}(L)$ defined by $A \sim B$ if and only if $\bigwedge A = \bigwedge B$ for every non empty subset A, B of L is a congruence. The quotient set $\mathcal{P}(L)/\sim$ is an inf-lattice isomorphic to L.*
If L is also a convexity algebra $\mathcal{P}(L)/\sim$ is a convexity algebra isomorphic to L.

Proof. Let us consider the map $F\colon \mathcal{P}(L)/\sim \rightarrow L$ such that $F([A]) = \bigwedge A$. The function F is well defined since obviously if $A \sim B$) then $F(A) = F(B)$.Being $F([A]) = F([B])$ if and only if $A \sim B$, F is an injective function and it is onto given that $F([a]) = a$. Moreover it preserves meets since we get
$F(\bigwedge[A_i]) = F([\bigcap A_i]) = \bigwedge(\bigcup A_i) = \bigwedge_i(\bigwedge A_i) = \bigwedge_i F(A_i)$. In the same way we can prove that if L is a convexity algebra then the map F preserves joins of totally ordered chain in $\mathcal{P}(L)/\mathcal{C}$. □

5 Inf-preserving maps, interior operators and congruences

Now, we define closure and interior operator in a partially ordered set and then we consider a link between interior operators and inf-preserving maps and dually between closure operators and sup-preserving maps (see [6]). If P is a partially ordered set, a closure operator on P is a map $c\colon P \rightarrow P$ which is extensive, monotone and idempotent this means the following conditions are satisfied for all $x, y \in P$:

(i) $x \leq c(x)$;

(ii) if $x \leq y$ then $c(x) \leq c(y)$;

(iii) $c(x) = c(c(x))$.

The notion of an interior operator is defined dually, i.e., an interior operator on P is a mapping $c\colon P \rightarrow P$ which is intensive (i.e., for $i(x) \leq x$ for all $x \in P$), monotone and idempotent.

Proposition 2. *If L_1, and L_2 are inf-lattices a function $f\colon L_1 \rightarrow L_2$ is an homomorphism from L_1, to L_2, if and only if there exists an interior operator $i\colon L_1 \rightarrow L_1$ such that*

$$f(x) \leq f(y) \iff i(x) \leq i(y)$$

Proof. Let $f\colon L_1 \rightarrow L_2$ be a homomorphism from L_1, to L_2 where L_1, are L_2 are inf-lattices. We define an operator $i\colon L_1 \rightarrow L_1$ by $i(x) = \bigwedge\{z : f(z) \geq f(x)\}$.
We get that for all $x \in L_1$, $i(x) \leq x$ by the definition of the operator i.
The monotonicity property is satisfied since if $x \leq y$ it can be easily proved that $\{z : f(z) \geq f(y)\} \subseteq \{z : f(z) \geq f(x)\}$. Therefore $\bigwedge\{z : f(z) \geq f(y)\} \geq \bigwedge\{z : f(z) \geq f(x)\}$ hence $i(x) \leq i(y)$.
Being f inf-preserving and also non decreasing we get that for all $x \in L_1$, $f(i(x)) \geq f(x)$ and since $i(x) \leq x$, $f(i(x)) \leq f(x)$. We can conclude that for all $x \in L_1$, $f(i(x)) = f(x)$. Next we can observe that $i(i(x)) = \bigwedge\{z : f(z) \geq f(i(x))\} = \bigwedge\{z : f(z) \geq f(x)\} = i(x)$. We can conclude that operator $i\colon L_1 \rightarrow L_1$ defined by $i(x) = \bigwedge\{z : f(z) \geq f(x)\}$ is an interior operator.
Conversely consider a map $f\colon L_1 \rightarrow L_2$ from L_1, to L_2 where L_1, are L_2 are inf-lattices and an interior operator $i\colon L_1 \rightarrow L_1$ such that $f(x) \leq f(y) \iff i(x) \leq i(y)$. For all $x \in L_1$, $i(i(x)) = i(x)$ and then we can easily prove that $f(i(x)) = f(x)$ then we get that $i(x) \geq \bigwedge\{z : f(z) \geq f(x)\}$. Moreover we note that if $f(z) \geq f(x)$ then $z \geq i(z) \geq i(x)$ and obviously we have $z \geq i(x)$. Hence $i(x) \leq \bigwedge\{z : f(z) \geq f(x)\}$ and then we proved that $i(x) = \bigwedge\{z : f(z) \geq f(x)\}$. This implies that if for every $x_i : i \in I$, $f(x_i) \geq y$ then $f(\bigwedge\{x_i : i \in I\}) \geq y$ hence $f(\bigwedge\{x_i : i \in I\}) \geq \bigwedge\{f(x_i) : i \in I\})$. The map f is non decreasing since the map i is non decreasing so we can also say that $\bigwedge\{f(x_i) : i \in I\}) \leq f(\bigwedge\{x_i : i \in I\})$. Then we get that $\bigwedge\{f(x_i) : i \in I\}) = f(\bigwedge\{x_i : i \in I\})$ and we proved that f is an homomorphism. □

Using the dual arguments the following result is proved.

Proposition 3. *If L_1, and L_2 are sup-lattices a function $f\colon L_1 \rightarrow L_2$ is an homomorphism from L_1, to L_2, if and only if there exists a closure operator $c\colon L_1 \rightarrow L_1$ such that*

$$f(x) \leq f(y) \iff c(x) \leq c(y)$$

The following result gives conditions for an homeomorphism between convexity algebras.

Proposition 4. *If L_1, and L_2 are convexity algebra a function $f\colon L_1 \rightarrow L_2$ is an homomorphism from L_1, to L_2, if and only if there exists an interior operator $i\colon L_1 \rightarrow L_1$ such that $i(\bigvee_{j\in I} x_j) = \bigvee_{j\in I} i(x_j)$ for every totally ordered set $\{x_j \in L_1 : j \in I\}$ and*

$$f(x) \leq f(y) \iff i(x) \leq i(y)$$

Proof. The proof follows directly from Proposition 2 by noting that $i(\bigvee_{j\in I} x_j) = \bigvee_{j\in I} i(x_j)$ for a totally ordered set $\{x_j \in L_1 : j \in I\}$ if and only if $f(\bigvee_{j\in I} x_j) = \bigvee_{j\in I} f(x_j)$. □

Note also that by Proposition 2 every inf-preserving function (or homeomorphism) from a inf-lattice L_1 to a inf-lattice L_2 defines a congruence in L_1 defined by xCy if and only if $f(x) = f(y)$ or if and only if $i(x) = i(y)$ where $i\colon L_1 \rightarrow L_1$ is an interior operator. Similar results can be obtained for convexity algebras and sup-lattices.

Now we focus on link between interior operators and congruences in an inf-lattice.

Proposition 5. *If $\sim$ is a congruence on a inf-lattice L then there is an interior operator on L such that*

$$x \sim y \quad \Longleftrightarrow \quad i(x) = i(y)$$

for every $x, y \in L$.If $\sim$ is a congruence on a sup-lattice L then there is a closure operator on L such that

$$x \sim y \quad \Longleftrightarrow \quad c(x) = c(y)$$

for every $x, y \in L$.

Proof. If $\sim$ is a congruence in an inf-lattice L (or in a sup-lattice) there is an homomorphism from L to the quotient space $L/\sim$ and then by Proposition 2 (or Proposition 3) we can prove our result. □

If an interior operator defined on an inf-lattice satisfies the properties considered in the following proposition then we can prove the converse of Proposition 5.

Proposition 6. *Let i be an interior operator defined on an inf-lattice L such that i is and homomorphism from L to L and*

$$\text{if} \quad x_1 \leq x_2 \quad \text{and} \quad i(x_1) = i(y_1) \quad \text{there is} \quad y_2 \in L,$$
$$i(x_2) = i(y_2) \quad \text{and} \quad y_1 \leq y_2.$$

Then there is a congruence $\sim$ on L such that

$$x \sim y \quad \Longleftrightarrow \quad i(x) = i(y).$$

Proof. It can be easily proved that the relation defined in L by $x \sim y \iff i(x) = i(y)$ is an equivalence relation. If $x_1 \leq x_2$ and $i(x_1) = i(y_1)$ there is $y_2 \in L$, $i(x_2) = i(y_2)$ so $x_2 \sim y_2$ and $y_1 \leq y_2$. Then property (i) in the definition of congruence is satisfied.
Let $x_1 \leq x_2$ and $x_2 \sim y_2$ hence $i(x_1) \leq i(x_2) = i(y_2)$. Then if we define $y_1 = x_1 \wedge y_2$ we have $i(y_1) = i(x_1) \wedge i(y_2) = i(x_1)$ and we can say that $x_1 \sim y_1$ and $y_1 \leq y_2$. This shows that property (ii) is satisfied. The operator i is an homomorphism so axioms (iii) and(iv) are verified and them $\sim$ is a congruence in L. □

A similar result can be obtained for closure operators defined on a sup-lattice.

Let us consider also the case of inf-preserving aggregation operators. If $(L, \leq)$ is an inf-lattice the product set L^n is an inf-lattice with respect to the product order and an aggregation function is a non decreasing map $f: L^n \to L$. If the poset $(L, \leq)$ has the least element 0 and the greatest element 1 as in the case of convexity algebra we assume that $f(0, \dots, 0) = 0$ and $f(1, \dots, 1) = 1$.
An aggregation function $f: L^n \to L$ is said to be inf-preserving if is an homomorphism from L^n to L.
Aggregations on posets and in algebraic structures is a topic of growing interest recently due also to applications in information sciences (see e.g [3] and [6]).
By Proposition 2 we can prove that inf-preserving aggregation functions are aggregation functions that can be represented by interior operators.

Proposition 7. *If L is an inf-lattice a function, $f: L^n \to L$ is an inf-preserving aggregation function if and only if there exists an interior operator $i: L^n \to L^n$ such that*

$$f(x) \leq f(y) \quad \text{if and only if} \quad i(x) \leq i(y)$$

We can also prove a dual result for sup- preserving aggregation operator defined in a sup-lattice that are represented by closure operators. Moreover we can characterize aggregation operators defined in a convexity algebra and that are homeomorphism between convexity algebras.

6 Conclusion

In this note we have focused on the link between closure and interior operators defined in a partially ordered set and homemorphisms between posets that are inf-preserving or sup-preserving. We also introduced congruences in inf- and sup- lattices. Moreover we have considered aggregation operators defined in inf-lattices, sup-lattices and convexity algebras. We plan to study other classes of aggregation operators in these algebraic structures, and to characterize in our framework aggregation operators which preserves congruences .

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