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Sup-T Compositions of Partial Fuzzy Relations

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Abstract

We present basic properties of the operation of sup-T composition of partial fuzzy relations, i.e., fuzzy relations defined on arbitrary subsets of the Cartesian square of the referential set. We show that for partial fuzzy relations, these properties are fairly similar to those for totally defined fuzzy relations. Furthermore, we show that the studied elementary properties of sup-T composition of partial fuzzy relations carry over to several composition-related operations on partial fuzzy relations and partial fuzzy sets, including the operations of image, preimage, and Cartesian product.

Keywords: Partial fuzzy relation, Partial fuzzy set, Fuzzy relational composition, Sup-T product.

1 Introduction

The operation of sup-T composition is a broadly applicable and well-studied notion in the theory of fuzzy relations [12, 4]. Well known results on fuzzy relational composition found in the literature usually regard either *homogeneous* fuzzy relations (i.e., fuzzy relations defined on the Cartesian square U^2 of a referential set U) or *heterogeneous* fuzzy relations (defined on Cartesian products of the form $V \times W$). Here we examine the properties of sup-T composition when it is applied to *partial* fuzzy relations, i.e., fuzzy relations defined on arbitrary subsets $X \subseteq U^2$ of the Cartesian square of the referential set U.

As represented in [2], partial fuzzy sets and partial fuzzy relations carry information about their domains of definition by means of an additional membership value $* \notin [0,1]$. Just like the regular membership degrees from $[0,1]$, the value $*$ can also be assigned to elements by membership functions, to represent an out-of-the-domain assignment error. The error value $*$ can be processed in various ways, depending on the intended application. The different $*$-processing methods are implemented by means of several families of partial fuzzy set operations, each designed to work with $*$ in a specific manner. Prominent among these families are the Bochvar and the Sobociński operations, which treat the value $*$, respectively, as a fatal and an ignorable error. The resulting formalism of partial fuzzy set theory, described in detail in [2], brings new possibilities to fuzzy systems and other applied fuzzy methods, providing them with a means for treating assignment errors internally in various desired ways, as specified by the choice of $*$-adapted operations [5, 8, 7].

In this paper, we investigate the sup-T composition along the lines of [1], in the more general setting of partial fuzzy relations. Moreover, similarly as in [1] we show that even in the setting of partial fuzzy sets and partial fuzzy relations, the studied elementary properties of sup-T composition automatically apply to several related operations, including those of image, preimage, and Cartesian product. This result is obtained by identifying partial fuzzy sets and membership degrees with partial fuzzy relations of certain forms; by this trick, the properties of relational composition are carried over to the lower-dimensional objects.

The paper is organized as follows. In Sections 2–4, we recall the prerequisite notions from the theory of partial fuzzy sets and relations (mostly from [3, 2]). Section 5 lists the basic properties of sup-T composition of partial fuzzy relations. The partial fuzzy relational representation of partial fuzzy sets and membership degrees along the lines of [1] and the resulting corollaries on composition-related operations are presented in Sections 6–7. Due to space limitation, most proofs are omitted here; they will be presented in an upcoming full paper (in preparation).

2 Membership Degrees for Partial Fuzzy Sets

In this section, we recall an algebraic structure commonly assumed for the degrees of membership in fuzzy sets and its extension for partial fuzzy sets. More details on the definitions can be found in the original papers [9, 3, 2].

The degrees of membership in fuzzy sets are commonly assumed to form a *complete linear MTL-algebra,* i.e., a lattice-complete linearly ordered commutative bounded integral residuated lattice

$$\boldsymbol{L} = \langle L, \vee, \wedge, \&, \rightarrow, 0, 1\rangle.$$

Principal examples of complete linear MTL-algebras are *standard MTL-algebras,* in which $L = [0,1]$ and & is a left-continuous *t-norm* (i.e., a commutative associative monotone binary operation with the unity element 1). From now on, let us fix a complete linear MTL-algebra $\boldsymbol{L}$.

In the following sections, we will consider $\boldsymbol{L}$-valued membership functions that are *partial,* i.e., that may have some membership values undefined. A convenient way to handle them is to extend the algebra $\boldsymbol{L}$ of membership degrees by an extra dummy element $* \notin \boldsymbol{L}$, intended to represent an undefined membership degree (see [3, 2]). For brevity, we will denote the extended set $\boldsymbol{L} \cup \{*\}$ of membership degrees by $\boldsymbol{L}_*$. Of the operations on $\boldsymbol{L}_*$ introduced in [3, 2], in this paper we only need the following ones:

Definition 2.1. Let $\boldsymbol{L} = \langle L, \vee, \wedge, \&, \rightarrow, 0, 1\rangle$ be a complete linear MTL-algebra and $* \notin \boldsymbol{L}$. We define the following operations on $\boldsymbol{L}_* = \boldsymbol{L} \cup \{*\}$:

- *Bochvar conjunction:*

$$\alpha \,\&_{\mathrm{B}}\, \beta = \begin{cases} \alpha \,\&\, \beta & \text{if } \alpha, \beta \neq * \\ * & \text{otherwise} \end{cases}$$

- *Bochvar infimum:*

$$\bigwedge_{i\in I}{}_{\mathrm{B}}\, \alpha_i = \begin{cases} \bigwedge\limits_{i\in I} \alpha_i & \text{if } \alpha_i \neq * \text{ for each } i \in I \\ * & \text{otherwise} \end{cases}$$

- *Bochvar supremum:*

$$\bigvee_{i\in I}{}_{\mathrm{B}}\, \alpha_i = \begin{cases} \bigvee\limits_{i\in I} \alpha_i & \text{if } \alpha_i \neq * \text{ for all } i \in I \\ * & \text{otherwise} \end{cases}$$

- *Sobociński infimum:*

$$\bigwedge_{i\in I}{}_{\mathrm{S}}\, \alpha_i = \begin{cases} \bigwedge\limits_{\substack{i\in I \\ \alpha_i \neq *}} \alpha_i & \text{if } \alpha_i \neq * \text{ for some } i \in I \\ * & \text{otherwise} \end{cases}$$

- *Sobociński supremum:*

$$\bigvee_{i\in I}{}_{\mathrm{S}}\, \alpha_i = \begin{cases} \bigvee\limits_{\substack{i\in I \\ \alpha_i \neq *}} \alpha_i & \text{if } \alpha_i \neq * \text{ for some } i \in I \\ * & \text{otherwise} \end{cases}$$

where I is a crisp index set and $\alpha, \beta, \alpha_i \in \boldsymbol{L}_*$ for all $i \in I$.

Note that the value $*$ acts as the neutral element for the Sobociński operations and as the absorbing element for the Bochvar operations.

Definition 2.2. We define the following orders on $\boldsymbol{L}_*$:

- $\alpha \le \beta$ in $\boldsymbol{L}_*$ iff $\alpha \le \beta$ in $\boldsymbol{L}$ or $\alpha = \beta = *$.
- $\alpha \le_{\mathrm{sub}} \beta$ in $\boldsymbol{L}_*$ iff $\alpha \le \beta$ in $\boldsymbol{L}$ or $\alpha = *$.

In other words, the ordering $\le$ of $\boldsymbol{L}_*$ extends the lattice order of $\boldsymbol{L}$ by making $*$ incomparable with any element of $\boldsymbol{L}$, while $\le_{\mathrm{sub}}$ extends the lattice order of $\boldsymbol{L}$ by making $*$ smaller than all elements of $\boldsymbol{L}$.

The results of the subsequent sections are based on the following properties of the Bochvar and Sobociński suprema and infima:

Proposition 2.3. *Let I be a crisp index set and let $\alpha, \beta_i \in \boldsymbol{L}_*$ for all $i \in I$. Then:*

$$\alpha \,\&_{\mathrm{B}} \bigvee_{i\in I}{}_{\mathrm{B}}\, \beta_i = \bigvee_{i\in I}{}_{\mathrm{B}} (\alpha \,\&_{\mathrm{B}}\, \beta_i) \tag{1}$$

$$\alpha \,\&_{\mathrm{B}} \bigvee_{i\in I}{}_{\mathrm{S}}\, \beta_i = \bigvee_{i\in I}{}_{\mathrm{S}} (\alpha \,\&_{\mathrm{B}}\, \beta_i) \tag{2}$$

$$\alpha \,\&_{\mathrm{B}} \bigwedge_{i\in I}{}_{\mathrm{B}}\, \beta_i \le \bigwedge_{i\in I}{}_{\mathrm{B}} (\alpha \,\&_{\mathrm{B}}\, \beta_i) \tag{3}$$

$$\alpha \,\&_{\mathrm{B}} \bigwedge_{i\in I}{}_{\mathrm{S}}\, \beta_i \le \bigwedge_{i\in I}{}_{\mathrm{S}} (\alpha \,\&_{\mathrm{B}}\, \beta_i) \tag{4}$$

If the algebra $\boldsymbol{L}$ of membership degrees is an MV-algebra or a finite MTL-algebra or a dense BL-algebra (which includes the case where $\boldsymbol{L} = [0,1]$ and & is a continuous t-norm), then (3)–(4) *hold even with equality:*

$$\alpha \,\&_{\mathrm{B}} \bigwedge_{i\in I}{}_{\mathrm{B}}\, \beta_i = \bigwedge_{i\in I}{}_{\mathrm{B}} (\alpha \,\&_{\mathrm{B}}\, \beta_i) \tag{5}$$

$$\alpha \,\&_{\mathrm{B}} \bigwedge_{i\in I}{}_{\mathrm{S}}\, \beta_i = \bigwedge_{i\in I}{}_{\mathrm{S}} (\alpha \,\&_{\mathrm{B}}\, \beta_i) \tag{6}$$

Proof. To illustrate, let us prove (4) and (6). The remaining proofs, omitted here due to space constraints, will be given in an upcoming full paper (in progress).

If $\alpha = *$, then both sides in (4) and (6) equal $*$. Let $\alpha \neq *$. Then $\alpha \,\&_{\mathrm{B}}\, \beta_i \neq *$ iff $\beta_i \neq *$. Thus, by the definition of $\bigwedge_S$ and by the distributivity law valid in complete MTL-algebras, we obtain (4) as follows:

$$\alpha \mathbin{\&_B} \bigwedge_{S\, i\in I} \beta_i = \alpha \mathbin{\&} \bigwedge_{\substack{i\in I\\ \beta_i\neq *}} \beta_i \leq \bigwedge_{\substack{i\in I\\ \beta_i\neq *}} (\alpha \mathbin{\&} \beta_i) = \bigwedge_{S\, i\in I} (\alpha \mathbin{\&_B} \beta_i)$$

In the MTL-algebras listed in the additional assumption for (6), equality holds in place of the middle inequality above: cf. [10, Rem. 5.1.19], [9, Prop. 1(30)], and [6, Tab. 3]. □

Proposition 2.4. *Let I,J be crisp index sets and let $\alpha_{i,j} \in \boldsymbol{L}_*$ for all $i \in I$ and $j \in J$. Then:*

$$\bigvee_{S\, i\in I} \bigvee_{S\, j\in J} \alpha_{i,j} = \bigvee_{S\, j\in J} \bigvee_{S\, i\in I} \alpha_{i,j} \tag{7}$$

$$\bigvee_{S\, i\in I} \bigwedge_{S\, j\in J} \alpha_{i,j} \leq \bigwedge_{S\, j\in J} \bigvee_{S\, i\in I} \alpha_{i,j} \tag{8}$$

$$\bigvee_{S\, i\in I} \bigvee_{B\, j\in J} \alpha_{i,j} \leq_{\text{sub}} \bigvee_{B\, j\in J} \bigvee_{S\, i\in I} \alpha_{i,j} \tag{9}$$

$$\bigvee_{S\, i\in I} \bigwedge_{B\, j\in J} \alpha_{i,j} \leq_{\text{sub}} \bigwedge_{B\, j\in J} \bigvee_{S\, i\in I} \alpha_{i,j} \tag{10}$$

3 Partial Fuzzy Set Theory

From now on, let us fix a crisp set $\boldsymbol{U}$ of atomic elements (the *universe of discourse*). As before, let $\boldsymbol{L}$ be a fixed lattice-complete linear MTL-algebra (of membership degrees).

By a *partial fuzzy set* on $\boldsymbol{U}$ we mean an $\boldsymbol{L}$-valued fuzzy set A on a crisp subset $X \subseteq \boldsymbol{U}$. Its membership function $\mu_A \in \boldsymbol{L}^X$ is thus a *partial* function from $\boldsymbol{U}$ to $\boldsymbol{L}$ with the *domain of definition* $X \subseteq \boldsymbol{U}$.

Following [2], we represent an $\boldsymbol{L}$-valued membership function μ_A on $X \subseteq \boldsymbol{U}$ by a (total) $\boldsymbol{L}_*$-valued membership function $\dot{\mu}_A$ on $\boldsymbol{U}$, which assigns the dummy value $*$ to the elements outside the domain X of μ_A:

$$\dot{\mu}_A(x) = \begin{cases} \mu_A(x) & \text{if } x \in X \\ * & \text{if } x \in \boldsymbol{U} \setminus X \end{cases}$$

As usual, we identify a partial fuzzy set A on $\boldsymbol{U}$ with its $\boldsymbol{L}_*$-valued membership function $\dot{\mu}_A \in \boldsymbol{L}_*^{\boldsymbol{U}}$ and write just Ax for $\dot{\mu}_A(x)$.

In Definition 3.1 we collect those of the operations introduced for partial fuzzy sets in [2] that will be needed later on. Note that for simplicity, we consider here, unlike in [1], only the unions and intersections of *crisp* systems of fuzzy sets (i.e., for crisp rather than fuzzy index sets I).

Definition 3.1. Let I be a crisp index set. We define the following operations for any partial fuzzy sets A, A_i, B on $\boldsymbol{U}$ (where $i \in I$ and $x \in \boldsymbol{U}$):

- *Domain of definition:* $(\mathrm{dom}A)x = \begin{cases} 1 & \text{if } Ax \neq * \\ 0 & \text{if } Ax = * \end{cases}$
- *Kernel:* $(\ker A)x = \begin{cases} 1 & \text{if } Ax = 1 \\ 0 & \text{if } Ax \notin \{1,*\} \\ * & \text{if } Ax = * \end{cases}$
- *Support:* $(\mathrm{supp}A)x = \begin{cases} 1 & \text{if } Ax \notin \{0,*\} \\ 0 & \text{if } Ax = 0 \\ * & \text{if } Ax = * \end{cases}$
- *Bochvar strong intersection:*

  $$(A \Cap_B B)x = Ax \mathbin{\&_B} Bx$$
- *Bochvar union:* $\left(\bigcup_{B\, i\in I} A_i\right)x = \bigvee_{B\, i\in I} A_i x$
- *Bochvar intersection:* $\left(\bigcap_{B\, i\in I} A_i\right)x = \bigwedge_{B\, i\in I} A_i x$
- *Sobociński union:* $\left(\bigcup_{S\, i\in I} A_i\right)x = \bigvee_{S\, i\in I} A_i x$
- *Sobociński intersection:* $\left(\bigcap_{S\, i\in I} A_i\right)x = \bigwedge_{S\, i\in I} A_i x$
- *Sobociński height:* $\mathrm{Hgt}_S A = \bigvee_{S\, x\in \boldsymbol{U}} Ax$

As usual, we identify the crisp set $\boldsymbol{U}$ with the fuzzy set U that assigns the value $U(x) = 1$ to all $x \in \boldsymbol{U}$. Additionally, we define the *empty-domain fuzzy set* λ on $\boldsymbol{U}$ by setting $\lambda(x) = *$ for all $x \in \boldsymbol{U}$.

Finally, we define the following relations between partial fuzzy sets on $\boldsymbol{U}$:

- *Strong equality:* $A = B$ iff $Ax = Bx$ for all $x \in \boldsymbol{U}$ (including the case when $Ax = Bx = *$).
- *Strong inclusion:* $A \subseteq B$ iff $Ax \leq Bx$ for all $x \in \boldsymbol{U}$ (where $\leq$ is the order on $\boldsymbol{L}_*$ from Definition 2.2).
- *Subfunctionality:* $A =_{\text{sub}} B$ iff $Ax = Bx$ for every $x \in \boldsymbol{U}$ such that $Ax \neq *$.
- *Subinclusion:* $A \subseteq_{\text{sub}} B$ iff $Ax \leq_{\text{sub}} Bx$ for all $x \in \boldsymbol{U}$.

Thus, besides the common requirement of pointwise (in)equality on $\mathrm{dom}A$, the strong relations $A = B$ and

$A \subseteq B$ require that $\mathrm{dom} A = \mathrm{dom} B$, while the relations $A =_{\mathrm{sub}} B$ and $A \subseteq_{\mathrm{sub}} B$ only require that $\mathrm{dom} A \subseteq \mathrm{dom} B$. Consequently, the property of A being a fuzzy set on $\boldsymbol{U}$ is characterized by the condition $A \subseteq \boldsymbol{U}$, while that of being a partial fuzzy set on $\boldsymbol{U}$ by $A \subseteq_{\mathrm{sub}} \boldsymbol{U}$.

4 Partial Fuzzy Relational Notions

Partial fuzzy relations on the universe of discourse $\boldsymbol{U}$ are partial fuzzy sets on the Cartesian square $\boldsymbol{U}^2$. The notions from Section 3 thus apply as well to partial fuzzy relations. In particular, partial fuzzy relations can be characterized by the condition $R \subseteq_{\mathrm{sub}} \boldsymbol{U}^2$, or $R \in \boldsymbol{L}_*^{\boldsymbol{U}^2}$. In other words, a partial fuzzy relation on $\boldsymbol{U}$ is just an $\boldsymbol{L}$-valued fuzzy set $R \in \boldsymbol{L}^X$ for $X \subseteq \boldsymbol{U}^2$.

If $\boldsymbol{U}$ is a finite set, $\boldsymbol{U} = \{x_1, \ldots, x_n\}$, the usual matrix notation can be employed as well for partial fuzzy relations:

$$R = \begin{pmatrix} Rx_1x_1 & \ldots & Rx_1x_n \\ \vdots & \ddots & \vdots \\ Rx_nx_1 & \ldots & Rx_nx_n \end{pmatrix} \in \boldsymbol{L}_*^{\,n \times n}$$

Several properties of partial fuzzy relations have already been given in [2, Sect. 6]. Here, we study them along the lines of [1, Sect. 4], which—from the point of view of the present paper—dealt with the special case of total homogeneous fuzzy relations on $\boldsymbol{U}$.

The central notion in [1, Sect. 4] was that of *sup-T composition* of fuzzy relation [12]. Recall the definition of the sup-T composition $R \circ S$ for total fuzzy relations $R, S \subseteq \boldsymbol{U}^2$:

$$(R \circ S)xz = \bigvee_{y \in \boldsymbol{U}} (Rxy \,\&\, Syz) \tag{11}$$

for all $x, z \in \boldsymbol{U}$.

In the setting of partial fuzzy sets, one of the most natural generalizations of sup-T composition is that which uses the *Bochvar* conjunction $\&_{\mathrm{B}}$ and the *Sobociński* supremum $\bigvee_{\mathrm{S}}$ in (11). In this way, the elements x, z are linked via y iff both $Rxy, Syz \neq *$; and the values are aggregated so that $*$-valued links are ignored. The resulting *Sobociński–Bochvar* sup-T composition (of Definition 4.1 below) has already been considered in [2, Sect. 6]. In this paper we restrict our attention to this single variant of the sup-T composition, although other partial fuzzy generalizations (such as the Bochvar–Sobociński composition studied in [5]) can also be useful for various specific purposes.

Definition 4.1. Let $R, S \subseteq_{\mathrm{sub}} \boldsymbol{U}^2$ be partial fuzzy relations on $\boldsymbol{U}$. The *(Sobociński–Bochvar sup-T) composition* $R \circ S \subseteq_{\mathrm{sub}} \boldsymbol{U}^2$ is defined by setting, for all $x, z \in \boldsymbol{U}$:

$$(R \circ S)xz = \bigvee\nolimits_{\mathrm{S}\; y \in \boldsymbol{U}} (Rxy \,\&_{\mathrm{B}}\, Syz)$$

The paper [1] also studied several operations that are closely related to the sup-T composition. Their Sobociński–Bochvar generalizations for partial fuzzy relations are defined as follows:

Definition 4.2. Let $R \subseteq_{\mathrm{sub}} \boldsymbol{U}^2$ be a partial fuzzy relation and $A, B \subseteq_{\mathrm{sub}} \boldsymbol{U}$ partial fuzzy sets. Then we define the following operations:

- The *image* $R^{\rightarrow} A$, by setting for all $y \in \boldsymbol{U}$:
$$(R^{\rightarrow} A)y = \bigvee\nolimits_{\mathrm{S}\; x \in \boldsymbol{U}} (Ax \,\&_{\mathrm{B}}\, Rxy)$$
- The *preimage* $R^{\leftarrow} A$, by setting for all $x \in \boldsymbol{U}$:
$$(R^{\leftarrow} A)x = \bigvee\nolimits_{\mathrm{S}\; y \in \boldsymbol{U}} (Ay \,\&_{\mathrm{B}}\, Rxy)$$
- The *Cartesian product* $A \times B$, by setting for all $x, y \in \boldsymbol{U}$:
$$(A \times B)xy = Ax \,\&_{\mathrm{B}}\, By$$
- The *active domain* $\mathrm{Dom}\, R$ and *active range* $\mathrm{Rng}\, R$, by setting for all $x, y \in \boldsymbol{U}$:
$$(\mathrm{Dom}\, R)x = \bigvee\nolimits_{\mathrm{S}\; y \in \boldsymbol{U}} Rxy, \quad (\mathrm{Rng}\, R)y = \bigvee\nolimits_{\mathrm{S}\; x \in \boldsymbol{U}} Rxy$$

We will also need the following defined notions:

Definition 4.3. Let $R, S \subseteq_{\mathrm{sub}} \boldsymbol{U}^2$ be partial fuzzy relations on $\boldsymbol{U}$. Then we define:

- The *transposition* R^{T} of R, by setting $R^{\mathrm{T}}xy = Ryx$ for all $x, y \in \boldsymbol{U}$.
- The partial fuzzy relation id^* of *identity,* by setting for all $x \in \boldsymbol{U}$:
$$\mathrm{id}^*\, xy = \begin{cases} 1 & \text{if } x = y \\ * & \text{otherwise,} \end{cases}$$
- The *empty-domain fuzzy relation* λ^2 (cf. Definition 3.1), by setting $\lambda^2 xy = *$ for all $x, y \in \boldsymbol{U}$.

Proposition 4.4. *The following properties of transposition are valid for all partial fuzzy relations:*

1. Double transposition: $R^{\mathrm{TT}} = R$.
2. Invariance: *Transposition commutes with all operations and relations of Section 3. Thus, e.g.:*
$$(R \Cap_{\mathrm{B}} S)^{\mathrm{T}} = R^{\mathrm{T}} \Cap_{\mathrm{B}} S^{\mathrm{T}}$$
$$\Big(\bigcup\nolimits_{\mathrm{S}\; i \in I} R_i\Big)^{\mathrm{T}} = \bigcup\nolimits_{\mathrm{S}\; i \in I} R_i^{\mathrm{T}}$$
$$R^{\mathrm{T}} \subseteq_{\mathrm{sub}} S^{\mathrm{T}} \text{ iff } R \subseteq_{\mathrm{sub}} S, \text{ etc.}$$

5 Basic Properties of Sup-T Composition for Partial Fuzzy Relations

Several properties of the Sobociński–Bochvar sup-T composition of partial fuzzy relations have already been given in [2, Sect. 6]. Here we present a more comprehensive list following the lines of [1, Sect. 4].

Proposition 5.1. *The following properties of the Sobociński–Bochvar sup-T composition hold for all partial fuzzy relations* $R, S, T \subseteq_{\text{sub}} \boldsymbol{U}^2$*:*

1. *Domain, kernel, and support:*

$$\operatorname{dom}(R \circ S) = \operatorname{dom} R \circ \operatorname{dom} S$$
$$\ker(R \circ S) \supseteq \ker R \circ \ker S$$
$$\operatorname{supp}(R \circ S) \subseteq \operatorname{supp} R \circ \operatorname{supp} S$$

2. *Neutral and absorbing elements:*

$$R \circ \mathrm{id}^* = \mathrm{id}^* \circ R = R$$
$$R \circ \lambda^2 = \lambda^2 \circ R = \lambda^2$$

3. *Transposition:* $(R \circ S)^{\mathrm{T}} = S^{\mathrm{T}} \circ R^{\mathrm{T}}$.
4. *Associativity:* $(R \circ S) \circ T = R \circ (S \circ T)$.
5. *Monotony:*

$$\text{If } R \subseteq S, \text{ then } R \circ T \subseteq S \circ T.$$
$$\text{If } R \subseteq_{\text{sub}} S, \text{ then } R \circ T \subseteq_{\text{sub}} S \circ T.$$

Remark 5.2. In Proposition 5.1.1, equality holds for kernels if R, S are finite and for supports if & has no zero divisors in $\boldsymbol{L}$. Both of the monotony properties in Proposition 5.1.5 hold as well for the left composition with T.

Proposition 5.3. *The following distributivity properties hold for any crisp index set I and any partial fuzzy relations* $R, S_i \subseteq_{\text{sub}} \boldsymbol{U}^2$ *for all* $i \in I$.

1. *Bochvar union:* $R \circ \bigcup_{i \in I}^{\mathrm{B}} S_i \subseteq_{\text{sub}} \bigcup_{i \in I}^{\mathrm{B}} (R \circ S_i)$.
2. *Sobociński union:* $R \circ \bigcup_{i \in I}^{\mathrm{S}} S_i = \bigcup_{i \in I}^{\mathrm{S}} (R \circ S_i)$.
3. *Bochvar intersection:* $R \circ \bigcap_{i \in I}^{\mathrm{B}} S_i \subseteq_{\text{sub}} \bigcap_{i \in I}^{\mathrm{B}} (R \circ S_i)$.
4. *Sobociński intersection:* $R \circ \bigcap_{i \in I}^{\mathrm{S}} S_i \subseteq \bigcap_{i \in I}^{\mathrm{S}} (R \circ S_i)$.

Remark 5.4. Analogous claims for right composition with R follow by Propositions 5.1.3 and 4.4.2. For total fuzzy relations (i.e., if $\operatorname{dom} R = \operatorname{dom} S_i = \boldsymbol{U}^2$ for all $i \in I$), the operations $\bigcup_{\mathrm{B}}, \bigcap_{\mathrm{B}}$ coincide with $\bigcup_{\mathrm{S}}, \bigcap_{\mathrm{S}}$, so in that case, Propositions 5.3.1 and 5.3.3 hold as well with $=$ and $\subseteq$, respectively. Without the totality assumption, however, the distributivity laws for $\bigcup_{\mathrm{B}}, \bigcap_{\mathrm{B}}$ cannot in general be strengthened to $\subseteq$ or $=_{\text{sub}}$ (as can be shown by two-element counterexamples).

6 Relational Representation of Partial Fuzzy Sets and Membership Degrees

Similarly as in [1, Sect. 3], we are going to represent partial fuzzy sets $A \subseteq_{\text{sub}} \boldsymbol{U}$ and membership degrees $\alpha \in \boldsymbol{L}_*$ by partial fuzzy relations $\mathrm{Rel}_A, \mathrm{Rel}_\alpha \subseteq_{\text{sub}} \boldsymbol{U}^2$ of certain specific forms. Analogously to [1, Sect. 4], this will allow us to apply the properties of sup-T composition to several related operations (in particular, those of Definition 4.2 and the Sobociński height of Definition 3.1).

Definition 6.1. From now on, we fix an arbitrary element $\mathbf{o} \in \boldsymbol{U}$. Furthermore, we define the (crisp) partial fuzzy set $\mathbf{1} \subseteq_{\text{sub}} \boldsymbol{U}$ by setting, for all $x \in \boldsymbol{U}$:

$$\mathbf{1}(x) = \begin{cases} 1 & \text{if } x = \mathbf{o} \\ * & \text{otherwise.} \end{cases}$$

Definition 6.2. For any partial fuzzy set $A \subseteq_{\text{sub}} \boldsymbol{U}$, we define the partial fuzzy relation $\mathrm{Rel}_A \subseteq_{\text{sub}} \boldsymbol{U}^2$ so that for all $x, y \in \boldsymbol{U}$:

$$\mathrm{Rel}_A\, xy = \begin{cases} Ax & \text{if } y = \mathbf{o} \\ * & \text{otherwise.} \end{cases}$$

Similarly, for every partial membership degree $\alpha \in \boldsymbol{L}_*$, we define the partial fuzzy relation $\mathrm{Rel}_\alpha \subseteq_{\text{sub}} \boldsymbol{U}^2$ so that for all $x, y \in \boldsymbol{U}$:

$$\mathrm{Rel}_\alpha\, xy = \begin{cases} \alpha & \text{if } x = y = \mathbf{o} \\ * & \text{otherwise.} \end{cases}$$

Example 6.3. Let $\boldsymbol{U} = \{x_1, x_2, x_3\}$, $\mathbf{o} = x_1$, $A \subseteq_{\text{sub}} \boldsymbol{U}$, and $\alpha \in \boldsymbol{L}_*$. Then:

$$\mathrm{Rel}_A = \begin{pmatrix} Ax_1 & * & * \\ Ax_2 & * & * \\ Ax_3 & * & * \end{pmatrix}, \quad \mathrm{Rel}_\alpha = \begin{pmatrix} \alpha & * & * \\ * & * & * \\ * & * & * \end{pmatrix}.$$

(Note that in this example, any of $Ax_1, Ax_2, Ax_3, \alpha \in \boldsymbol{L}_*$ can equal $*$ too. Thus, e.g., $\mathrm{Rel}_* = \mathrm{Rel}_\lambda = \lambda^2$.)

Observe that $\mathrm{Rel}_A \subseteq_{\text{sub}} \boldsymbol{U} \times \mathbf{1}$ and $\mathrm{Rel}_\alpha \subseteq_{\text{sub}} \mathbf{1} \times \mathbf{1}$; and moreover that every $R \subseteq_{\text{sub}} \boldsymbol{U} \times \mathbf{1}$ represents some $A \subseteq_{\text{sub}} \boldsymbol{U}$ so that $R = \mathrm{Rel}_A$, and analogously for every $R \subseteq_{\text{sub}} \mathbf{1} \times \mathbf{1}$ and $\alpha \in \boldsymbol{L}_*$. Because of this correspondence, partial fuzzy sets and membership degrees can be identified with the corresponding partial fuzzy relations, as done in Convention 6.4:

Convention 6.4. We identify:

- A partial fuzzy set $A \subseteq_{\text{sub}} \boldsymbol{U}$ with the partial fuzzy relation $\text{Rel}_A \subseteq_{\text{sub}} \boldsymbol{U} \times \mathbf{1}$ (writing just A for Rel_A and calling Rel_A a partial fuzzy set); and similarly,
- A partial membership degree $\alpha \in \boldsymbol{L}_*$ with the partial fuzzy relation $\text{Rel}_\alpha \subseteq_{\text{sub}} \mathbf{1} \times \mathbf{1}$.

It can be observed that under this identification, several partial fuzzy relational notions reduce to special cases of the composition of partial fuzzy relations:

Proposition 6.5. *Let $R \subseteq_{\text{sub}} \boldsymbol{U}^2$ and $A, B \subseteq_{\text{sub}} \boldsymbol{U}$. Then:*

$$R^{\rightarrow}A = R^{\mathrm{T}} \circ A \tag{12}$$
$$R^{\leftarrow}A = R \circ A \tag{13}$$
$$A \times B = A \circ B^{\mathrm{T}} \tag{14}$$
$$\mathrm{Dom}\, R = R \circ \boldsymbol{U} \tag{15}$$
$$\mathrm{Rng}\, R = R^{\mathrm{T}} \circ \boldsymbol{U} \tag{16}$$
$$\mathrm{Hgt}_{\mathrm{S}}\, A = A^{\mathrm{T}} \circ \boldsymbol{U} \tag{17}$$

Proof. (12): By Convention 6.4, we need to prove:

$$\text{Rel}_{R^{\rightarrow}A} = R^{\mathrm{T}} \circ \text{Rel}_A\,.$$

By expanding the definitions, we obtain for all $x, z \in \boldsymbol{U}$:

$$(R^{\mathrm{T}} \circ \text{Rel}_A)xz = \bigvee_{y\in\boldsymbol{U}}{}_{\mathrm{S}} (Ryx \mathbin{\&_{\mathrm{B}}} \text{Rel}_A\, yz) =$$
$$= \begin{cases} \bigvee_{y\in\boldsymbol{U}}{}_{\mathrm{S}} (Ryx \mathbin{\&_{\mathrm{B}}} Ay) = \text{Rel}_{R^{\rightarrow}A}\, xz & \text{if } z = \mathbf{o} \\ \bigvee_{y\in\boldsymbol{U}}{}_{\mathrm{S}} (Ryx \mathbin{\&_{\mathrm{B}}} *) = * = \text{Rel}_{R^{\rightarrow}A}\, xz & \text{if } z \neq \mathbf{o}. \end{cases}$$

The proofs of (13)–(17) are similar. □

7 Derived Properties of Partial Fuzzy Relational Notions

By Convention 6.4, partial fuzzy sets and membership degrees can be regarded as special partial fuzzy relations; and by Proposition 6.5, several partial fuzzy relational notions reduce to partial fuzzy sup-T composition. Consequently, the properties of partial fuzzy composition listed in Section 5 automatically apply to these notions as well. Thus, we obtain the following immediate corollaries of Propositions 5.1 and 5.3:

Corollary 7.1. *The following properties of partial fuzzy relations hold for any $R, R_1, R_2, S \subseteq_{\text{sub}} \boldsymbol{U}^2$ and $A, A_1, A_2, B \subseteq_{\text{sub}} \boldsymbol{U}$:*

1. *Domain, kernel, and support:*

$$\mathrm{dom}(R^{\rightarrow}A) = (\mathrm{dom}\, R)^{\rightarrow}(\mathrm{dom}\, A)$$
$$\mathrm{dom}(R^{\leftarrow}A) = (\mathrm{dom}\, R)^{\leftarrow}(\mathrm{dom}\, A)$$
$$\mathrm{dom}(A \times B) = (\mathrm{dom}\, A) \times (\mathrm{dom}\, B)$$
$$\mathrm{dom}(\mathrm{Dom}\, R) = \mathrm{Dom}(\mathrm{dom}\, R)$$
$$\mathrm{dom}(\mathrm{Rng}\, R) = \mathrm{Rng}(\mathrm{dom}\, R)$$
$$\ker(R^{\rightarrow}A) \supseteq (\ker R)^{\rightarrow}(\ker A),\ \textit{etc.}$$
$$\mathrm{supp}(R^{\rightarrow}A) \subseteq (\mathrm{supp}\, R)^{\rightarrow}(\mathrm{supp}\, S),\ \textit{etc.}$$

2. *Neutral and absorbing elements:*

$$(\mathrm{id}^*)^{\rightarrow}A = (\mathrm{id}^*)^{\leftarrow}A = A$$
$$\lambda^{\rightarrow}A = R^{\rightarrow}\lambda = \mathrm{Dom}\, \lambda = \lambda$$
$$A \times \lambda = \lambda \times A = \lambda$$

3. *Transposition:*

$$R^{\rightarrow}A = (R^{\mathrm{T}})^{\leftarrow}A$$
$$\mathrm{Dom}\, R = \mathrm{Rng}(R^{\mathrm{T}})$$
$$(A \times B)^{\mathrm{T}} = B \times A$$

4. *Associativity:*

$$R^{\rightarrow}(S^{\rightarrow}A) = (S \circ R)^{\rightarrow}A$$
$$R^{\leftarrow}(S^{\leftarrow}A) = (R \circ S)^{\leftarrow}A$$
$$R \circ (A \times B) = (R^{\leftarrow}A) \times B$$
$$(A \times B) \circ R = A \times (R^{\rightarrow}B)$$
$$\mathrm{Dom}(R \circ S) = R^{\leftarrow}\mathrm{Dom}\, S$$
$$\mathrm{Rng}(R \circ S) = S^{\rightarrow}\mathrm{Rng}\, R$$
$$\mathrm{Hgt}_{\mathrm{S}}(\mathrm{Dom}\, R) = \mathrm{Hgt}_{\mathrm{S}}(\mathrm{Rng}\, R)$$

5. *Monotony:*

If $R_1 \subseteq R_2$, then:
$$R_1{}^{\rightarrow}A \subseteq R_2{}^{\rightarrow}A$$
$$R_1{}^{\leftarrow}A \subseteq R_2{}^{\leftarrow}A$$
$$\mathrm{Dom}(R_1) \subseteq \mathrm{Dom}(R_2)$$
$$\mathrm{Rng}(R_1) \subseteq \mathrm{Rng}(R_2)$$
$$\mathrm{Hgt}_{\mathrm{S}}(R_1) \leq \mathrm{Hgt}_{\mathrm{S}}(R_2)$$

If $A_1 \subseteq A_2$, then:
$$R^{\rightarrow}A_1 \subseteq R^{\rightarrow}A_2$$
$$R^{\leftarrow}A_1 \subseteq R^{\leftarrow}A_2$$
$$A_1 \times B \subseteq A_2 \times B$$
$$\mathrm{Hgt}_{\mathrm{S}}(A_1) \leq \mathrm{Hgt}_{\mathrm{S}}(A_2)$$

Analogously for $\subseteq_{\text{sub}}$, i.e.:

If $R_1 \subseteq_{\text{sub}} R_2$, then $R_1{}^{\rightarrow}A \subseteq_{\text{sub}} R_2{}^{\rightarrow}A$.

⋮

If $A_1 \subseteq_{\text{sub}} A_2$, then $\mathrm{Hgt}_{\mathrm{S}}(A_1) \leq_{\text{sub}} \mathrm{Hgt}_{\mathrm{S}}(A_2)$.

Moreover, some of the properties in Corollary 7.1 can be extended or strengthened by applying the claims of Remark 5.2.

Corollary 7.2. *The following distributivity properties hold for any crisp index set I, partial fuzzy relations $R, R_i \subseteq_{\rm sub} \boldsymbol{U}^2$, and partial fuzzy sets $A, A_i, B, B_i \subseteq_{\rm sub} \boldsymbol{U}$, for all $i \in I$.*

1. *Bochvar union:*

$$R^{\rightarrow} \bigcup_{i\in I}{}_{\rm B} A_i \subseteq_{\rm sub} \bigcup_{i\in I}{}_{\rm B} (R^{\rightarrow} A_i)$$
$$\Big(\bigcup_{i\in I}{}_{\rm B} R_i\Big)^{\rightarrow} A \subseteq_{\rm sub} \bigcup_{i\in I}{}_{\rm B} (R_i^{\rightarrow} A)$$
$$\operatorname{Dom} \bigcup_{i\in I}{}_{\rm B} R_i \subseteq_{\rm sub} \bigcup_{i\in I}{}_{\rm B} \operatorname{Dom} R_i$$
$$\operatorname{Hgt}_{\rm S} \bigcup_{i\in I}{}_{\rm B} A_i \leq_{\rm sub} \bigvee_{i\in I}{}_{\rm B} \operatorname{Hgt}_{\rm S} A_i$$

2. *Sobociński union:*

$$R^{\rightarrow} \bigcup_{i\in I}{}_{\rm S} A_i = \bigcup_{i\in I}{}_{\rm S} (R^{\rightarrow} A_i)$$
$$\Big(\bigcup_{i\in I}{}_{\rm S} R_i\Big)^{\rightarrow} A = \bigcup_{i\in I}{}_{\rm S} (R_i^{\rightarrow} A)$$
$$A \times \bigcup_{i\in I}{}_{\rm S} B_i = \bigcup_{i\in I}{}_{\rm S} (A \times B_i)$$
$$\operatorname{Dom} \bigcup_{i\in I}{}_{\rm S} R_i = \bigcup_{i\in I}{}_{\rm S} \operatorname{Dom} R_i$$
$$\operatorname{Hgt}_{\rm S} \bigcup_{i\in I}{}_{\rm S} A_i = \bigvee_{i\in I}{}_{\rm S} \operatorname{Hgt}_{\rm S} A_i$$

3. *Bochvar intersection:*

$$R^{\rightarrow} \bigcap_{i\in I}{}_{\rm B} A_i \subseteq_{\rm sub} \bigcap_{i\in I}{}_{\rm B} (R^{\rightarrow} A_i)$$
$$\Big(\bigcap_{i\in I}{}_{\rm B} R_i\Big)^{\rightarrow} A \subseteq_{\rm sub} \bigcap_{i\in I}{}_{\rm B} (R_i^{\rightarrow} A)$$
$$A \times \bigcap_{i\in I}{}_{\rm B} B_i \subseteq_{\rm sub} \bigcap_{i\in I}{}_{\rm B} (A \times B_i)$$
$$\operatorname{Dom} \bigcap_{i\in I}{}_{\rm B} R_i \subseteq_{\rm sub} \bigcap_{i\in I}{}_{\rm B} \operatorname{Dom} R_i$$
$$\operatorname{Hgt}_{\rm S} \bigcap_{i\in I}{}_{\rm B} A_i \leq_{\rm sub} \bigwedge_{i\in I}{}_{\rm B} \operatorname{Hgt}_{\rm S} A_i$$

4. *Sobociński intersection:*

$$R^{\rightarrow} \bigcap_{i\in I}{}_{\rm S} A_i \subseteq \bigcap_{i\in I}{}_{\rm S} (R^{\rightarrow} A_i)$$
$$\Big(\bigcap_{i\in I}{}_{\rm S} R_i\Big)^{\rightarrow} A \subseteq \bigcap_{i\in I}{}_{\rm S} (R_i^{\rightarrow} A)$$
$$A \times \bigcap_{i\in I}{}_{\rm S} B_i \subseteq \bigcap_{i\in I}{}_{\rm S} (A \times B_i)$$
$$\operatorname{Dom} \bigcap_{i\in I}{}_{\rm S} R_i \subseteq \bigcap_{i\in I}{}_{\rm S} \operatorname{Dom} R_i$$
$$\operatorname{Hgt}_{\rm S} \bigcap_{i\in I}{}_{\rm S} A_i \leq \bigwedge_{i\in I}{}_{\rm S} \operatorname{Hgt}_{\rm S} A_i$$

Moreover, the distributivity laws analogous to those of $^{\rightarrow}$ *hold for* $^{\leftarrow}$ *and those of* Dom *for* Rng.

The claims of Corollary 7.2 cannot be further strengthened to hold with $\subseteq$, $=_{\rm sub}$, or $=$, since the counterexamples for $\circ$ (see Remark 5.4) apply to them as well. Some of the distributivity laws for Cartesian products, however, are valid in stronger forms than what would follow directly from Proposition 5.3:

Proposition 7.3. *Let I be a crisp index set and let $A, B_i \subseteq_{\rm sub} \boldsymbol{U}$ for all $i \in I$. Then:*

1. $A \times \bigcup_{i\in I}{}_{\rm B} B_i = \bigcup_{i\in I}{}_{\rm B} (A \times B_i)$
2. *If the algebra $\mathbf{L}$ of membership degrees is an MV-algebra or a finite MTL-algebra or a linear dense BL-algebra (this includes the case when $\mathbf{L} = [0,1]$ and & is a continuous t-norm), then:*

$$A \times \bigcap_{i\in I}{}_{\rm B} B_i = \bigcap_{i\in I}{}_{\rm B} (A \times B_i)$$
$$A \times \bigcap_{i\in I}{}_{\rm S} B_i = \bigcap_{i\in I}{}_{\rm S} (A \times B_i)$$

Proof. Let us prove just the last claim; the other proofs are similar. For all $x, z \in \boldsymbol{U}$ we have:

$$\Big(A \times \bigcap_{i\in I}{}_{\rm S} B_i\Big) xz = Ax \,\&_{\rm B} \bigwedge_{i\in I}{}_{\rm S} B_i z =$$
$$\bigwedge_{i\in I}{}_{\rm S} (Ax \,\&_{\rm B}\, B_i z) = \Big(\bigcap_{i\in I}{}_{\rm S} (A \times B_i)\Big) xz,$$

where the middle equality holds in the MTL-algebras listed in the assumption by Proposition 2.3(6). □

8 Conclusion

We presented basic properties of the sup-T composition for partial fuzzy relations. In the partial fuzzy relational setting, the domains of definition of the input partial fuzzy relations have to be taken into account; consequently, the well-known properties of fuzzy relational composition do not automatically carry over to partial fuzzy relations. For example, as seen in Proposition 5.3, the sup-T composition does distribute over the Sobociński union in the same manner as for total fuzzy relations, but only subinclusion holds in general for the distributivity over the Bochvar union.

Furthermore, employing the representation described in Section 6, we showed that the studied elementary properties of sup-T composition apply as well to a family of related operations on lower-dimensional objects (namely, partial fuzzy sets and membership degrees). Of these, particularly the results on images under partial fuzzy relations can be employed for proving the properties of fuzzy inference systems, as in many models of fuzzy IF–THEN rules, the domains of definition play a significant role. For example, real-world applications often deal with hierarchical fuzzy systems [11], in which the involved fuzzy rule bases have generally different domains. Representing inputs outside the domain of definition by $*$ then provides the means for treating such inputs uniformly and systematically within the inference system. Even so, the results presented in this paper are only applicable to the theory of fuzzy inference based on sup-T composition. An elaboration of relational compositions suitable for a broader spectrum of fuzzy systems is left for future work.

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