Atlantis Studies in Uncertainty Modelling, volume 3
Joint Proceedings of the 19th World Congress of the International Fuzzy Systems Association (IFSA), the 12th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT), and the 11th International Summer School on Aggregation Operators (AGOP)

Order Based on Non-Associative Operations

Kavit Nanavati [a] and *__Balasubramaniam Jayaram__[a]

[a]Department of Mathematics, Indian Institute of Technology Hyderabad, INDIA.
ma20resch01004@iith.ac.in,jbala@math.iith.ac.in

Abstract

Recently there have been many works studying orders obtained from fuzzy logic connectives, using the relation proposed in [12]. However, almost all of these works have dealt only with associative operations. In this work, we investigate the conditions under which the above relation leads to a partial order even when the operation is non-associative, i.e., in the setting of a groupoid instead of a semigroup. We begin by presenting the necessary and sufficient conditions towards the same. Following this we study the major non-associative aggregation operations, viz., semi-copulas, fuzzy implications, overlapping and grouping functions. Our investigations show that even appropriate non-associative functions can lead to interesting order-theoretic structures on the underlying set.

Keywords: Non-Associative Operations, Posets, Lattices, Copulas, Fuzzy Implications.

1 Introduction

The study of algebraic structures from an order-theoretic perspective often offers a hitherto hidden insight. Such studies have been classically undertaken intensively on semigroups, see for instance, [15, 16, 17]. Inspired by this, such studies have been conducted on fuzzy logic connectives, where the underlying set is the totally ordered $[0,1]$ or an arbitrary bounded lattice. The first such work was that of [12] wherein the following relation was explored.

Definition 1. *Let $\mathbb{P} \neq \emptyset$ and $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$. Consider the relation $\sqsubseteq_F$ defined as follows:*

$$x \sqsubseteq_F y \iff \text{there exists } \ell \text{ s.t. } F(\ell, y) = x\,. \qquad (1)$$

Studying this for t-norms [12] showed that not only all t-norms, but also all t-conorms, give rise to a partial order. This relation was further generalised to the setting of uninorms [7] and nullnorms [1] suitably. See also the work of [11] that discusses this relation in its generality for any associative operation, without making use of the monotonicity of the operations considered.

1.1 Motivation for this work

Largely all the works, so far, have considered associative aggregation operations on either $[0,1]$ or a bounded lattice. The only works that have dealt with non-associative operations, in this context, are those of [13, 9, 10] which deal with fuzzy implications (considering appropriately the dual of (1)) and that of [18] dealing with Overlapping and Grouping Functions.

However, so far, there exists no independent study dealing with obtaining an order from (1) for general binary operations, irrespective of whether they are associative or not. This forms the main motivation for this work.

1.2 Highlights of this work:

The following are the highlights of this work:

1. Following the work of [9, 10], we do not assume the monotonicity of the considered operations nor that the set $\mathbb{P}$ has an existing order.

2. Our study shows that non-associative operations can also give rich order-theoretic structures just as their associative counterparts.

3. Our study also subsumes the works of [9, 10, 18], in that the operations or the setting considered by them implies the necessary and sufficient conditions proposed in this work.

2 Order from Non-Associative Binary Operations

In this section, we consider a set $\mathbb{P} \neq \emptyset$ with no other structure on it.

Definition 2. *Let $\mathbb{P} \neq \emptyset$ and $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$. F is said to satisfy the*

1. ***Local left identity*** *property, if for every $x \in \mathbb{P}$, there exists $\ell \in \mathbb{P}$ such that*

$$F(\ell, x) = x. \qquad \text{(LLI)}$$

2. ***Quasi-Projection*** *property, if for any $x, y, z \in \mathbb{P}$,*

$$F(x, F(y,z)) = z \implies F(y,z) = z. \qquad \text{(QP)}$$

3. ***Generalised Quasi-Projection*** *property, if for any $x, y, z, w \in \mathbb{P}$,*

$$F(x, F(y,z)) = w \implies \exists\, \ell \in \mathbb{P} \text{ s.t. } F(\ell, z) = w. \qquad \text{(GQP)}$$

Theorem 1. *Let $\mathbb{P} \neq \emptyset$ and $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ be a binary function. Let $\sqsubseteq_F$ be a binary relation defined on $\mathbb{P}$ as in* (1)*. Then the following are equivalent:*

(i) $(\mathbb{P}, \sqsubseteq_F)$ *is a poset.*

(ii) F *satisfies* (LLI), (QP), *and* (GQP).

Proof. Suppose, $(\mathbb{P}, \sqsubseteq_F)$ is a poset. That the reflexivity of $\sqsubseteq_F$ is equivalent to (LLI) and that its antisymmetry is equivalent to (QP) follows from Theorem 2.4 in [11].

Thus it suffices to show the equivalence between the transitivity of $\sqsubseteq_F$ and (GQP).

Let us suppose that $\sqsubseteq_F$ is transitive, and that $F(x, F(y,z)) = w$ for some $x, y, z, w \in \mathbb{P}$. Thus from (1), we have $w \sqsubseteq_F F(y,z)$. Since $F(y,z) = F(y,z)$, we have that $F(y,z) \sqsubseteq_F z$, and by transitivity of $\sqsubseteq_F$, we have $w \sqsubseteq_F z$. Clearly from (1), we have $F(\ell, z) = w$. Thus, F satisfies (GQP).

Suppose F satisfies (GQP), and that $x \sqsubseteq_F y$, and $y \sqsubseteq_F z$. Thus there exist $\ell, m \in \mathbb{P}$ such that $F(m,z) = y$ and

$$F(\ell, y) = x \Longrightarrow F(\ell, F(m,z)) = x .$$

Now by (GQP), we have an $n \in \mathbb{P}$ such that $F(n,z) = x$ and hence $x \sqsubseteq_F z$, i.e., $\sqsubseteq_F$ is transitive. ☐

We present in the following examples some binary functions on finite sets, satisfying the above mentioned properties and the corresponding Hasse diagrams.

F_1	x	y	z	1
x	x	y	z	1
y	x	y	z	1
z	x	y	z	1
1	x	y	z	1

F_2	x	y	z	1
x	x	1	1	1
y	1	y	1	1
z	1	1	z	1
1	1	1	1	1

F_3	x	y	z	1
x	1	1	1	1
y	z	1	1	1
z	y	1	1	1
1	x	y	z	1

F_4	x	y	z	1
x	x	y	x	x
y	x	y	z	y
z	x	y	z	1
1	x	y	z	z

Table 1: Functions F_i for $i = 1, 2, 3$ in Example 1

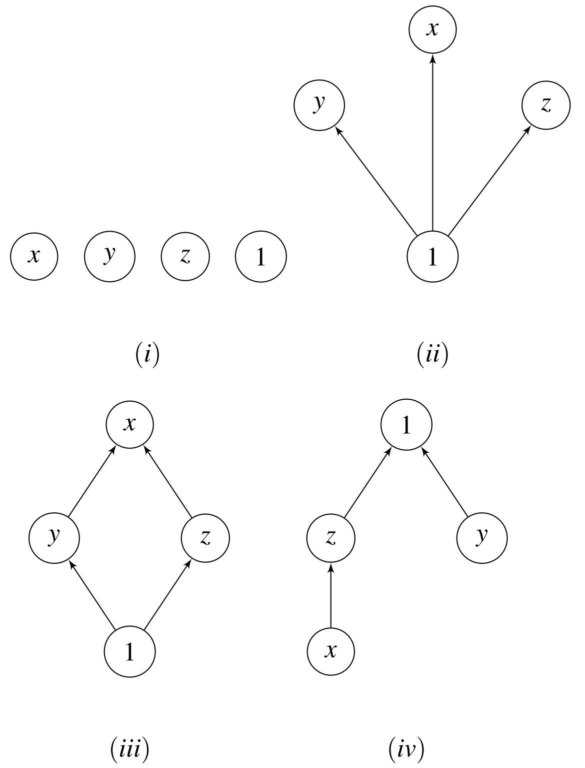

Figure 1: Hasse Diagrams of Posets obtained in Table 1

Example 1. *Let $\mathbb{P} = \{x, y, z, 1\}$. The functions given in Table 1 satisfy* (LLI), (QP), *and* (GQP) *and hence, yield a poset. These posets are visualized in figure 1.*

- *Note that the poset obtained from F_1 is an antichain, i.e every element is both maximal and minimal and there are no greatest and least elements.*
- *The poset obtained by F_2 is bounded below by the element* 1 *and has* 3 *maximal elements.*
- *The poset obtained from F_3 is, in fact, a bounded*

F'_1	0	x	y	z	1
0	x	x	x	x	x
x	x	x	y	x	x
y	x	x	x	x	x
z	x	x	x	x	x
1	x	x	x	x	x

F'_2	0	x	y	z	1
0	z	1	x	y	0
x	1	y	z	0	x
y	x	z	0	1	y
z	y	0	1	x	z
1	0	x	y	z	1

F'_3	0	x	y	z	1
0	0	x	y	z	1
x	0	x	1	z	1
y	0	x	y	y	1
z	0	x	y	z	1
1	0	x	y	z	1

Table 2: Functions F'_i for $i = 1,2,3$ in Remark 1

lattice.

- *The poset obtained by F_4 is bounded above by the element* 1 *and has* 2 *minimal elements.*

Remark 1. *Consider $\mathbb{P} = \{0,x,y,z,1\}$. The functions F'_i defined in Table 2 show that the three properties mentioned above are mutually independent of each other.*

(i) Clearly F'_1 does not satisfy (LLI)*, since there exists no $\ell \in \mathbb{P}$ such that $F(\ell,0) = 0$. However, F_1 satisfies* (QP) *and* (GQP).

(ii) Clearly F'_2 satisfies (LLI) *and* (GQP)*. However $F'_2(y,F'_2(z,x)) = x \neq 0 = F'_2(z,x)$, i.e. F'_2 does not satisfy* (QP).

(iii) Note that F'_3 satisfies (LLI) *and* (QP)*. However, $F'_3(x,F'_3(y,z)) = 1 \neq F'_3(\ell,z)$ for any $\ell \in \mathbb{P}$.*

Remark 2. *If F is associative then clearly it satisfies* (GQP)*. However, associativity is only sufficient and not necessary for* (GQP)*. Note that while the functions F_1 and F_2 are associative, F_3 and F_4 are not, since*

$$F_3(x,F_3(y,z)) = 1 \neq z = F_3(F_3(x,y),z) \ ,$$
$$F_4(y,F_4(z,1)) = y \neq 1 = F_4(F_4(y,z),1) \ .$$

3 Posets induced on $[0,1]$ by Non-Associative Operations

Among the non-associative fuzzy logic connectives, the well studied classes are that of semi-copulas- and their variants, viz., quasi-copulas and copulas - fuzzy implications, overlapping and grouping functions.

In this section, we deal with each of them and discuss if and when they lead to a poset under the relation given by (1), i.e., essentially studying the satisfaction of the properties of (LLI), (QP) and (GQP) by these operations.

We begin by giving a general sufficiency result for an operation F to satisfy (LLI), (QP) and (GQP).

Lemma 1. *Let $F : [0,1]^2 \to [0,1]$.*

(i) If F has a left neutral element $e \in [0,1]$ then it satisfies (LLI).

(ii) Further if e is also the right neutral element of F, and $e = 0$ or $e = 1$, and F is monotonic, then it satisfies (QP).

(iii) Further, if F is continuous then it satisfies (GQP).

Proof. (i) is obvious and (ii) follows from Proposition 4.1 in [11].

To see (iii), let us consider the case $e = 1$. The case $e = 0$ can be shown similarly. Suppose $F(x,F(y,z)) = w$. Then $F(x,F(y,z)) = w \leq F(y,z) = F(1,F(y,z))$. Since $F(1,0) = 0$, by monotonicity we have

$$0 = F(0,z) \leq w \leq F(y,z) \ .$$

Now, by the continuity of F, there exists $\ell \in [0,1]$ such that $F(\ell,z) = w$, i.e., F satisfies (GQP). □

3.1 Posets from Semi-copulas

A semi-copula can be seen as a generalisation of the classical notion of intersection. They generalise T-norms as they are not required to be commutative or associative. Quasi-copulas and copulas form a subclass of semi-copulas. We investigate the orders obtained from them in this section.

Definition 3. *[14] A function S is said to be a*

- *semi-copula if for all $x \in [0,1]$,*
  - $S(0,x) = S(x,0) = 0$,
  - $S(1,x) = S(x,1) = x$,
  - *S is increasing in both variables.*
- *quasi-copula if it is a semi-copula and if for all $x_1,x_2,y_1,y_2 \in [0,1]$,*
  - $|S(x_1,y_1) - S(x_2,y_2)| \leq |x_1 - x_2| + |y_1 - y_2|$.
- *copula if it is a semi-copula and if for all $x,x_1,x_2,y_1,y_2 \in [0,1]$,*
  - $S(x_1,y_1) + S(x_2,y_2) \geq S(x_1,y_2) + S(x_2,y_1)$.

Remark 3. • *Quasi-copulas and copulas are continuous functions.*

- *Every copula is a quasi-copula but the converse is not true.*
- *We will use the notation Q for a quasi-copula and C for a copula.*

From the definition of a semi-copula S and Lemma 1 we see that they always satisfy (LLI) and (QP). However, they may not satisfy (GQP) and thus may not always give rise to a poset on $[0,1]$.

For instance, consider the semi-copula

$$S(x,y) = \begin{cases} \min(x,y), & \text{if } \max(x,y) = 1 \\ 0, & \text{if (x,y)} \in [0,0.4]^2 \\ \frac{1}{2}\min(x,y), & \textit{else} \end{cases} .$$

While $S(0.7, S(0.5,0.4)) = S(0.7,0.2) = 0.1$, there does not exist any $\ell \in [0,1]$ s.t. $S(\ell,0.4) = 0.1$. Thus S does not satisfy (GQP) and the relation $\sqsubseteq_S$ does not give rise to a poset on $[0,1]$.

In the following example, we present a couple of semi-copulas that give rise to a poset.

Example 2. *Consider $S:[0,1] \to [0,1]$. The following functions satisfy* (LLI), (QP), *and* (GQP) *and hence, yield a poset.*

$$S_1(x,y) = \begin{cases} \min(x,y), & \textit{if } \max(x,y) = 1 \\ 0, & \textit{if } \min(x,y) \le 0.4 \textit{ and} \\ & \quad \max(x,y) < 1 \\ 0.4, & \textit{otherwise} \end{cases} .$$

$$S_2(x,y) = \begin{cases} \min(x,y), & \textit{if } \max(x,y) = 1 \textit{ or} \\ & \quad \min(x,y) \le 0.4 \\ 0.4, & \textit{otherwise} \end{cases} .$$

The Hasse diagrams of the posets obtained from S_1, S_2 are given in Fig. 2. As can be seen, both are bounded lattices.

From Lemma 1 we see that every quasi-copula Q and a copula C give rise to a poset. In fact, due to their continuity, they give rise to a totally ordered complete lattice on $[0,1]$.

Lemma 2. *If $F:[0,1]^2 \to [0,1]$ is either a quasi-copula or a copula, it always satisfies* (LLI), (QP) *and* (GQP).

Lemma 3. *Given a quasi-copula Q, the poset obtained from $\sqsubseteq_Q$ on $[0,1]$ is a chain, and is in fact, the usual order on $[0,1]$, i.e., $([0,1], \sqsubseteq_Q) = ([0,1], \le)$.*

Proof. Let, $x \sqsubseteq_Q y$. Hence, there exists $\ell \in [0,1]$ such that $Q(\ell,y) = x$. By the monotonicity of Q, we have $x = Q(\ell,y) \le Q(1,y) = y$.

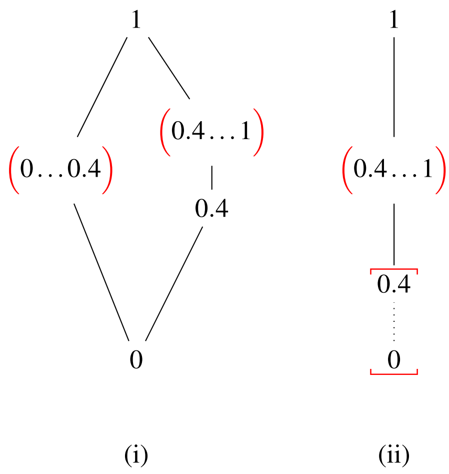

Figure 2: Hasse Diagrams of Posets obtained from Semi-Copulas in example 2

Let $x \le y$. Since $0 = Q(0,y) \le Q(1,y) = y$ by continuity of Q, there exists an $\ell \in [0,1]$ such that $Q(\ell,y) = x$, i.e., $x \sqsubseteq_Q y$. □

Corollary 1. *Given a copula C, the poset obtained from $\sqsubseteq_C$ on $[0,1]$ is a chain, and is in fact, the usual order on $[0,1]$.*

3.2 Posets from Overlap & Grouping Functions

Introduced by Bustince et al. [19, 20], overlap and grouping functions also generalise the notion of classical intersection and union, respectively. They play an important role in image processing and have been theoretically investigated in recent works. Since they aren't required to be associative in nature, we investigate the order induced from them and compare it with the results of [18].

Definition 4. *Let $F:[0,1]\times[0,1] \to [0,1]$ be a commutative and a continuous function that is non-decreasing in both variables. F is said to be*

- *an overlap function if, further, for any $x,y \in [0,1]$*
  - $F(x,y) = 0 \iff xy = 0;$
  - $F(x,y) = 1 \iff xy = 1;$
- *a grouping function if, further, for any $x,y \in [0,1]$*
  - $F(x,y) = 0 \iff x = y = 0;$
  - $F(x,y) = 1 \iff x = 1$ *or* $y = 1;$

We denote an overlap function by O and a grouping function by G.

Remark 4. *From [18], we have the result that an overlapping function O yields a poset w.r.t the order defined in* (1) *if and only if O has a neutral element 1, i.e.*

$O(1,x) = x$ *for every* $x \in [0,1]$. *We show in the following result that the above condition is equivalent to an overlap function satisfying* (LLI), (QP), *and* (GQP).

Proposition 1. *Given an overlap function* $O : [0,1]^2 \to [0,1]$, *the following are equivalent:*

(i) $O(1,x) = x$, *for all* $x \in [0,1]$.

(ii) O *satisfies* (LLI), (QP), *and* (GQP).

Proof. (i) $\implies$ (ii): Since 1 is the neutral element of O, it immediately follows from Lemma 1 and the properties of O that O satisfies (LLI), (QP), and (GQP).

(ii) $\implies$ (i): Let $a \in [0,1]$ be arbitrary. By (LLI), we have that there exists an $x \in [0,1]$ such that $O(x,a) = a$. However, by monotonicity of O, we have $O(1,a) \geq O(x,a) = a$.

Suppose that $O(1,a) = b > a$. By continuity of O, we have a $c \in [0,1]$ such that $O(c,b) = a$. Now, $O(1,O(c,b)) = b$ implies, by (QP), that $O(c,b) = b$, leading to $a = b$. Hence, $O(1,a) = a$ for all $a \in [0,1]$.

□

Once again, by the continuity of an overlap function, we have the following result:

Lemma 4. *Given an overlap function O such that* 1 *is the left neutral element of O, the poset obtained from* $\sqsubseteq_O$ *on* $[0,1]$ *is a chain, and is in fact, the usual order on* $[0,1]$, *i.e.,* $([0,1], \sqsubseteq_O) = ([0,1], \leq)$.

The results for a grouping function follow along similar lines and hence we present only the statements without their proofs.

Proposition 2. *Given a grouping function* $G : [0,1]^2 \to [0,1]$, *the following are equivalent:*

(i) $G(0,x) = x$, *for all* $x \in [0,1]$.

(ii) G *satisfies* (LLI), (QP), *and* (GQP).

Lemma 5. *Given a grouping function G such that* 0 *is the left neutral element of G, the poset obtained from* $\sqsubseteq_G$ *on* $[0,1]$ *is a chain, and is in fact, the usual order on* $[0,1]$, *i.e.,* $([0,1], \sqsubseteq_G) = ([0,1], \leq)$.

3.3 Posets from Fuzzy Implications

Fuzzy implications generalise the notion of classical implication and are one of the important fuzzy logic connectives. They play a vital role in fuzzy control, fuzzy logic systems, and approximate reasoning [4]. Unlike semi-copulas, overlap and grouping functions, fuzzy implications do not include any associative versions of them. We investigate the posets obtained from them below.

Definition 5. *An* $I : [0,1]^2 \to [0,1]$ *is said to be a fuzzy implication if:*

- *I is decreasing in the first variable, i.e.*

$$x_1 \leq x_2 \implies I(x_1,y) \geq I(x_2,y) \qquad \textbf{(I1)}$$

- *I is increasing in the second variable, i.e.*

$$y_1 \leq y_2 \implies I(x,y_1) \leq I(x,y_2) \qquad \textbf{(I2)}$$

- *I satisfies the boundary conditions:*

$$I(0,0) = I(1,1) = 1, I(1,0) = 0$$

Definition 6. *[5] A fuzzy implication I is said to satisfy*

- ***left neutrality*** *property if*

$$I(1,y) = y, \qquad y \in [0,1]\ . \qquad \textbf{(NP)}$$

- ***the law of importation*** *w.r.t. C if there exists a binary operation C such that*

$$I(x,I(y,z)) = I(C(x,y),z), \qquad x,y,z \in [0,1]\ . \qquad (\textbf{LI}(C))$$

Remark 5. *In the literature, if an I satisfies* (**LI**(C)) *w.r.t. a t-norm, i.e.,* $C = T$, *it is called the law of importation, and in case C is increasing, conjunctive and commutative, then it is called the the weak law of importation.*

We see in the following lemma that that a fuzzy implication I satisfying (**NP**) and (**LI**(C)) are sufficient to yield a poset.

Lemma 6. *A fuzzy implication I satisfying* (**NP**) *and* (**LI**(C))*, satisfies* (LLI), (QP), *and* (GQP).

Proof. (LLI) is directly implied by (**NP**).

By (**I1**) we have for all $a,b \in [0,1], I(a,b) \geq I(1,b) = b$. Let $I(x,I(y,z)) = z$. Now, suppose $I(y,z) = a > z$. Then, $I(x,I(y,z)) = I(x,a) \geq a > z$, which is a contradiction. Hence, I satisfies (QP).

Let $I(x,I(y,z)) = w$. By (**LI**(C)), we know $I(x,I(y,z)) = I(C(x,y),z)$. Letting $\ell = C(x,y)$, we see that (GQP) is valid. □

There exist many families of fuzzy implications. Among the major ones, it is well-known that the families of (S,N)-, R-, f-, and g-implications satisfy both (**NP**) and (**LI**(C)) w.r.t. an appropriate t-norm, see for instance, [3, 2, 6]. Thus when these families are considered, $\sqsubseteq_I$ do give rise to posets. This has been extensively dealt with in [9, 10].

However, the family of *QL*-implications is not known to always satisfy (**LI**(C)). Thus it is interesting to study if fuzzy implications I belonging to this family give rise to a poset w.r.t. the relation $\sqsubseteq_I$ as defined in (1). Of course, the QL implications which also belong to the families of (S,N)-, and R- implications, yield posets by the relation in (1). In the following, we present some positive examples of QL implications that aren't (S,N)- or R- implications.

Definition 7. *[4] A function $I : [0,1]^2 \to [0,1]$ is called a QL-implication if there exist a t-norm T, a t-conorm S and a fuzzy negation N such that $I(x,y) = S(N(x),T(x,y)), x,y \in [0,1]$ is a fuzzy implication. We will denote such an I by $I_{T,S,N}$.*

Note that it is well known that if a fuzzy implication I does not satisfy the exchange principle (EP),

$$I(x,I(y,z)) = I(y,I(x,z)), \qquad x,y,z \in [0,1] , \quad \text{(EP)}$$

then it does not satisfy (**LI**(C)) with any commutative C [4] .

Example 3. *The following QL-implications do not satisfy* (**LI**(C)) *with any commutative C. However, the relations $\sqsubseteq_{I_{\mathbf{PC}}}, \sqsubseteq_{I_{\mathbf{PR}}}$ define a poset on $[0,1]$. In fact, because of their continuity, they impose a total but the dual of the natural order on $[0,1]$:*

$$I_{\mathbf{PR}}(x,y) = 1 - \max(x(1+xy^2-2y),0)^{\frac{1}{2}} ,$$
$$I_{\mathbf{PC}}(x,y) = 1 - \max(x(x+xy^2-2y),0)^{\frac{1}{2}} .$$

From the following it is clear that $I_{\mathbf{PR}}$ does not satisfy the exchange principle

$$I_{\mathbf{PR}}(0.1, I_{\mathbf{PR}}(0.3,0)) = I_{\mathbf{PR}}(0.1,0.453) = 0.89 ,$$
$$I_{\mathbf{PR}}(0.3, I_{\mathbf{PR}}(0.1,0) = I_{\mathbf{PR}}(0.3,0.684) = 1 ,$$

and hence it cannot satisfy (**LI**(C)) *with any commutative C.*

Similarly, the following calculations show that $I_{\mathbf{PC}}$ does not satisfy the exchange principle

$$I_{\mathbf{PC}}(0.4, I_{\mathbf{PC}}(0.8,0)) = I_{\mathbf{PC}}(0.4,0.2) = 0.92 ,$$
$$I_{\mathbf{PC}}(0.8, I_{\mathbf{PC}}(0.4,0) = I_{\mathbf{PC}}(0.8,0.6) = 1 ,$$

and hence it cannot satisfy (**LI**(C)) *with any commutative C either.*

4 Concluding Remarks

In this work, we have discussed the conditions under which a non-associative operation gives rise to a partial order when the relation (1) is considered. Note that the motivation for considering the relation (1) comes from the work of [15], wherein it was shown that it leads an order on any semigroup. Note, however, that in this work, the operations considered are non-associative and hence do not lead to a semigroup. Thus this work can also be seen as a generalisation of Mitsch's order relation to that of groupoids.

As we have shown, the current work, at once both generalises and subsumes the following works, viz., [9, 10, 18], when seen in the particular context of obtaining orders.

This submission contains only the nascent works in this direction. For instance, in the case of semi-copulas or some families of fuzzy implications, only some sufficient conditions under which (GQP) is satisfied is known. However, a deeper study of the necessary conditions and the type of special posets one could obtain from them, viz., lattice, distributive or modular lattices, etc., is yet to be done. Of course, in the case the underlying operations are continuous, we get the usual order on $[0,1]$ which is both complete and totally ordered, and hence is bounded and distributive too.

Further, many of the aggregation functions have been suitably generalised to a bounded lattice and hence do not enjoy all the properties they do on $[0,1]$. Despite being defined on sets that have an underlying order, it is worthwhile to study the orders obtained, if any, from (1), since it can provide hitherto unknown insights. A case in point was the investigation of uninorms w.r.t. (1) on bounded lattices in [8], wherein it was shown that a uninorm U on a bounded lattice $(\mathbb{L},\leq)$ can also be seen as a t-norm on the alternate lattice $(\mathbb{L},\sqsubseteq_U)$.

References

[1] E. Aşıcı, An order induced by nullnorms and its properties, Fuzzy Sets and Systems 325 (2017) 35–46.

[2] M. Baczyński, B. Jayaram, On the characterizations of (S, N)-implications, Fuzzy Sets and Systems 158 (15) (2007) 1713–1727.

[3] M. Baczyński, B. Jayaram, Yager's classes of fuzzy implications: Some properties and intersections, Kybernetika 43 (2007) 157 – 182.

[4] M. Baczyński, B. Jayaram, Fuzzy Implications, Vol. 231 of Studies in Fuzziness and Soft Computing, Springer-Verlag, Berlin Heidelberg, 2008.

[5] M. Baczynski, B. Jayaram, Fuzzy Implications: Some Recently Solved Problems, Springer Berlin

Heidelberg, Berlin, Heidelberg, 2013, pp. 177–204.

[6] J. Balasubramaniam, Yager's new class of implications J_f and some classical tautologies, Information Sciences 177 (3) (2007) 930–946.

[7] Ü. Ertuğrul, M. N. Kesicioğlu, F. Karacal, Ordering based on uninorms, Information Sciences 330 (2016) 315–327.

[8] V. K. Gupta, B. Jayaram, Orders from uninorms on bounded lattices: Some perspectives, Submitted to AGOP 2021.

[9] V. K. Gupta, B. Jayaram, Importation algebras, in: International Summer School on Aggregation Functions, Springer, 2019, pp. 83–94.

[10] V. K. Gupta, B. Jayaram, Importation lattices, Fuzzy Sets and Systems 405 (2021) 1 – 17.

[11] V. K. Gupta, B. Jayaram, Order based on associative operations, Information Sciences 566 (2021) 326–346.

[12] F. Karaçal, M. N. Kesicioğlu, A T-partial order obtained from t-norms, Kybernetika 47 (2) (2011) 300–314.

[13] M. N. Kesicioğlu, R. Mesiar, Ordering based on implications, Information Sciences 276 (2014) 377–386.

[14] R. Mesiar, A. Kolesárová, Copulas and fuzzy implications, International Journal of Approximate Reasoning 117 (2020) 52–59.

[15] H. Mitsch, A natural partial order for semigroups, Proceedings of the American Mathematical Society 97 (3) (1986) 384–388.

[16] H. Mitsch, Semigroups and their natural order, Mathematica Slovaca 44 (4) (1994) 445–462.

[17] K. S. Nambooripad, The natural partial order on a regular semigroup, Proceedings of the Edinburgh Mathematical Society 23 (3) (1980) 249–260.

[18] J. Qiao, On binary relations induced from overlap and grouping functions, International Journal of Approximate Reasoning 106 (2019) 155–171.

[19] H. B. Sola, J. Fernández, R. Mesiar, J. Montero, R. Orduna, Overlap index, overlap functions and migrativity, in: J. P. Carvalho, D. Dubois, U. Kaymak, J. M. da Costa Sousa (Eds.), Proceedings of the Joint 2009 International Fuzzy Systems Association World Congress and 2009 European Society of Fuzzy Logic and Technology Conference, Lisbon, Portugal, July 20-24, 2009, 2009, pp. 300–305.

[20] H. B. Sola, M. Pagola, R. Mesiar, E. Hüllermeier, F. Herrera, Grouping, overlap, and generalized bientropic functions for fuzzy modeling of pairwise comparisons, IEEE Trans. Fuzzy Syst. 20 (3) (2012) 405–415.