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Solution to the Advection Equation with Fuzzy Initial Condition via Sup-J Extension Principle

Vinícius F. Wasques[a,b], Estevão Esmi[c], Laécio C. Barros[d]

[a]Dept. Math, São Paulo State University, Brazil vwasques@outlook.com

[b]Dept. Int. Sci. Teach. Center, National Center for Research in Energy and Materials, Brazil

[c]Dept. Appl. Math, University of Campinas, Brazil eelaureano@gmail.com

[d]Dept. Appl. Math, University of Campinas, Brazil laeciocb@ime.unicamp.br

Abstract

This paper presents a study on the advection equation with an uncertain condition given by a fuzzy-number-valued function. A fuzzy solution to the problem is presented. This solution is obtained from the sup-J extension principle, which is a generalization of the Zadeh's extension principle. In this case, the extension principle is used to extend the classical solution to the problem. The extension principle incorporates the relationship of interactivity, which in this work is associated with a family of parameterized joint possibility distributions. A comparison with the solution obtained from the Zadeh's extension principle is also presented, in order to illustrate the advantages of considering interactivity in the approach.

Keywords: Interactive fuzzy numbers, Sup-J extension principle, Advection equation, Zadeh's extension principle.

1 Introduction

The concept of advection is widely used in the fields of physics and engineering, to describe the transport of a substance or quantity by a bulk flux. For example, the transport of pollutants in a river by a water flow can be expressed by an advection process.

The transport $u(x,t)$ of this quantity depends on the initial condition $u(x,0)$, in every location x, and the law governing the flux from the initial point to another. This law can be mathematically represented by a Partial Differential Equation (PDE), given by [9]

$$\begin{cases} u_t + \kappa u_x = 0 \\ u(x,0) = \eta(x) \end{cases}, \tag{1}$$

where κ represents the velocity constant, $\eta(x)$ is the initial condition, u_t and u_x stand for the partial derivatives with respect to time and space, respectively.

Although the advection equation depends on two different partial derivatives, the analytical solution to (1) can be given in terms of one initial condition $\eta(x)$, that is,

$$u(x,t) = \eta(x - \kappa t). \tag{2}$$

As it can be seen in (2), the classical solution for (1) depends directly from the initial condition given by a real function $\eta(x)$. Determining the exact position of the studied substance at $t = 0$, in every position x, can be a complicated task. In fact, the initial position can be considered as uncertain. In this case, fuzzy sets theory can be used to determine the behavior of the phenomenon, in a more precisely way.

In this paper, the initial condition of the advection equation will be given by a fuzzy-number-valued function, that is, for each position x, a fuzzy number is assigned. Moreover, knowing that there is a time-dependence in the process, the fuzzy relation of interactivity will be considered.

The relationship of interactivity arises from the notion of Joint Possibility Distributions (JPDs) [7]. This relation is similar, but not equivalent, to dependence in the case of random variables. Several authors have considered this concept to provide solutions to Fuzzy Differential Equations (FDEs) [5, 15] and Fuzzy Optimization Problems (FOPs) [10, 11].

The proposed fuzzy solution consists in an extension of the classical solution to the problem via sup-J extension principle, which is a generalization of the Zadeh extension principle [7]. The sup-J extension takes the interactivity between fuzzy numbers into account, in contrast to the Zadeh's extension.

Two examples will be provided to compare the fuzzy solutions. In one of them, it will be illustrated when the two extension principles produce the same solution. In the other example, the advantages of considering the notion of interactivity in the proposed approach will be highlighted.

2 Mathematical Background

This section presents the basic concepts of fuzzy sets theory required for this paper.

A fuzzy set A of a universe X is characterized by a function

$$\mu_A : X \to [0,1]$$

called membership function, where $\mu_A(x)$ represents the membership degree of x in A, for all $x \in X$. For convenience, the symbol $A(x)$ is considered instead of $\mu_A(x)$.

The class of fuzzy subsets of X is denoted by $\mathcal{F}(X)$. Recall that each classical subset of X can be uniquely identified with the fuzzy set whose membership function is given by its characteristic function.

A fuzzy set can be determined by a family of classical subsets, called α-cuts. The α-cut of a fuzzy set $A \subseteq X$, denoted by $[A]^\alpha$, is defined by

$$[A]^\alpha = \{x \in X : A(x) \geq \alpha\}, \ \forall \alpha \in (0,1]. \tag{3}$$

In addition, if X is also a topological space, then the 0-cut of A is defined by $[A]^0 = cl\{x \in X : A(x) > 0\}$, where cl Y, $Y \subseteq X$, denotes the closure of Y [1]. Recall that $A \subseteq B \Leftrightarrow [A]^\alpha \subseteq [B]^\alpha$, for all $\alpha \in [0,1]$ and for all $A, B \in \mathcal{F}(X)$.

An important subclass of $\mathcal{F}(\mathbb{R})$, denoted by $\mathbb{R}_\mathcal{F}$, is the class of fuzzy numbers which includes the set of the real numbers as well as the set of the bounded closed intervals of $\mathbb{R}$.

Definition 1 *[5] A fuzzy set A of $\mathbb{R}$ is said to be a fuzzy number if all α-cuts are bounded, closed and non-empty nested intervals, for all $\alpha \in [0,1]$.*

From the above definition, it is possible to denote the α-cuts of a fuzzy number A by $[A]^\alpha = [a^-_\alpha, a^+_\alpha]$.

A fuzzy number A is called triangular, if its α-cuts are given by $[A]^\alpha = [a + \alpha(b-a), c - \alpha(c-b)]$, $\forall \alpha \in [0,1]$, for some $a, b, c \in \mathbb{R}$, such that $a \leq b \leq c$. A triangular fuzzy number is denoted by the triple $(a;b;c)$.

The class of fuzzy numbers such that a^-_α and a^+_α are continuous functions with respect to α, is denoted by $\mathbb{R}_{\mathcal{F}_C}$. Note that every triangular fuzzy number is an example of element that is contained in $\mathbb{R}_{\mathcal{F}_C}$.

Let $A \in \mathbb{R}_{\mathcal{F}_C}$ such that $[A]^\alpha = [a^-_\alpha, a^+_\alpha]$, for all $\alpha \in [0,1]$. The corresponding translated fuzzy number $A^{(\overline{a})}$ is defined by $A^{(\overline{a})}(x) = A(x - \overline{a})$ for all $x \in \mathbb{R}$, where $\overline{a} = 0.5(a^-_1 + a^+_1)$ [12].

Definition 2 *The width (or* diameter*) of a fuzzy number $A \in \mathbb{R}_\mathcal{F}$ is defined by*

$$width(A) = a^+_0 - a^-_0. \tag{4}$$

The fuzzy number A is said to be more specific than B, if $width(A) \leq width(B)$. In particular, if $A \subseteq B$, then A is more specific than B.

A t-norm is an associative, commutative and increasing operator $t : [0,1]^2 \to [0,1]$ that satisfies $t(x,1) = x$, for all $x \in [0,1]$. An example of a t-norm is the minimum operator $t = \wedge$.

The Zadeh's extension principle is a mathematical method to extend classical functions to functions that have fuzzy variables as arguments.

Definition 3 *Let $A_1, \ldots, A_n$ be fuzzy numbers. The Zadeh's extension of a given function $f : \mathbb{R}^n \to \mathbb{R}$ is the fuzzy function $\widehat{f} : \mathcal{F}(X_1) \times \cdots \times \mathcal{F}(X_n) \to \mathcal{F}(Z)$ defined for each fuzzy set $A_i \in \mathcal{F}(X_i)$, for $i = 1, \ldots, n$. The membership function of the fuzzy number $\widehat{f}(A_1, \ldots, A_n)$ is given by*

$$\widehat{f}(A_1, \ldots, A_n)(z) = \sup_{f(x_1, \ldots, x_n) = z} A_1(x_1) \wedge \ldots \wedge A_n(x_n) \tag{5}$$

where sup *represents the supremum operator.*

Next subsection presents the definition of joint possibility distribution, which gives raise to the notion of interactivity.

2.1 Joint Possibility Distribution

An n-ary relation on $X = X_1 \times \ldots \times X_n$ is defined by a fuzzy (sub)set of X, where $R(x_1, \ldots, x_n) \in [0,1]$ stands for the degree of relationship among $x_1 \in X_1, \ldots, x_n \in X_n$.

A fuzzy relation $J \in \mathcal{F}(\mathbb{R}^n)$ is said to be a joint possibility distribution (JPD) among the fuzzy numbers $A_1, \ldots, A_n \in \mathbb{R}_\mathcal{F}$ if

$$A_i(y) = \sup_{(x_1, \ldots, x_n) : x_i = y} J(x_1, \ldots, x_n), \ \forall y \in \mathbb{R}, \tag{6}$$

for all $i = 1, \ldots, n$.

An example of JPD is the t-norm-based joint possibility distribution J_t, which is defined by

$$\begin{aligned} J_t(x_1, \ldots, x_n) &= t(A_1(x_1), \ldots, A_n(x_n)) \\ &= A_1(x_1) \ t \ \ldots \ t \ A_n(x_n), \end{aligned} \tag{7}$$

where t is a t-norm.

Definition 4 *The fuzzy numbers $A_1,...,A_n$ are said to be* non-interactive*, if the JPD among them is given by*

$$J_\wedge(x_1,...,x_n) = A_1(x_1) \wedge \ldots \wedge A_n(x_n).$$

Otherwise, that is, if J satisfies (6) *and $J \neq J_\wedge$, then $A_1,...,A_n$ are called* interactive.

The above definition clarifies that the notion of interactivity among the fuzzy numbers $A_1,...,A_n$ arises from a given JPD. This means that in order to deal with interactivity, it is necessary to predetermine which JPD is being considered.

The sup-J extension principle is also a mathematical tool that extends classical functions to fuzzy functions. In addition, the sup-J extension principle takes the interactivity between fuzzy numbers into account. The definition is given as follows [3, 7].

Definition 5 (Sup-J extension principle) *Let $J \in \mathcal{F}(\mathbb{R}^n)$ be a joint possibility distribution of $(A_1,...,A_n) \in \mathbb{R}^n_{\mathcal{F}}$ and $f : \mathbb{R}^n \to \mathbb{R}$. The sup-J extension of f at $(A_1,...,A_n) \in \mathbb{R}^n_{\mathcal{F}}$, denoted $f_J(A_1,...,A_n)$, is the fuzzy set defined by:*

$$f_J(A_1,...,A_n)(y) = \sup_{f(x_1,...,x_n)=y} J(x_1,...,x_n). \quad (8)$$

Both sup-J and Zadeh's extension principles can be used to provide arithmetic for fuzzy numbers [15, 17]. They can also be used to extend classical solutions of FOPs [10, 11] and FDEs analytically [5, 14]. In these cases, the function f presenting in the definition of the sup-J and Zadeh's extension principles plays the role of the classical solution of the problem.

Coroianu and Fullér [4] established the necessary and sufficient conditions for the equality between these two types of extensions. Wasques et al. [13] established the criteria for comparison between the arithmetics obtained from these extension principles in numerical methods.

Now, let us provide some remarks about the sup-J and Zadeh's extensions principles.

Remark 1 *If only one fuzzy number A is considered as an argument, then the sup-J extension of a function f at A is always the same, regardless of the chosen JPD. In particular, for one fuzzy argument the sup-J extension and the Zadeh's extension are equivalent.*

Remark 2 *If $A_1,\ldots,A_n$ are* non-interactive *fuzzy numbers, then the sup-J extension principle boils down to the Zadeh's extension principle, since in this case the JPD is given by $J = J_\wedge$.*

There are several types of interactive fuzzy numbers. This paper focuses on the interactivity that is associated with a family of parametrized JPDs [6], which is defined as follows.

Let $A, B \in \mathbb{R}_{\mathcal{F}_C}$, for each $\gamma \in [0,1]$ and for each $(x_1,x_2) \in \mathbb{R}^2$, consider J_γ given by

$$J_\gamma(x_1,x_2) = \begin{cases} A(x_1) \wedge B(x_2) & \text{, if } (x_1,x_2) \in P(\gamma) \\ 0 & \text{, otherwise} \end{cases}, \quad (9)$$

where the set $P(\gamma) := P_A(\gamma) \cup P_B(\gamma)$ is defined as the region where the fuzzy relation J_γ satisfies $J_\gamma(u,v) > 0$. Recall that

$$P_A(\gamma) = \left(\bigcup_{\alpha \in [0,1)} \left(\bigcup_{x \in \{a^-_\alpha, a^+_\alpha\}} \{x\} \times I_A(x,\alpha,\gamma) \right) \right) \cup \left(\bigcup_{x \in [A]^1} \{x\} \times I_A(x,1,\gamma) \right)$$

and

$$P_B(\gamma) = \left(\bigcup_{\alpha \in [0,1)} \left(\bigcup_{x \in \{b^-_\alpha, b^+_\alpha\}} \{x\} \times I_B(x,\alpha,\gamma) \right) \right) \cup \left(\bigcup_{x \in [B]^1} \{x\} \times I_B(x,1,\gamma) \right),$$

with

$$I_A(x,\alpha,\gamma) = [\overline{b} + f^\alpha_A(x) + \gamma((b^{(\overline{b})})^-_\alpha - f^\alpha_A(x)), \overline{b} + f^\alpha_A(x) + \gamma((b^{(\overline{b})})^+_\alpha - f^\alpha_A(x))],$$

and

$$I_B(x,\alpha,\gamma) = [\overline{a} + f^\alpha_B(x) + \gamma((a^{(\overline{a})})^-_\alpha - f^\alpha_B(x)), \overline{a} + f^\alpha_B(x) + \gamma((a^{(\overline{a})})^+_\alpha - f^\alpha_B(x))],$$

where the functions f^α_A and f^α_B are respectively given by

$$f^\alpha_A(x) = (-(x - \overline{a}) \vee ((b^{(\overline{b})})^-_\alpha)) \wedge ((b^{(\overline{b})})^+_\alpha)$$

and

$$f^\alpha_B(x) = (-(x - \overline{b}) \vee ((a^{(\overline{a})})^-_\alpha)) \wedge ((a^{(\overline{a})})^+_\alpha).$$

The fuzzy relation J_γ is a JPD between A and B, for all $A, B \in \mathbb{R}_{\mathcal{F}_C}$ and for all $\gamma \in [0,1]$ [6]. Moreover, this family of JPDs satisfies the following property $J_\gamma(x_1,x_2) \leq J_\lambda(x_1,x_2)$, for all $0 \leq \gamma \leq \lambda \leq 1$ and for all $(x_1,x_2) \in \mathbb{R}^2$.

Remark 3 *It is interesting to observe that for $\gamma = 1$, it follows that $J_1 = J_\wedge$. Moreover, J_0 resembles the JPD denoted by I proposed in [16], which gives raise to the generalized difference [2]. This result implies that the Hukuhara difference and its generalizations incorporate the notion of interactivity.*

The next section presents the proposed solution to the advection equation, where the initial condition is given by a fuzzy-number-valued function.

3 Fuzzy solution to the advection equation

The classical advection equation given by (1) can be easily solved analytically by $u(x,t) = \eta(x - \kappa t)$, where $\eta(x)$ is the initial condition.

From the classical solution u and for each $(x,t) \in U \subseteq \mathbb{R}^2$, where U is an open set, let us define the operator $H : \mathbb{R}^{n+2} \to \mathbb{R}$ given by

$$H(x,t,a_1,\ldots,a_n) = u_{(a_1,\ldots,a_n)}(x,t), \qquad (10)$$

where $a_1,\ldots,a_n$ are the parameters implicitly given in function $u(x,t)$ that will be considered as uncertain.

Now, let $A_1,\ldots,A_n$ be interactive fuzzy numbers with respect to J. The fuzzy solution for (1), via sup-J extension principle, is defined by the function

$$(H(x,t,A_1,\ldots,A_n))_J(y) = u_{(A_1,\ldots,A_n)_J}(x,t)(y), \qquad (11)$$

$$= \sup_{u_{(a_1,\ldots,a_n)}(x,t)=y} J(a_1,\ldots,a_n)$$

for all $(x,t) \in U \subseteq \mathbb{R}^2$.

In terms of α-cuts, the fuzzy solution (11) can be given by

$$[(H(x,t,A_1,\ldots,A_n))_J]^\alpha = \left[u_{(A_1,\ldots,A_n)_J}(x,t)\right]^\alpha$$

$$= \left[\inf_{(a_1,\ldots,a_n)\in[J]^\alpha} u_{(a_1,\ldots,a_n)}(x,t)\,, \sup_{(a_1,\ldots,a_n)\in[J]^\alpha} u_{(a_1,\ldots,a_n)}(x,t)\right],$$

where the symbol inf stands for the infimum operator.

In the case that $A_1,\ldots,A_n$ are considered as non-interactive, that is, $J = J_\wedge$, the fuzzy solution via sup-J extension, which coincides with Zadeh's extension, is denoted by $\hat{H}(x,t,A_1,\ldots,A_n)$.

For illustration, consider the following examples.

3.1 Example 1

Consider the advection equation given by [8]

$$\begin{cases} u_t + 0.5u_x = 0 \\ u(x,0) = 0.5exp(-(x-2)^2) \end{cases}. \qquad (12)$$

Note that in this case the initial condition $\eta(x) = 0.5exp(-(x-2)^2)$ and $\kappa = 0.5$. Therefore, the classical solution for (12) is given by

$$u(x,t) = 0.5exp(-(x-0.5t-2)^2),$$

which is depicted in Figure 1.

Now, suppose that the initial condition is given by $u(x,0) = Aexp(-(x-2)^2)$, where A is the triangular fuzzy number $A = (0;0.5;1)$. In this case, the fuzzy solution obtained from the sup-J extension principle is given by

$$\begin{aligned}(H(x,t,A))_J(y) &= \sup_{u_a(x,t)=y} J(a) \\ &= \sup_{a\cdot exp(-(x-0.5t-2))^2=y} A(a).\end{aligned}$$

Note that for this particular case, the sup-J extension principle produces the same fuzzy solution to the problem regardless of the chosen JPD, since there is only one fuzzy argument (see Remark 1). In other words, for one fuzzy argument, it follows that $H_J = \hat{H}$, for all JPD J. The fuzzy solution H_J is depicted in Figure 2.

Note that the extension principle produces a fuzzy solution qualitatively similar to the solution of the classical problem. It is also interesting to observe that the extension principle propagates the uncertainty throughout its evolution.

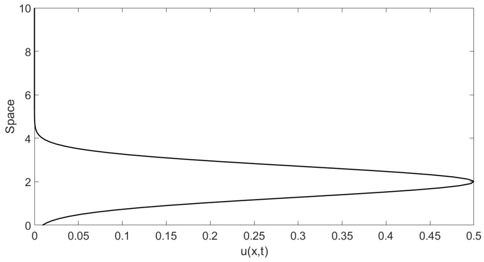
Figures 1 and 2 also illustrate that the location of greatest uncertainty occurs at the extreme value of the classical solution. For example, in Subfigure 2(a) the greatest uncertainty occurs at $x = 2$, which is exactly the maximum value of the classical solution at $x = 2$ and $t = 0$ (see Subfigure 1(a)).

In this example, the initial condition was considered as a fuzzy-number-valued function of the form $u(x,0) = Af(x)$, where A is a fuzzy number and $f(x)$ is a classical real function. The next example will consider a fuzzy function of the form $u(x,0) = Af(x) +_J B$, where A and B are interactive fuzzy numbers.

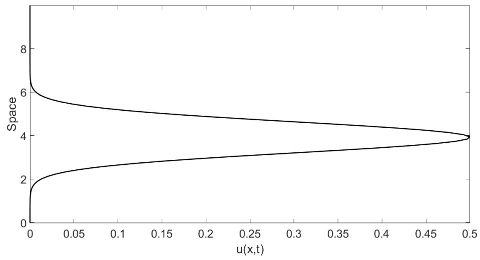
3.2 Example 2

Consider the advection equation given by

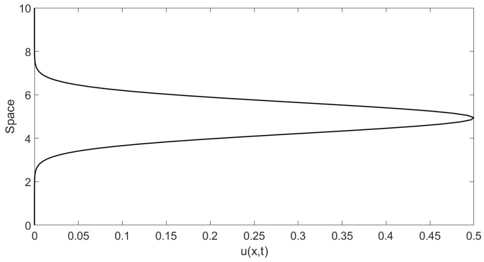
$$\begin{cases} u_t + 0.5u_x = 0 \\ u(x,0) = 0.5exp(-(x-2)^2) + 0.25 \end{cases}. \qquad (13)$$



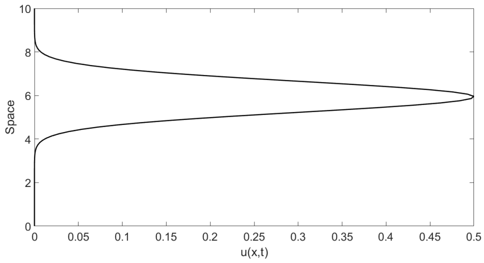
(a) Classical solution for $t = 0$.



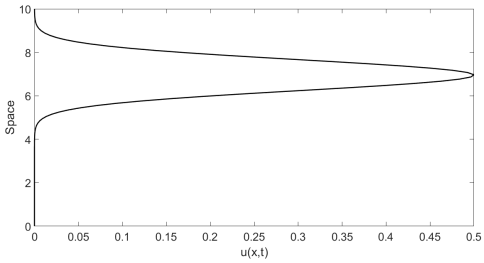
(b) Classical solution for $t = 3.8$.



(c) Classical solution for $t = 5.8$.

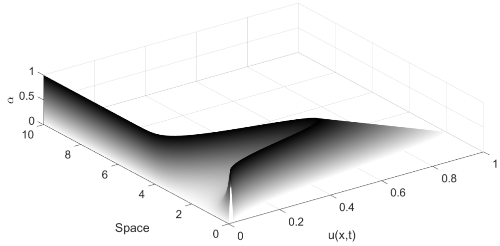


(d) Classical solution for $t = 7.8$.

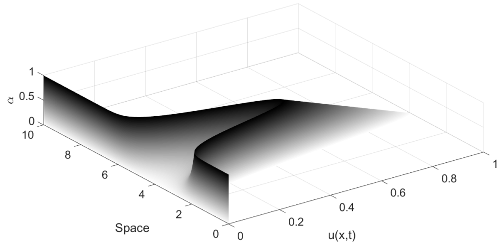


(e) Classical solution for $t = 9.8$.

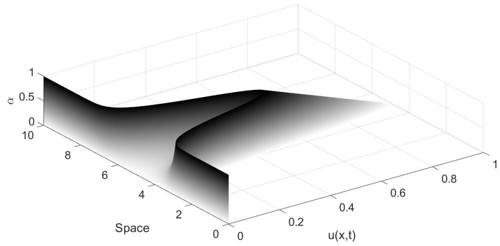
Figure 1: Classical solution for (12), for $t = 0$, $t = 3.8$, $t = 5.8$, $t = 7.8$ and $t = 9.8$.



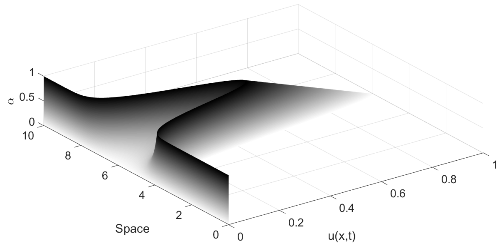
(a) Tri-dimensional view of the solution for $t = 0$.



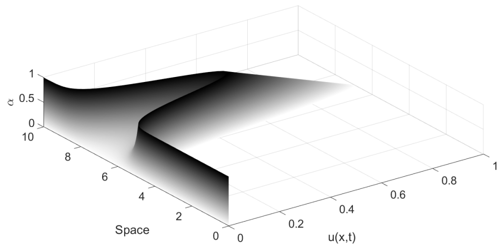
(b) Tri-dimensional view of the solution for $t = 3.8$.



(c) Tri-dimensional view of the solution for $t = 5.8$.



(d) Tri-dimensional view of the solution for $t = 7.8$.



(e) Tri-dimensional view of the solution for $t = 9.8$.

Figure 2: The Zadeh extension of the classical solution of advection equation (12). The gray lines represent the α-cuts of the fuzzy solutions, for gray-scale lines for α varying from 0 to 1.

Although this initial condition is only a translation of $u(x,0)$ given in the previously example, in the context of fuzzy sets theory there are different solutions for this problem.

Since the initial condition is $\eta(x) = 0.5exp(-(x-2)^2) + 0.25$ and $\kappa = 0.5$, the classical solution for (13) is given by

$$u(x,t) = 0.5exp(-(x-0.5t-2)^2) + 0.25$$

For this example, the fuzzy solution obtained from the sup-J extension principle is given by

$$(H(x,t,A,B))_J(y) = \sup_{u_{a,b}(x,t)=y} J(a,b)$$

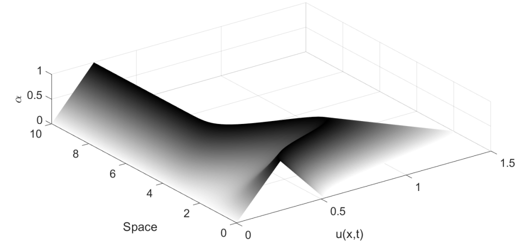
$$= \sup_{a\cdot exp(-(x-0.5t-2))^2+b=y} J(a,b).$$

The fuzzy solution H_J depends on the JPD previously chosen. Here, the JPDs J_0, $J_{0.5}$ and J_1 will be considered. In particular, for J_1 the sup-J extension is equivalent to the Zadeh's extension, since $J_1 = J_\wedge$ (see Remark 2 and 3).

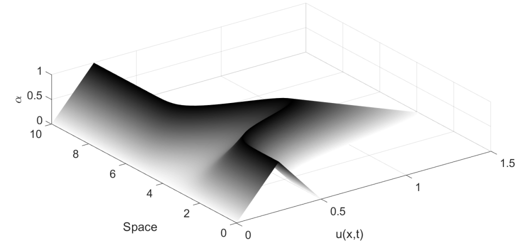
Figures 3, 4 and 5 depict the fuzzy solutions for J_1, $J_{0.5}$ and J_0, respectively. All the fuzzy solutions present the same behaviour that the classical solution, qualitatively. For this simulation, the parameters 0.5 and 0.25, given in the initial condition of (13), were fuzzified by $A = (0; 0.5; 1)$ and $B = (0; 0.25; 0.5)$, respectively.

Figures 3, 4 and 5 also illustrate that the location of greatest uncertainty occurs at the extreme value of the classical solution, as observed in Example 1. This uncertainty can be controlled through the JPDs J_γ, as it can be seen in the width of the fuzzy solutions given in Subfigures 3 (a), 4 (a) and 5 (a). In fact, the width of the fuzzy solutions, for each x, is decreasing as the value of γ decreases. This property is associated with the fact that $J_\lambda \subseteq J_\gamma$, for all $0 \le \lambda \le \gamma \le 1$. Consequently, the fuzzy solution via J_0 is more specific than $J_{0.5}$ and J_1. In other words, the use of interactivity results in propagation of less uncertainty, than in the case of non-interactivity. Moreover, among the values of γ, J_0 produces the least uncertainty.

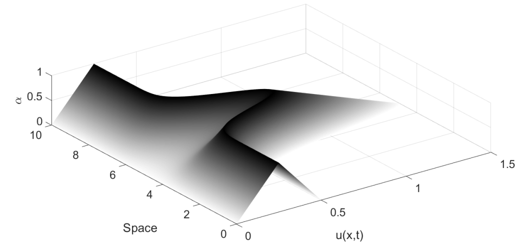
The behaviour of this proposed fuzzy solution, with respect to the width and the propagation of uncertainty, can be observed in numerical methods as well [13, 15, 17]. In these papers, the arithmetic operations in the classical numerical methods were adapted for interactive fuzzy numbers, via sup-J extension principle. Hence, the notation $+_J$ mentioned in Example 1 makes more sense, since before performing an arithmetic operation between fuzzy numbers, it is necessary to establish the JPD between them.



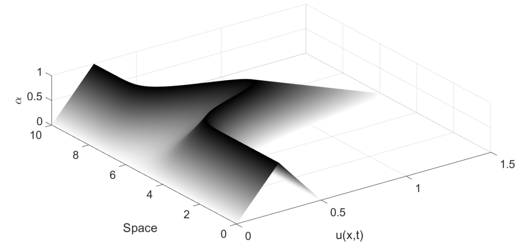
(a) Tri-dimensional view of the solution for $t = 0$.



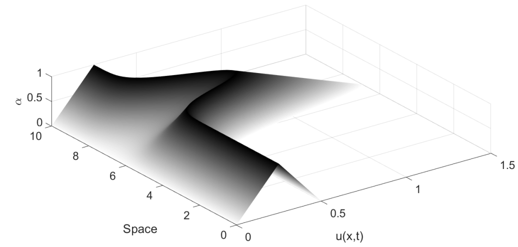
(b) Tri-dimensional view of the solution for $t = 3.8$.



(c) Tri-dimensional view of the solution for $t = 5.8$.

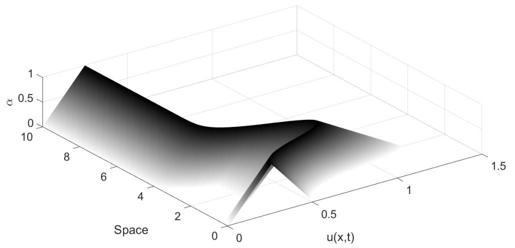


(d) Tri-dimensional view of the solution for $t = 7.8$.

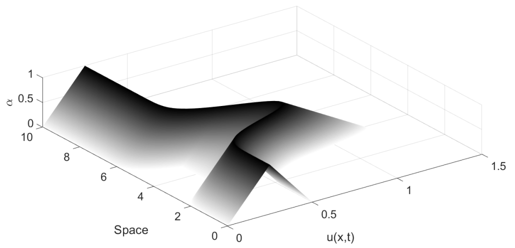


(e) Tri-dimensional view of the solution for $t = 9.8$.

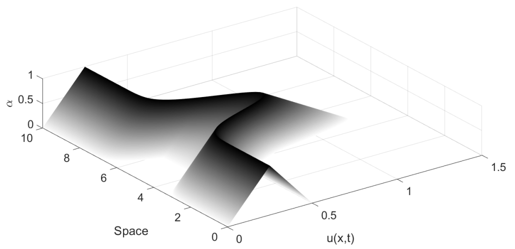
Figure 3: The fuzzy solution to (13) for $J = J_1$. The gray lines represent the α-cuts of the fuzzy solutions, for gray-scale lines for α varying from 0 to 1.



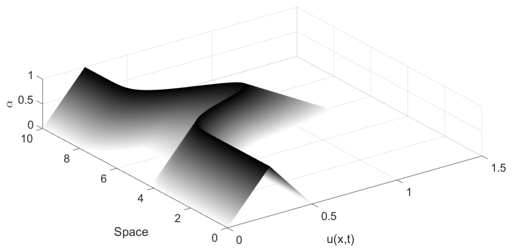
(a) Tri-dimensional view of the solution for $t = 0$.



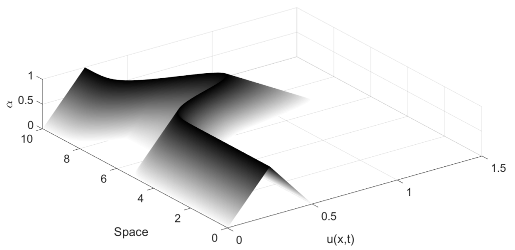
(b) Tri-dimensional view of the solution for $t = 3.8$.



(c) Tri-dimensional view of the solution for $t = 5.8$.

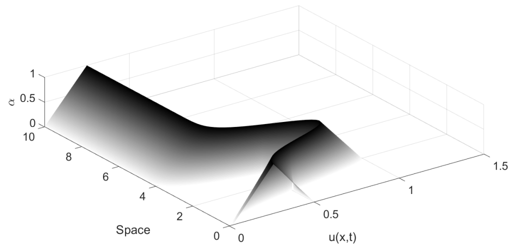


(d) Tri-dimensional view of the solution for $t = 7.8$.

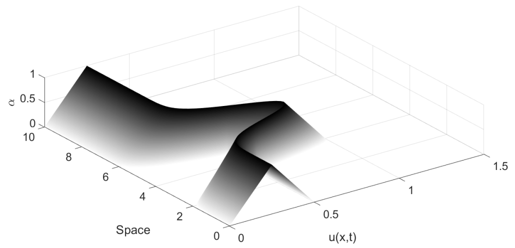


(e) Tri-dimensional view of the solution for $t = 9.8$.

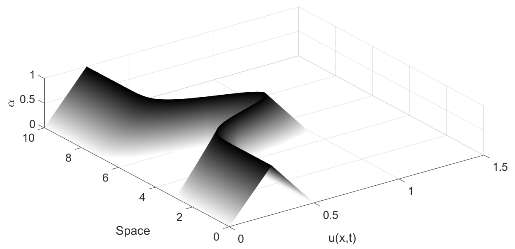
Figure 4: The fuzzy solution to (13) for $J = J_{0.5}$. The gray lines represent the α-cuts of the fuzzy solutions, for gray-scale lines for α varying from 0 to 1.



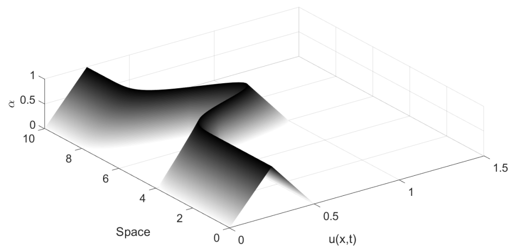
(a) Tri-dimensional view of the solution for $t = 0$.



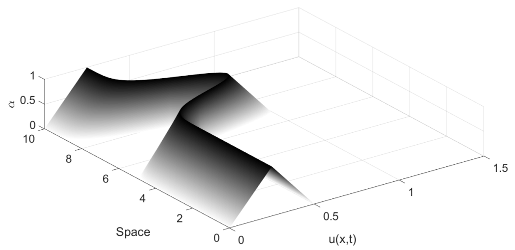
(b) Tri-dimensional view of the solution for $t = 3.8$.



(c) Tri-dimensional view of the solution for $t = 5.8$.



(d) Tri-dimensional view of the solution for $t = 7.8$.



(e) Tri-dimensional view of the solution for $t = 9.8$.

Figure 5: The fuzzy solution to (13) for $J = J_0$. The gray lines represent the α-cuts of the fuzzy solutions, for gray-scale lines for α varying from 0 to 1.

4 Final remarks

This paper studied the advection equation described by a partial differential equation, where the initial condition is given by a fuzzy-number-valued function. The fuzzy solution to this problem was obtained from the sup-J extension of the classical solution. The interactivity considered here is the one associated with a parametrized family of joint possibility distributions J_γ, with $\gamma \in [0,1]$. From this family, several fuzzy solutions were presented. Among the possible fuzzy solutions via J_γ, the J_0 is the one that produces the solution with the smallest possible width, corroborating theoretical results [6]. A comparison with the Zadeh's extension principle was also provided, illustrating that the Zadeh's extension propagates more uncertainty throughout the advection process, than the method that considers interactivity. This approach can be used for any Fuzzy Differential Equation (Partial or Ordinary) with initial/boundary conditions, that has a classical solution.

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