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An Alternative Characterization of t-norms and t-conorms on An Appropriate Bounded Lattice

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Abstract

In this paper, we present new methods for constructing triangular norms and triangular conorms on an arbitrary bounded lattice under some constraints. Also, we give some illustrative examples for the clarity. Then, we investigate the relation between introduced methods and present methods given in Theorem 1 and Theorem 2. So, we obtain some important results from this investigation.
Keywords: Bounded lattice; triangular norm; triangular conorm; ordinal sum

1 Introduction

1.1 A brief review on the development of triangular norms and triangular conorms

The triangular norms (t-norms for short) with 1 as neutral element and triangular conorms (t-conorms for short) with 0 as neutral element were introduced by Schweizer and Sklar in [28]. These operators are extensively used in many applications in fuzzy set theory, fuzzy logics, multicriteria decision support and several branches of information sciences.

As an important tool in the construction of operators, ordinal sums can be used to construct new t-norms and t-conorms on bounded lattices. In the following, we give a brief introduction for some existing studies of t-norms and t-conorms on bounded lattices by means of ordinal sums.

The notion of ordinal sum of semigroups in Clifford's sense [11] was further developed by Mostert and Shields [21] and later used for introducing new t-norms and conorms on the unit interval $[0,1]$, see [19]. Note that there is a minor difference in ordinal sum construction for triangular norms (based on min operator) with those for triangular conorms (based on max operator). Since Goguen' s [17] generalization of the classical fuzzy sets (with membership values from $[0,1]$) to L-fuzzy sets (with membership values from a bounded lattice L), there is a growing interest in t-norms and t-conorms on bounded lattices, in particular in ordinal sum constructions. Saminger [27] focused on ordinal sums of t-norms acting on some particular bounded lattice which is not necessarily a chain or an ordinal sum of lattices. Also, she provided necessary and sufficient conditions for an ordinal sum operation yielding again a t-norm on some bounded lattice whereas the operation is determined by an arbitrary selection of subintervals as carriers for arbitrary summand t-norms. Medina [20] presented several necessary and sufficient conditions for ensuring whether an ordinal sum on a bounded lattice of arbitrary t-norms is a t-norm.

Ertuğrul, Karaçal, Mesiar [16] showed a modification of ordinal sums of t-norms and conorms resulting to t-norms and conorms on particularly bounded lattice. Also, they presented a new method for constructing the t-norms and t-conorms on special bounded lattices L by using the existence of t-norms on a sublattice $[a,1]$ and t-conorms on a sublattice $[0,a]$, respectively, where $a \in L \setminus \{0,1\}$.

1.2 The motivation

Recently, ordinal sums of t-norms and t-conorms on bounded lattices have been studied intensively. Unlike the various classes of t-norms and t-conorms on the unit interval, nowadays, the classes of t-norms and t-conorms on bounded lattices are not very clearly yet, even though there are many studies for t-norms and t-conorms on bounded lattices stated in Subsection 1.1. Therefore, as a supplement of this research topic from the theoretical viewpoint, the study of the construction of t-norms and t-conorms on bounded lattices can be regarded as one of the most important and meaningful work. In 2015, a modification of ordinal sums of t-norms and t-conorms resulting to a t-norms and t-conorms on an arbitrary bounded lattice was shown by Ertuğrul, Karaçal, Mesiar [16]. Further modifications were proposed by Çaylı [13, 12], Aşıcı and Mesiar [4, 5], Aşıcı [2, 3], Ouyang, Zhang, Baets [22] and Dan, Hu, Qiao [14]. In 2020, a new ordinal sum construction of t-norms and t-conorms on bounded lattices based on interior and closure operators was proposed by Dvořák, Holčapek [15]. Also, the proposed method generalized several known constructions and provided a simple tool to introduce new classes of t-norms and t-conorms.

In this paper, we present ordinal sum construction of t-norms and t-conorms on an arbitrary bounded lattice satisfying some constraints for a fixed element $a \in L \setminus \{0,1\}$, by using the existence of t-norms on the sublattice $[0,a]$ and of t-conorms on the sublattice $[a,1]$, respectively.

This paper is organized as follows. In Section 2, some basic notions are shortly presented. In Section 3, we present ordinal sum construction of t-norms and t-conorms on an appropriate bounded lattice under some constraints, respectively. We should add these constraints to satisfy commutativity of t-norms and t-conorms on bounded lattices. Also, we give some illustrative examples. Then, we investigate the relation between introduced methods and present methods proposed by Ertuğrul, Karaçal, Mesiar [16] and Aşıcı, Mesiar [4]. Then, we provide some examples. Finally, some concluding remarks are added.

2 Preliminaries

In this section, we recall some concepts based on bounded lattices, t-norms, t-conorms and some properties related to them. A bounded lattice $(L, \leq, 0_L, 1_L)$ is a lattice that has the top element and bottom element denoted as 1_L and 0_L , respectively.

Let $c, d \in L$. We use the notation $c \parallel d$ to denote that c and d are incomparable. I_c denotes the family of all incomparable elements with c , that is, $I_c = \{x \in L \mid x \parallel c\}$.

Given a bounded lattice $(L, \leq, 0_L, 1_L)$ and $c, d \in L$, $c \leq d$, a subinterval $[c,d]$ of L is defined as

$[c,d] = \{x \in L \mid c \leq x \leq d\}$
Similarly, $[c,d) = \{x \in L \mid c \leq x < d\}$, $(c,d] = \{x \in L \mid c < x \leq d\}$ and $(c,d) = \{x \in L \mid c < x < d\}$ (see [1, 10, 18]).

Definition 1 *[6, 27] Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice. A triangular norm T (t-norm) is a binary operation on L which is commutative, associative, increasing with respect to both variables and it satisfies $T(x, 1_L) = x$ for all $x \in L$.*

Example 1 *[19] The following are the four basic t-norms T_M, T_P, T_L and T_D on real unit interval $[0,1]$ given by, respectively,*
$T_M(x,y) = \min(x,y)$,
$T_P(x,y) = xy$,
$T_L(x,y) = \max(x+y-1, 0)$,
$$T_D(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1)^2 , \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Definition 2 *[7, 27] Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice. A triangular conorm S (t-conorm) is a binary operation on L which is commutative, associative, increasing with respect to both variables and it satisfies $S(x, 0_L) = x$ for all $x \in L$.*

Example 2 *[8, 19] The following are the four basic t-conorms S_M, S_P, S_L and S_D on real unit interval $[0,1]$ given by, respectively,*
$S_M(x,y) = \max(x,y)$,
$S_P(x,y) = x + y - xy$,
$S_L(x,y) = \min(x+y, 1)$,
$$S_D(x,y) = \begin{cases} 1 & \text{if } (x,y) \in (0,1]^2 , \\ \max(x,y) & \text{otherwise.} \end{cases}$$

The t-norms T_{inf} and T_W on an arbitrary bounded lattice L are defined as follows, respectively:
$T_{inf}(x,y) = inf\{x,y\}$
$$T_W(x,y) = \begin{cases} inf\{x,y\} & if\ 1_L \in \{x,y\} , \\ 0_L & \text{otherwise.} \end{cases}$$
Similarly, the t-conorms S_{sup} and S_W on L are defined as follows, respectively:
$S_{sup}(x,y) =$
$$S_W(x,y) = \begin{cases} sup\{x,y\} & if\ 0_L \in \{x,y\} , \\ 1_L & \text{otherwise.} \end{cases}$$

Theorem 1 *[16] Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice and $a \in L \setminus \{0_L, 1_L\}$. If V_T is a t-norm on $[a, 1_L]$ and W_T is a t-conorm on $[0_L, a]$, then the functions $T^* : L^2 \to L$ and $S^* : L^2 \to L$ are a t-norm and a t-conorm on L, respectively, where*

$$T^*(x,y) = \begin{cases} inf\{x,y\} & if\ x = 1_L\ or\ y = 1_L , \\ V_T(x,y) & if\ x,y \in [a, 1_L) , \\ inf\{x,y,a\} & otherwise. \end{cases}$$

$$S^*(x,y) = \begin{cases} sup\{x,y\} & if\ x = 0_L\ or\ y = 0_L , \\ W_T(x,y) & if\ x,y \in (0_L, a] , \\ sup\{x,y,a\} & otherwise. \end{cases}$$

Theorem 2 *[4] Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice and $a \in L \setminus \{0_L, 1_L\}$. Given t-norm V on $[0_L, a]$ and t-conorm W on $[a, 1_L]$.*

i) If $x \parallel y$ for all $x \in I_a$ and $y \in (0_L, a]$, then the function $T^{\sim} : L^2 \to L$ defined as follows is a t-norm on L

$$T^{\sim}(x,y) = \begin{cases} V(x,y) & if\ (x,y) \in [0_L,a]^2\ , \\ 0_L & if\ (x,y) \in [0_L,a) \times I_a \\ & \cup I_a \times [0_L,a) \cup [a,1_L) \times I_a \\ & \cup I_a \times [a,1_L) \cup I_a \times I_a\ , \\ inf\{x,y\} & otherwise. \end{cases}$$

ii) If $x \parallel y$ for all $x \in I_a$ and $y \in [a, 1_L)$, then the function $S^{\sim} : L^2 \to L$ defined as follows is a t-conorm on L

$$S^{\sim}(x,y) = \begin{cases} W(x,y) & if\ (x,y) \in [a,1_L]^2\ , \\ 1_L & if\ (x,y) \in (a,1_L] \times I_a \\ & \cup I_a \times (a,1_L] \cup (0_L,a] \times I_a \\ & \cup I_a \times (0_L,a] \cup I_a \times I_a\ , \\ sup\{x,y\} & otherwise. \end{cases}$$

3 New methods to construct t-norms and t-conorms on appropriate bounded lattices

In this section, we propose new construction methods for t-norms and t-conorms on an appropriate bounded lattice L in Theorem 3 and Theorem 4, respectively, where $a \in L \setminus \{0_L, 1_L\}$, V is t-norm on $[0_L, a]$ and W is t-conorm on $[a, 1_L]$, respectively. Also, we give some illustrative examples for clarity.

Theorem 3 *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice and $a \in L \setminus \{0_L, 1_L\}$. If $x \parallel y$ for all $x \in I_a$ and $y \in (0_L, a]$, and $x < y$ for all $x \in I_a$ and $y \in (a, 1_L]$, then the function $T : L^2 \to L$ defined as follows is a t-norm on L, where V is a t-norm on $[0_L, a]$.*

$$T(x,y) = \begin{cases} V(x,y) & if\ (x,y) \in [0_L,a]^2\ , \\ y & if\ (x,y) \in (a,1_L] \times I_a\ , \\ x & if\ (x,y) \in I_a \times (a,1_L]\ , \\ 0_L & if\ (x,y) \in [0_L,a] \times I_a \\ & \cup I_a \times [0_L,a]\ , \\ inf\{x,y\} & otherwise. \end{cases}$$

Remark 1 *The t-norm T introduced in Theorem 3 can be alternatively described as follows:*

$$T(x,y) = \begin{cases} V(x,y) & if\ (x,y) \in [0_L,a]^2\ , \\ y & if\ (x,y) \in (a,1_L] \times I_a \\ & \cup [a,1_L] \times [0_L,a]\ , \\ x & if\ (x,y) \in I_a \times (a,1_L] \\ & \cup [0_L,a] \times [a,1_L]\ , \\ 0_L & if\ (x,y) \in [0_L,a] \times I_a \\ & \cup I_a \times [0_L,a]\ , \\ inf\{x,y\} & if\ (x,y) \in [a,1_L]^2 \cup I_a \times I_a. \end{cases}$$

Corollary 1 *If we take $V = T_{inf}$ on $[0_L, a]$ given in Theorem 3, then we obtain the following t-norm on L.*

$$T(x,y) = \begin{cases} y & if\ (x,y) \in (a,1_L] \times I_a\ , \\ x & if\ (x,y) \in I_a \times (a,1_L]\ , \\ 0_L & if\ (x,y) \in [0_L,a] \times I_a \\ & \cup I_a \times [0_L,a]\ , \\ inf\{x,y\} & otherwise. \end{cases}$$

Example 3 *Consider the lattice $(L_1 = \{0_{L_1}, p, a, m, q, d, 1_{L_1}\}, \leq, 0_{L_1}, 1_{L_1})$ in Figure 1 satisfies the constraints of Theorem 3 (for element $a \in L_1$). Consider the t-norm $V : [0_{L_1}, a]^2 \to [0_{L_1}, a]$ as follows:*

$$V(x,y) = \begin{cases} inf\{x,y\} & if\ a \in \{x,y\}\ , \\ 0_{L_1} & otherwise. \end{cases}$$

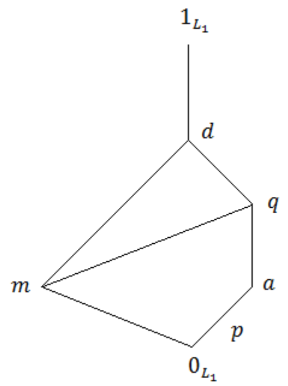

Figure 1: The lattice L_1

It is clear that the function T on L_1 defined by Table 1 is a t-norm.

Table 1: The t-norm T on L_1

T	0_{L_1}	p	a	m	q	d	1_{L_1}
0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}
p	0_{L_1}	0_{L_1}	p	0_{L_1}	p	p	p
a	0_{L_1}	p	a	0_{L_1}	a	a	a
m	0_{L_1}	0_{L_1}	0_{L_1}	m	m	m	m
q	0_{L_1}	p	a	m	q	q	q
d	0_{L_1}	p	a	m	q	d	d
1_{L_1}	0_{L_1}	p	a	m	q	d	1_{L_1}

Remark 2 *Let $(L, \leq, 0_L, 1_L)$ be a bounded lattice and $a \in L \setminus \{0_L, 1_L\}$. In Theorem 3, observe that the condition for all $x \in I_a$ and $y \in (0_L, a]$ it holds $x \parallel y$ can not be omitted, in general. The following example illustrates the fact that the function $T : L^2 \to L$ defined by Theorem 3 is not a t-norm.*

Example 4 *Consider the lattice* $(L_2 = \{0_{L_2}, p, a, m, q, d, 1_{L_2}\}, \leq, 0_{L_2}, 1_{L_2})$ *in Figure 2 does not satisfy (for* $a \in L_2$*) one of the constraints of Theorem 3. That is, there is the element* $p \in L_2$ *such that* $p < m$ *for* $m \in I_a$ *and* $p \in (0_{L_2}, a)$*. Consider the t-norm* $V : [0_{L_2}, a]^2 \to [0_{L_2}, a]$, $V(x,y) = inf\{x,y\}$.

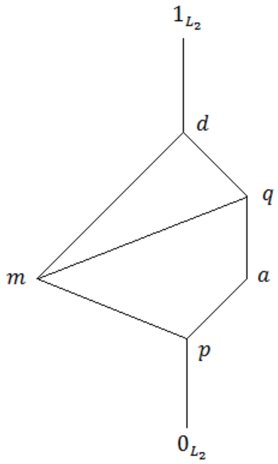

Figure 2: The lattice L_2

Then, the function T *on* L_2 *defined by Table 2 is not a t-norm. Indeed, it does not satisfy monotonicity. Clearly,* $p < m$ *and* $T(p,p) = p \nleq 0_{L_2} = T(m,p)$.

Table 2: The function T on L_2

T	0_{L_2}	p	a	m	q	d	1_{L_2}
0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}
p	0_{L_2}	p	p	0_{L_2}	p	p	p
a	0_{L_2}	p	a	0_{L_2}	a	a	a
m	0_{L_2}	0_{L_2}	0_{L_2}	m	m	m	m
q	0_{L_2}	p	a	m	q	q	q
d	0_{L_2}	p	a	m	q	d	d
1_{L_2}	0_{L_2}	p	a	m	q	d	1_{L_2}

Remark 3 *Let* $(L, \leq, 0_L, 1_L)$ *be a bounded lattice and* $a \in L \setminus \{0_L, 1_L\}$*. In Theorem 3, observe that the condition for all* $x \in I_a$ *and* $y \in (a, 1_L]$ *it holds* $x < y$ *can not be omitted, in general. The following example illustrates the fact that the function* $T : L^2 \to L$ *defined by Theorem 3 is not a t-norm.*

Example 5 *Consider the lattice* $(L_3 = \{0_{L_3}, p, a, m, q, d, 1_{L_3}\}, \leq, 0_{L_3}, 1_{L_3})$ *in Figure 3 does not satisfy (for* $a \in L_3$*) one of the constraints of Theorem 3. Namely, there is the element* $m \in L_3$ *such that* $m \parallel q$ *for* $m \in I_a$ *and* $q \in (a, 1_{L_3})$*. Consider the t-norm* $V : [0_{L_3}, a]^2 \to [0_{L_3}, a]$, $V(x,y) = inf\{x,y\}$.

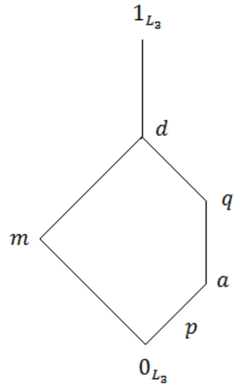

Figure 3: The lattice L_3

Then, the function T *on* L_3 *defined by Table 3 is not a t-norm. Indeed, it does not satisfy monotonicity. Clearly,* $m < d$ *and* $T(m,q) = m \nleq q = T(d,q)$.

Table 3: The function T on L_3

T	0_{L_3}	p	a	m	q	d	1_{L_3}
0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}
p	0_{L_3}	p	p	0_{L_3}	p	p	p
a	0_{L_3}	p	a	0_{L_3}	a	a	a
m	0_{L_3}	0_{L_3}	0_{L_3}	m	m	m	m
q	0_{L_3}	p	a	m	q	q	q
d	0_{L_3}	p	a	m	q	d	d
e	0_{L_3}	p	a	m	q	d	e
1_{L_3}	0_{L_3}	p	a	m	q	d	1_{L_3}

Proposition 1 *Let* $(L, \leq, 0_L, 1_L)$ *be a bounded lattice with* $a \in L \setminus \{0_L, 1_L\}$ *such that* $x \parallel y$ *for all* $x \in I_a$ *and* $y \in (0_L, a]$*, and* $x < y$ *for all* $x \in I_a$ *and* $y \in (a, 1_L]$*. Suppose* T *and* $T^{\sim}$ *are the t-norms on* L *defined as in Theorem 3 and Theorem 2 with underlying t-norm* V *on* $[0_L, a]$*. Then,* $T^{\sim} \leq T$.

Example 6 *Consider the lattice* $(L_1 = \{0_{L_1}, p, a, m, q, d, 1_{L_1}\}, \leq, 0_{L_1}, 1_{L_1})$ *which is depicted by Hasse diagram in Figure 1. Consider the t-norm* V *on* $[0_{L_1}, a]$ *defined as follows:*

$$V(x,y) = \begin{cases} inf\{x,y\} & if\ a \in \{x,y\}, \\ 0_{L_1} & otherwise. \end{cases}$$

By using Theorem 3 and Theorem 2 define the corresponding t-norms T *and* $T^{\sim}$ *as given in Table 1 and Table 4, respectively.*

According to Table 1 and Table 4, it is clear that $T^{\sim} \leq T$.

Table 4: The t-norm $T^{\sim}$ on L_1

$T^{\sim}$	0_{L_1}	p	a	m	q	d	1_{L_1}
0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}
p	0_{L_1}	0_{L_1}	p	0_{L_1}	p	p	p
a	0_{L_1}	p	a	0_{L_1}	a	a	a
m	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	m
q	0_{L_1}	p	a	0_{L_1}	q	q	q
d	0_{L_1}	p	a	0_{L_1}	q	d	d
1_{L_1}	0_{L_1}	p	a	m	q	d	1_{L_1}

Remark 4 *Let $(L,\leq,0_L,1_L)$ be a bounded lattice with $a\in L\setminus\{0_L,1_L\}$ such that $x \parallel y$ for all $x\in I_a$ and $y\in(0_L,a]$, and $x<y$ for all $x\in I_a$ and $y\in(a,1_L]$. Suppose T and $T^\star$ are the t-norms on L defined as in Theorem 3 and Theorem 1 with underlying t-norms V and V_T on $[0_L,a]$ and $[a,1_L]$, respectively. One can wonder if the t-norms T and $T^\star$ can comparable on any bounded lattice. To illustrate this question we shall give the following Examples and Remarks. In Example 7 and Example 8, we show that $T^\star\leq T$ and $T\leq T^\star$, respectively. In Remark 5, we show that the t-norms T and $T^\star$ cannot be compared. In Remark 6, we show that the t-norms T and $T^\star$ coincide with each other.*

Example 7 *Consider the lattice $(L_1=\{0_{L_1},p,a,m,q,d,1_{L_1}\},\leq,0_{L_1},1_{L_1})$ which is depicted by Hasse diagram in Figure 1. Consider the t-norm V on $[0_{L_1},a]$, $V(x,y)=inf\{x,y\}$ and the t-norm V_T on $[a,1_{L_1}]$, $V_T(x,y)=inf\{x,y\}$.*

By using Theorem 3 and Theorem 1 define the corresponding t-norms T and $T^\star$ as given in Table 5 and Table 6, respectively.

Table 5: The t-norm T on L_1

T	0_{L_1}	p	a	m	q	d	1_{L_1}
0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}
p	0_{L_1}	p	p	0_{L_1}	p	p	p
a	0_{L_1}	p	a	0_{L_1}	a	a	a
m	0_{L_1}	0_{L_1}	0_{L_1}	m	m	m	m
q	0_{L_1}	p	a	m	q	q	q
d	0_{L_1}	p	a	m	q	d	d
1_{L_1}	0_{L_1}	p	a	m	q	d	1_{L_1}

According to Table 5 and Table 6, it is clear that $T^\star\leq T$.

Remark 5 *In Example 7, if we take the t-norm V on $[0_{L_1},a]$ defined as follows:*

$$V(x,y)=\begin{cases} inf\{x,y\} & if\ a\in\{x,y\}\ , \\ 0_L & otherwise. \end{cases}$$

Table 6: The t-norm $T^\star$ on L_1

$T^\star$	0_{L_1}	p	a	m	q	d	1_{L_1}
0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}
p	0_{L_1}	p	p	0_{L_1}	p	p	p
a	0_{L_1}	p	a	0_{L_1}	a	a	a
m	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	m
q	0_{L_1}	p	a	0_{L_1}	q	q	q
d	0_{L_1}	p	a	0_{L_1}	q	d	d
e	0_{L_1}	p	a	0_{L_1}	q	d	e
1_{L_1}	0_{L_1}	p	a	m	q	d	1_{L_1}

Then, we can not obtain $T^\star\leq T$ since $T(p,p)=0_{L_1}$. So, we can not compare these t-norms on every case.

Example 8 *Consider the lattice $(L_4=\{0_{L_4},t,a,s,k,p,1_{L_4}\},\leq,0_{L_4},1_{L_4})$ which is depicted by Hasse diagram in Figure 4. Consider the t-norm V_T on $[a,1_{L_4}]$, $V_T(x,y)=inf\{x,y\}$ and the t-norm V on $[0_{L_4},a]$ defined as follows:*

$$V(x,y)=\begin{cases} inf\{x,y\} & if\ a\in\{x,y\}\ , \\ 0_{L_4} & otherwise. \end{cases}$$

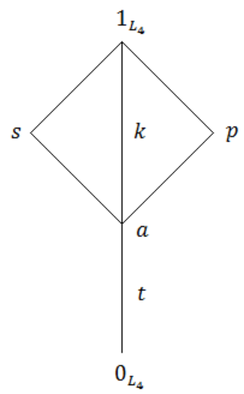

Figure 4: The lattice L_4

By using Theorem 3 and Theorem 1 define the corresponding t-norms T and $T^\star$ as given in Table 7 and Table 8, respectively.

According to Table 7 and Table 8, it is clear that $T\leq T^\star$.

Remark 6 *In Example 8, if we take the t-norm V on $[0_{L_4},a]$, $V(x,y)=inf\{x,y\}$, then we obtain $T=T^\star$ since $T(t,t)=t$.*

Next, we propose new construction method for t-conorms on an arbitrary bounded lattice L with some properties related to an element $a\in L\setminus\{0_L,1_L\}$.

Table 7: The t-norm T on L_4

T	0_{L_4}	t	a	s	k	p	1_{L_4}
0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}
t	0_{L_4}	0_{L_4}	t	t	t	t	t
a	0_{L_4}	t	a	a	a	a	a
s	0_{L_4}	t	a	s	a	a	s
k	0_{L_4}	t	a	a	k	a	k
p	0_{L_4}	t	a	a	a	p	p
1_{L_4}	0_{L_4}	t	a	s	k	p	1_{L_4}

Table 8: The t-norm $T^\star$ on L_4

$T^\star$	0_{L_4}	t	a	s	k	p	1_{L_4}
0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}
t	0_{L_4}	t	t	t	t	t	t
a	0_{L_4}	t	a	a	a	a	a
s	0_{L_4}	t	a	s	a	a	s
k	0_{L_4}	t	a	a	k	a	k
p	0_{L_4}	t	a	a	a	p	p
1_{L_4}	0_{L_4}	t	a	s	k	p	1_{L_4}

Theorem 4 *Let $(L,\leq,0_L,1_L)$ be a bounded lattice and $a\in L\setminus\{0_L,1_L\}$. If $x \parallel y$ for all $x\in I_a$ and $y\in[a,1_L)$, and $x>y$ for all $x\in I_a$ and $y\in[0_L,a)$, then the function $S:L^2\to L$ defined as follows is a t-conorm on L, where W is a t-conorm on $[a,1_L]^2$.*

$$S(x,y)=\begin{cases} W(x,y) & if\ (x,y)\in[a,1_L]^2\ , \\ y & if\ (x,y)\in[0_L,a)\times I_a\ , \\ x & if\ (x,y)\in I_a\times[0_L,a)\ , \\ 1_L & if\ (x,y)\in[a,1_L]\times I_a \\ & \cup I_a\times[a,1_L]\ , \\ sup\{x,y\} & otherwise. \end{cases}$$

The proof of this Theorem is the same as the related proof of Theorem 3. This argument is based on the the fact that exchanging, in original bounded lattice $\mathscr{L}=(L,\wedge,\vee,0_L,1_L)$, $\wedge$ and $\vee$, and 0_L and 1_L, i.e., considering $\overline{\mathscr{L}}=(L,\overline{\wedge},\overline{\vee},\overline{0_L},\overline{1_L})$ with $\overline{\wedge}=\vee$, $\overline{\vee}=\wedge$, $\overline{0_L}=1_L$, $\overline{1_L}=0_L$, we obtain a dual lattice $\overline{\mathscr{L}}$, in this duality, t-norms on $\mathscr{L}$ are linked to t-conorms on $\overline{\mathscr{L}}$. Also, we omitted the examples related to t-conorms from this duality.

Remark 7 *The t-conorm S introduced in Theorem 4 can be described alternatively as follows:*

$$S(x,y)=\begin{cases} W(x,y) & if\ (x,y)\in[a,1_L]^2\ , \\ y & if\ (x,y)\in[0_L,a)\times I_a \\ & \cup[0_L,a]\times[a,1_L]\ , \\ x & if\ (x,y)\in I_a\times[0_L,a) \\ & \cup[a,1_L]\times[0_L,a]\ , \\ 1_L & if\ (x,y)\in[a,1_L]\times I_a\cup \\ & I_a\times[a,1_L]\ , \\ sup\{x,y\} & if\ (x,y)\in[0_L,a]^2\cup I_a\times I_a. \end{cases}$$

Corollary 2 *If we take $W=S_{sup}$ on $[a,1_L]^2$ given in Theorem 4, then we obtain the following t-conorm on L.*

$$S(x,y)=\begin{cases} y & if\ (x,y)\in[0_L,a)\times I_a\ , \\ x & if\ (x,y)\in I_a\times[0_L,a)\ , \\ 1_L & if\ (x,y)\in[a,1_L]\times I_a \\ & \cup I_a\times[a,1_L]\ , \\ sup\{x,y\} & otherwise. \end{cases}$$

4 Concluding remarks

We have introduced new constructions method for building t-norms and t-conorms on an arbitrary bounded lattice with some constraints. Based on these methods, we have introduced a new class of t-norms T and t-conorms S on an arbitrary relevant bounded lattice, by using the existence of a t-norm V on a sublattice $[0,a]$ and a t-conorm W on a sublattice $[a,1]$, respectively. In order to better understand the introduced t-norms T and t-conorms S, we have given some illustrative examples. Also, we investigate the relation between introduced methods and present methods proposed by Ertuğrul, Karaçal, Mesiar [16] and Aşıcı, Mesiar [4]. Then, we provide some examples. Our methods allow to construct t-norms and t-conorms with the unitary subsets of $[0,1]$ playing the role of its identity in the bounded lattices frequently considered in several branches of uncertainty modeling and information systems such as group decision making problems [26] and computing with words [24, 25], including set-valued fuzzy sets, hesitant fuzzy sets and typical hesitant fuzzy sets [9, 29].

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