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On Preservation of Residuated Lattice Properties for Partial Algebras

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Abstract

This paper concentrates on investigating the preservation of the axioms and essential properties of residuated lattices in partial fuzzy set theory. We consider seven most-known partial algebras dealing with undefined values such as the Bochvar, Bochvar external, Sobociński, McCarthy, Nelson, Kleene, Łukasiewicz, and additional, two recent ones, namely the Lower estimation algebra and the Dragonfly algebra. We provide the sketch of proofs in details for the preservation of considered axioms and properties in these algebras. The paper concludes with the tables summarizing the results, which visibly show how close is a partial algebra to a residuated lattice.

Keywords: Residuated lattice structures, Properties, Partial fuzzy set theory, Partial fuzzy logic, Undefined values.

1 Introduction and preliminaries

1.1 Introduction

Residuated lattices are algebraic structures of truth values for many-valued (fuzzy) logics. They were initiated by Ward and Dilworth in [17] as a generalization of ideals lattices of a ring. Later on, this topic became crucial and was discussed by numerous authors in distinct directions, both for pure mathematical interest and for the sake of computing application [7, 3]. Several interesting algebraic and logical properties of residuated lattice have been taken into account and a reduction of the set of axioms defining each residuated lattice was proposed [8]. Classes of residuated lattice are rich and covering, for instance, BL (Basic Logic) algebras, MV-algebras, Heyting algebras, MTL-algebras, Wajsberg algebra, De Morgan residuated lattices, or involutive residuated lattice. For details and relationships between such algebras, we refer readers to e.g., [13, 9]. It is worth mentioning that the topic is still actual and elaborated, see [14, 15].

Partial fuzzy logic and the related partial fuzzy set theory have been proposed and developed recently, see [2, 4, 12]. They generalized three-valued logic, i.e., they enable dealing with undefined values (which can be interpreted as "indeterminable", "meaningless", "irrelevant", or "missing") in many-valued logics. Let us mention that three-valued logic (and consequently their extensions such as four-, five-, or six-valued logics) belong to classical topics which has been explored by numerous logicians over the past decades, since their initiation in 1920's by the work of Łukasiewicz [11] (cf. [10, 1]). Several algebras in three-valued logic such as Bochvar, Sobociński, Kleene, McCarthy, Nelson, etc. were designed in order to capture different semantics of undefined values. Later on, they have been generalized into their partial variants in order to be employed in the partial fuzzy set theory. Furthermore, the authors have proposed two algebras employing the so-called lower-approximation strategy in order to deal with the "missing values" type of undefinedness, namely *Lower estimation* [6] algebra and the *Dragonfly* algebra [16]. These algebras were shown to meet the expectations in the real application of the dragonfly classification problem in biology.

Note, that partial algebras have been taken into account from various perspectives [2, 5] and it seems that there is no generally accepted agreement on the choice on which algebras are the most appropriate for given tasks. However, their usefulness for distinct fields and applications is unquestionable. Hence, considering several partial algebras for particular studies is a natural and desirable step we follow in this work. We consider nine algebras, namely *Bochvar, Bochvar external, Sobociński, Kleene, McCarthy, Nelson, Łukasiewicz, Dragonfly*, and the *Lower estimation* algebra. Without

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distinguishing the underlying semantics of the undefined values in those algebras, we will call them *undefined values* and denote them by a dummy value $\star$.

Consequently, the operations of particular partial algebras are defined on a support $L^\star = L \cup \{\star\}$ standing for the extension of the support L of a residuated lattice by the dummy value $\star$. Note that the operations on $L^\star$ extend those on L that is, when they are identical to the underlying residuated lattice operations whenever being applied to elements from L. We should emphasize that the structure $\langle L^\star, \wedge_\theta, \vee_\theta, \otimes_\theta, \rightarrow_\theta, 0, 1\rangle$, where θ denotes a particular partial algebra, is no more a residuated lattice. Studying the preservation of properties preserved in residuated structures is an indispensable part toward the development of the partial algebras in partial fuzzy set theory. Several previous studies considered the preservation of some chosen properties of residuated lattices in partial algebras. For example, an exhaustive study for the Dragonfly algebra can be found in [16]. In this paper, we focus on the complete set of axioms of a residuated lattice and additional essential properties, and study their validity in various partial algebras, in particular, in the nine above-mentioned ones.

1.2 Background algebraic structure of truth values

In this section, we recall some basic notions related to residuated lattices which are used in the sequel.

Definition 1. [13] An algebraic structure $\mathscr{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ is a *residuated lattice* if

1. $\langle L, \wedge, \vee, 0, 1\rangle$ is a lattice with the least element 0 and the greatest element 1
2. $\langle L, \otimes, 1\rangle$ is a commutative monoid such that $\otimes$ is isotone in both arguments
3. the operation $\rightarrow$ is a residuation with respect to $\otimes$, i.e.

$$a \otimes b \leq c \quad \text{iff} \quad a \rightarrow c \geq b\,. \tag{1}$$

Note, that the residuated implication is given by

$$a \rightarrow b = \bigvee \{c \mid a \otimes c \leq b\}\,. \tag{2}$$

The following theorem published in [8] presents an alternative way to define a residuated lattice.

Theorem 1. *Let $\mathscr{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, \rangle$ be a structure such that $1 := 0 \rightarrow 0$. Then, $\mathscr{L}$ is residuated lattice if the following identities are satisfied for all $a, b, c \in L$:*

(R_1) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

(R_2) $a \vee (b \vee c) = (a \vee b) \vee c$

(R_3) $a \vee (b \wedge a) = a$

(R_4) $0 \wedge a = 0$

(R_5) $a \otimes (b \otimes c) = (a \otimes b) \otimes c$

(R_6) $a \otimes 1 = a$

(R_7) $a \wedge (b \rightarrow ((a \otimes b) \vee c)) = a$

(R_8) $((a \rightarrow b) \otimes a) \vee b = b$

(R_9) $(a \vee b) \otimes c = (a \otimes c) \vee (b \otimes c)$

Let us recall the definitions of additional operations in a residuated lattice, namely the negation and the addition:

$$\neg a = a \rightarrow 0\,, \qquad a \oplus b = \neg(\neg a \otimes \neg b)\,. \tag{3}$$

We recall also two notions that will narrow the classes of residuated lattice to preserve some desirable properties.

Definition 2. Let $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ be a residuated lattice. We say that an element $a \in L \setminus \{0,1\}$ is a *zero divisor* of $\otimes$ if there exists some $b \in L \setminus \{0,1\}$ such that $a \otimes b = 0$.

Definition 3. Let $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ be a residuated lattice. We say that the negation $\neg$ is *strict*, if it holds that $\neg a = 1$ if $a = 0$, and $\neg a = 0$, otherwise.

The relation between the two above-recalled notions is formulated in the following lemma.

Lemma 1. *Let $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ be a residuated lattice. The negation $\neg$ given by (3) is strict if and only if $\otimes$ is without zero divisors.*

Proof. Let $\otimes$ be without zero divisors. Substituting to (2), we get the definition of the negation:

$$\neg a = \bigvee \{c \mid a \otimes c \leq 0\}\,.$$

As $\otimes$ is without zero divisors, for $a \neq 0$ the equality $a \otimes c = 0$ implies $c = 0$ and thus, $\neg a = 0$. In case of $a = 0$, it is easy to imply $\neg a = 1$.

The opposite implication can be proved analogously by inverse steps. □

1.3 Various kinds of algebras dealing with undefined values

Let us consider $\mathscr{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, \rangle$ and let us consider the operations of distinct partial algebras defined on the support $L^{\star}$. In the Bochvar algebra (B), the operations dealing with $\star$ always result $\star$ as this dummy value plays the annihilator role. In the Bochvar external algebra (Be), $\star$ acts like 0 in the underlying residuated lattice. In the Sobociński algebra (S), $\star$ works like the neutral element for the conjunction and disjunction operations. The operations of the Kleene algebra (K) that combine $\star$ and $0,1$ comply with the ordering $0 \leq \star \leq 1$, however, whenever $\star$ is combined with $a \notin \{0,1\}$, the operations coincide the Bochvar ones. The McCarthy (Mc) operations correspond to the Bochvar ones when $\star$ appears in the first argument, and correspond to the Kleene ones when $\star$ is placed in the second argument. In the Nelson (N) and the Łukasiewicz (L) algebras, the conjunction and disjunction operations are identical with the Kleene ones while the difference occurs at some positions of the implication operation. In particular, $\star \to_L \star = 1$ and $\star \to_N 0 = \star \to_N \star = 1$ while the Kleene implication yields $\star$. The Lower estimation algebra (Le) and the Dragonfly algebra (D) are identical regarding the conjunction and disjunction operations – for the combination of $\star$ with $a \notin \{0,1\}$, they mirror the Bochvar ones and Sobociński operations, respectively.

Notice that in the Bochvar, Bochvar external and Sobociński algebras, $\star$ is incomparable to any $a \in L$ and thus, only elements in L are comparable. In the Kleene, Nelson, McCarthy, Łukasiewicz, and the Lower estimation algebra, apart from operating on elements of L, the ordering $\leq$ can be applied to compare $\star$ and the extreme points in the following way: $0 \leq \star \leq 1$. Otherwise, $\star$ is not comparable to any $a \notin \{0,1\}$.

Tables 1-3 provide the definitions of the above-mentioned operations.

$\wedge,\otimes$		B	Be	S	K	Mc	N	L	Le	D
α	$\star$	$\star$	0	α	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
$\star$	β	$\star$	0	β	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
$\star$	$\star$	$\star$	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
$\star$	0	$\star$	0	0	0	$\star$	0	0	0	0
0	$\star$	$\star$	0	0	0	0	0	0	0	0

Table 1: Conjunctive operations of distinct algebras ($\alpha,\beta \in (0,1]$).

Apart from the above operations, we may define the operations of negation and addition as $\neg_{\theta} a = a \to_{\theta} 0$ and $a \oplus_{\theta} b = \neg_{\theta}(\neg_{\theta} a \otimes_{\theta} \neg_{\theta} b)$ where $\theta \in \{\mathrm{B},\mathrm{Be},\mathrm{S},\mathrm{K},\mathrm{Mc},\mathrm{N},\mathrm{L},\mathrm{Le},\mathrm{D}\}$.

$\vee,\oplus$		B	Be	S	K	Mc	N	L	Le	D
α	$\star$	$\star$	α	α	$\star$	$\star$	$\star$	$\star$	α	α
$\star$	β	$\star$	β	β	$\star$	$\star$	$\star$	$\star$	β	β
$\star$	$\star$	$\star$	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
$\star$	1	$\star$	1	1	1	$\star$	1	1	1	1
1	$\star$	$\star$	1	1	1	1	1	1	1	1
$\star$	0	$\star$	0	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
0	$\star$	$\star$	0	0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$

Table 2: Disjunctive operations of distinct algebras ($\alpha,\beta \in (0,1)$).

$\to$		B	Be	S	K	Mc	N	L	Le	D
α	$\star$	$\star$	$\neg\alpha$	$\neg\alpha$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
$\star$	β	$\star$	1	β	$\star$	$\star$	$\star$	$\star$	β	β
$\star$	$\star$	$\star$	1	$\star$	$\star$	$\star$	1	1	$\star$	1
$\star$	1	$\star$	1	1	1	$\star$	1	1	1	1
0	$\star$	$\star$	1	1	1	1	1	1	1	1
$\star$	0	$\star$	1	0	$\star$	$\star$	1	$\star$	0	$\star$

Table 3: Implicative operations of distinct algebras ($\alpha \in (0,1]$, $\beta \in (0,1)$).

2 Preservation of axioms

In this section, we focus on investigating whether axioms and some essential properties that are valid in several classes of residuated lattices are still valid in the considered partial algebras. In particular, we check the validity of axioms (R_1)-(R_9) from Theorem 1, and additional important properties of residuated lattices such as monotonic properties of implications or the first De Morgan law.

Proposition 1. ($\wedge-$Associativity (R_1)) *The following property holds for any $a,b,c \in L^{\star}$:*

$$a \wedge_{\theta} (b \wedge_{\theta} c) = (a \wedge_{\theta} b) \wedge_{\theta} c \tag{4}$$

where $\theta \in \{\mathrm{B},\mathrm{Be},\mathrm{S},\mathrm{K},\mathrm{Mc},\mathrm{N},\mathrm{L},\mathrm{Le},\mathrm{D}\}$.

Sketch of the proof: Let a or b or c equals to $\star$. Then both sides of (4) equal to $\star$ for $\theta = \mathrm{B}$, and equal to 0 for $\theta = \mathrm{Be}$. Let e.g. $a = \star$ then for any $b,c \in L^{\star}$, both sides of (4) equal to $b \wedge_{\theta} c$ for $\theta = \mathrm{S}$. The validity of (4) for $\theta \in \{\mathrm{K},\mathrm{Mc},\mathrm{N},\mathrm{L},\mathrm{Le},\mathrm{D}\}$ can be showed as follows. Let $a = \star$. Then, one can check easily that both sides of (4) equal to $\star$ if $b \wedge_{\theta} c \neq 0$, and equal to 0, otherwise. Let $b = \star$. Then both sides of (4) result in 0 if $a = 0$ or $c = 0$, and result in $\star$, otherwise. The case that $c = 0$ can be checked similarly. □

Proposition 2. ($\vee-$Associativity (R_2)) *The following* *property holds for any* $a,b,c \in L^\star$*:*

$$a \vee_\theta (b \vee_\theta c) = (a \vee_\theta b) \vee_\theta c \tag{5}$$

where $\theta \in \{\mathrm{B},\mathrm{Be},\mathrm{S},\mathrm{K},\mathrm{Mc},\mathrm{N},\mathrm{L},\mathrm{Le},\mathrm{D}\}$.

Sketch of the proof: Let a or b or c equals to $\star$. Then both sides of (5) equal to $\star$ for $\theta = \mathrm{B}$. Let e.g. $a = \star$ then both sides of (5) equal to $b \vee_\theta c$ for $\theta \in \{\mathrm{Be},\mathrm{S}\}$. Consider $\theta \in \{\mathrm{K},\mathrm{Mc},\mathrm{N},\mathrm{L}\}$ then both sides of (5) equal to 1 if $b \vee_\theta c = 1$, otherwise they equal to $\star$. Let $b = \star$. Then, both sides of (5) result in 1 if $a = 1$ or $c = 1$, and result in $\star$, otherwise. The case that $c = \star$ can be verified analogously. Now, consider $\theta \in \{\mathrm{Le},\mathrm{D}\}$. Let $a = \star$. Then both sides of (5) equal to $\star$ if $b \vee_\theta c \in \{0,\star\}$, otherwise they equal to $b \vee_\theta c$. The case of $b = \star$ or $c = \star$ can be checked similarly. □

Proposition 3. (Absorption (R_3)) *The following property holds for any* $a,b \in L^\star$*:*

$$a \vee_\theta (a \wedge_\theta b) = a \tag{6}$$

where $\theta \in \{\mathrm{Le},\mathrm{D}\}$.

Sketch of the proof: Let $a = \star$. Then for all $b \in L^\star$, the left hand side of (6) equals to $\star$. Let $b = \star$. Then, both hand sides of (6) equal to 0 if $a = 0$, and equal to a otherwise. □

Remark 1. *Note that the other absorption law*

$$a \wedge_\theta (a \vee_\theta b) = a$$

holds for $\theta \in \{\mathrm{Le},\mathrm{D}\}$ *as well.*

Let us show the invalidity of (6) for $\theta \in \{\mathrm{B},\mathrm{Be},\mathrm{S},\mathrm{K},\mathrm{Mc},\mathrm{L},\mathrm{N}\}$ in the following proposition.

Proposition 4. *The absorption property expressed in (6) cannot be guaranteed for arbitrary* $a,b \in L^\star$ *whenever* $\theta \in \{\mathrm{B},\mathrm{Be},\mathrm{S},\mathrm{K},\mathrm{Mc},\mathrm{L},\mathrm{N}\}$ *is considered.*

Sketch of the proof: By finding the counterexamples. Consider $\theta = \mathrm{B}$ and $b = \star, a \neq \star$. Then $a \vee_\theta (a \wedge_\theta b) = \star \neq a$. Consider $\theta = \mathrm{Be}$ and $a = \star$. Then for all $b \in L^\star$, $a \vee_\theta (a \wedge_\theta b) = 0 \neq a$. Similarly, when $\theta = \mathrm{S}$, we may take $a = \star, b \neq \star$ then it is clear that $a \vee_\theta (a \wedge_\theta b) = b \neq a$. For $\theta \in \{\mathrm{K},\mathrm{Mc},\mathrm{N},\mathrm{L}\}$, we may consider $b = \star, a \notin \{0,\star,1\}$ then $a \vee_\theta (a \wedge_\theta b) = \star \neq a$. □

Proposition 5. (Annihilating 0 (R_4)) *The following property holds for any* $a \in L^\star$*:*

$$a \wedge_\theta 0 = 0 \tag{7}$$

where $\theta \in \{\mathrm{Be},\mathrm{S},\mathrm{K},\mathrm{N},\mathrm{L},\mathrm{Le},\mathrm{D}\}$.

Sketch of the proof: It is clear that $\star \wedge_\theta 0 = 0$ for $\theta \in \{\mathrm{Be},\mathrm{S},\mathrm{K},\mathrm{N},\mathrm{L},\mathrm{Le},\mathrm{D}\}$. □

Note that (7) does not hold for $\theta \in \{\mathrm{B},\mathrm{Mc}\}$.

Proposition 6. *Property (7) cannot be guaranteed for arbitrary* $a \in L^\star$ *whenever* $\theta \in \{\mathrm{B},\mathrm{Mc}\}$ *is considered.*

Sketch of the proof: Indeed, $\star \wedge_\theta 0 = \star$ for $\theta \in \{\mathrm{B},\mathrm{Mc}\}$. □

It is important to mention that if the positions of arguments a and 0 in the left-hand side of (7) exchange then it holds that $0 \wedge_\theta a = 0$ for $\theta = \mathrm{Mc}$. We obtain the following proposition.

Proposition 7. *For any* $a \in L^\star$*:*

$$0 \wedge_\theta a = 0$$

where $\theta \in \{\mathrm{Be},\mathrm{S},\mathrm{K},\mathrm{Mc},\mathrm{N},\mathrm{L},\mathrm{Le},\mathrm{D}\}$.

Sketch of the proof: The proof for $\theta \in \{\mathrm{Be},\mathrm{S},\mathrm{K},\mathrm{N},\mathrm{L},\mathrm{Le},\mathrm{D}\}$ follows from the commutativity of $\wedge_\theta$ and from the validity of Proposition 5. From the definition of $\wedge_{\mathrm{Mc}}$, it follows that $0 \wedge_{\mathrm{Mc}} \star = 0$. □

Proposition 8. ($\otimes-$Associativity (R_5)) *The following property holds for any* $a,b,c \in L^\star$*:*

$$a \otimes_\theta (b \otimes_\theta c) = (a \otimes_\theta b) \otimes_\theta c \tag{8}$$

where $\theta \in \{\mathrm{B},\mathrm{Be},\mathrm{S}\}$.

Sketch of the proof: Let, e.g., $\theta = \mathrm{S}$ and $a = \star$ then both sides of (8) equal to $(b \otimes_{\mathrm{S}} c)$ for any $b,c \in L^\star$. For $\theta = \mathrm{B}/\mathrm{Be}$, the annihilator role of $\star$ ensures the equality. □

If we investigate the associativity of $\otimes$ in the remaining algebras, the positive result is only for the restricted class of the underlying residuated lattices without zero divisors.

Proposition 9. ($\otimes-$ restricted associativity) *Let* $\mathscr{L}$ *be a residuated lattice such that* $\otimes$ *is without zero divisors. Then the following holds for any* $a,b,c \in L^\star$*:*

$$a \otimes_\theta (b \otimes_\theta c) = (a \otimes_\theta b) \otimes_\theta c \tag{9}$$

where $\theta \in \{\mathrm{K},\mathrm{Mc},\mathrm{N},\mathrm{L},\mathrm{Le},\mathrm{D}\}$.

Sketch of the proof: As an example, let $\theta = \mathrm{K}$ and $a = \star$ and $b,c \neq \star$. Then, if b or c is equal to 0 then both sides of (9) are trivially 0 too. So, let $b,c > 0$. Since $\otimes$ is without zero divisors, $b \otimes c \neq 0$ and thus, both sides of (9) result in $\star$. If we allow also b and/or c to be equal to $\star$, we again obtain both sides of (9) equal to $\star$. All combinations are checked analogously for all algebras. □

Proposition 10. (Neutral-1 (R_6)) *The following property holds for any $a \in L^\star$:*

$$a \otimes_\theta 1 = a \tag{10}$$

where $\theta \in \{\mathrm{B}, \mathrm{K}, \mathrm{Mc}, \mathrm{N}, \mathrm{L}, \mathrm{Le}, \mathrm{D}\}$.

Sketch of the proof: By the definition of $\otimes_\theta$ for $\theta \in \{\mathrm{B}, \mathrm{K}, \mathrm{Mc}, \mathrm{N}, \mathrm{L}, \mathrm{Le}, \mathrm{D}\}$. □

Proposition 11. *Property (10) cannot be guaranteed for arbitrary $a \in L^\star$ whenever $\theta \in \{\mathrm{Be}, \mathrm{S}\}$ is considered.*

Sketch of the proof: Indeed, $\star \wedge_{\mathrm{Be}} 1 = 0$ and $\star \wedge_{\mathrm{S}} 1 = 1$. □

Proposition 12. (R_7) *Property*

$$a \wedge_\theta (b \rightarrow_\theta ((a \otimes_\theta b) \vee_\theta c)) = a \tag{11}$$

cannot be guaranteed for arbitrary $a, b, c \in L^\star$ and for any $\theta \in \{\mathrm{B}, \mathrm{Be}, \mathrm{S}, \mathrm{K}, \mathrm{Mc}, \mathrm{N}, \mathrm{L}, \mathrm{Le}, \mathrm{D}\}$.

Sketch of the proof: By finding the counterexamples. Consider $\theta = \mathrm{B}$ and $b = \star, a \neq \star$. Then the left-hand side of (11) equals to $\star$. Consider $\theta = \mathrm{Be}$ and $a = \star$. Then for any $b, c \in L^\star$ the left-hand side of (11) equals to 0. Consider $\theta = \mathrm{S}$ and $a = \star, b = c = 1$. Then the left-hand side of (11) equals to 1. Let us consider $\theta \in \{\mathrm{K}, \mathrm{Mc}, \mathrm{L}, \mathrm{N}\}$ and let $a \notin \{0, \star, 1\}, b = 1, c = \star$. Then the left-hand side of (11) equals to $a \wedge (1 \rightarrow_\theta (a \vee_\theta \star)) = \star$. Finally, consider $\theta \in \{\mathrm{Le}, \mathrm{D}\}$. Let $a, c \notin \{0, \star, 1\}, b = \star$. Then the left-hand side of (11) equals to $a \wedge_\theta c$ which can be strictly smaller than a for the choice of $c < a$. □

Proposition 13. (R_8) *The following property holds for any $a, b \in L^\star$:*

$$((a \rightarrow_\theta b) \otimes_\theta a) \vee_\theta b = b \tag{12}$$

where $\theta = \mathrm{Le}$.

Sketch of the proof: Let $a = \star$. We consider three cases of $b \in L^\star$. If $b = 0$ then both sides of (12) equal to 0. If $b = \star$ then both sides of (12) equal to $\star$. If $b \notin \{0, \star\}$, then both sides of (12) equal to b. Now, let $b = \star$. Then for any $a \in L^\star$, we may check that both sides of (12) equal to $\star$. □

Note that property (12) does not hold for the remaining algebras.

Proposition 14. *Property (12) cannot be guaranteed for arbitrary $a, b, c \in L^\star$ and for arbitrary $\theta \in \{\mathrm{B}, \mathrm{Be}, \mathrm{S}, \mathrm{K}, \mathrm{Mc}, \mathrm{N}, \mathrm{L}, \mathrm{D}\}$.*

Sketch of the proof: By finding the counterexamples. Consider $\theta = \mathrm{B}$ and $a = \star, b \neq \star$. Then the left-hand side of (12) equals to $\star$. Consider $\theta = \mathrm{Be}$ and $a = b = \star$. Then the left-hand side of (12) equals to 0. Consider $\theta = \mathrm{S}$ and $a = 1, b = \star$. Then the left-hand side of (12) equals to 0. Consider $\theta \in \{\mathrm{K}, \mathrm{Mc}, \mathrm{N}, \mathrm{L}\}$. Let $a = \star, b \notin \{0, \star, 1\}$, then $((a \rightarrow_\theta b) \otimes_\theta a) \vee_\theta b = \star \vee_\theta b = \star$. Consider $\theta = \mathrm{D}$ and $a = \star, b = 0$. Then one may check that $((a \rightarrow_\theta b) \otimes_\theta a) \vee_\theta b = \star \vee_\theta 0 = \star$. □

Proposition 15. (Distributivity of the multiplication over the supremum (R_9)) *For any $a, b, c \in L^\star$:*

$$a \otimes_\theta (b \vee_\theta c) = (a \otimes_\theta b) \vee_\theta (a \otimes_\theta c) \tag{13}$$

where $\theta \in \{\mathrm{B}, \mathrm{Be}\}$.

Sketch of the proof: Let a, b, or c equals to $\star$. Then both sides of (13) are equal to $\star$ for the Bochvar algebra. Let $a = \star$. Then both sides of (13) are equal to 0 for the Bochvar external algebra. The same result is obtain for $a \in L$ and $b, c = \star$. The equality is obtained also for only $b \in L$ (or for only $c \in L$). □

Proposition 16. (Distributivity of the multiplication over the supremum (R_9) – restricted) *Let $\mathscr{L}$ be a residuated lattice such that $\otimes$ is without zero divisors. For any $a, b, c \in L^\star$:*

$$a \otimes_\theta (b \vee_\theta c) = (a \otimes_\theta b) \vee_\theta (a \otimes_\theta c)$$

where $\theta \in \{\mathrm{D}, \mathrm{Le}\}$.

Sketch of the proof: In [16], the proposition has been proved for the Dragonfly algebra. The proof for the Lower estimation algebra is analogous. □

It is worth mentioning that the distributivity expressed in (13) does not hold for the remaining considered algebras.

Proposition 17. *The distributivity of $\otimes$ over the supremum expressed in (13) cannot be guaranteed for arbitrary $a, b, c \in L^\star$ whenever $\theta \in \{\mathrm{S}, \mathrm{K}, \mathrm{Mc}, \mathrm{N}, \mathrm{L}\}$ is considered.*

Sketch of the proof: By finding the counterexamples. For example, consider the Kleene algebra. Let $a \notin \{0, \star, 1\}, b = \star, c = 1$. Then, $a \otimes_{\mathrm{K}} (\star \vee_{\mathrm{K}} 1) = a$ which is different from the right hand side $(a \otimes_{\mathrm{K}} \star) \vee_{\mathrm{K}} (a \otimes_{\mathrm{K}} 1) = \star$. The proofs for the other algebras are analogous. □

3 Preservation of additional properties

Proposition 18. (Isotonicity of the implication) *For any $a,b,c \in L^\star$:*

$$a \leq b \Rightarrow c \rightarrow_\theta a \leq c \rightarrow_\theta b \tag{14}$$

where $\theta \in \{\mathrm{B},\mathrm{Be},\mathrm{S}\}$.

Sketch of the proof: Let us consider $\theta = \mathrm{S}$. Let $a,b \neq \star$ and $a \leq b$ then for $c = \star$ we get $\star \rightarrow a = a$ and $\star \rightarrow b = b$ holds and thus, the implication is preserved. Let $a = b = \star$ then the right inequality holds trivially. □

Proposition 19. (Isotonicity of the implication – restricted) *Let $\mathscr{L}$ be a residuated lattice such that $\neg$ is strict. For any $a,b,c \in L^\star$:*

$$a \leq b \Rightarrow c \rightarrow_\theta a \leq c \rightarrow_\theta b \tag{15}$$

where $\theta \in \{\mathrm{K},\mathrm{Mc},\mathrm{L},\mathrm{Le}\}$.

Sketch of the proof: Let us demonstrate the proof for $\theta = \mathrm{K}$. Let $a = 0$, $b = \star$, $c \notin \{0,\star,1\}$ then we may observe that $c \rightarrow_\mathrm{K} 0 \leq c \rightarrow_\mathrm{K} b$ if $c \rightarrow_\mathrm{K} 0 = 0$. The proofs for the remaining algebras are analogous. □

Proposition 20. *The isotonicity the implication $\rightarrow$ in the second argument expressed in (14) cannot be guaranteed for arbitrary $a,b,c \in L^\star$ whenever $\theta \in \{\mathrm{N},\mathrm{D}\}$ is considered.*

Sketch of the proof: By finding the counterexamples. Let $\theta = \mathrm{N}$ and let $a = 0$, $b \notin \{0,\star,1\}$, $c = \star$. Then we may check that $1 = \star \rightarrow_\mathrm{N} 0 \nleq \star \rightarrow_\mathrm{N} b = \star$. The proof for $\theta = \mathrm{D}$ is analogous. □

Proposition 21. (Antitonicity of the implication) *For any $a,b,c \in L^\star$:*

$$a \leq b \Rightarrow b \rightarrow_\theta c \leq a \rightarrow_\theta c \tag{16}$$

where $\theta \in \{\mathrm{B},\mathrm{Be},\mathrm{S},\mathrm{Le},\mathrm{D}\}$.

Sketch of the proof: Let us consider $\theta = \mathrm{S}$. Let $a,b \neq \star$ and $a \leq b$ then for $c = \star$ we get $b \rightarrow_\theta \star = \neg b$ and $a \rightarrow_\theta \star = \neg a$. Since $\neg b \leq \neg a$, the property is preserved. Let $a = b = \star$ then the right-hand side holds trivially.

In [16], we proved that (16) holds for the use of the Dragonfly algebra. In a similar way, we may show that (16) is preserved for the Lower estimation strategy as well. The proof for $\theta \in \{\mathrm{B},\mathrm{Be}\}$ can be added in an analogous way too. □

Proposition 22. *The antitonicity the implication $\rightarrow$ in the first argument expressed in (16) cannot be guaranteed for arbitrary $a,b,c \in L^\star$ whenever $\theta \in \{\mathrm{K},\mathrm{Mc},\mathrm{N},\mathrm{L}\}$ is considered.*

Sketch of the proof: By finding the counterexamples. Let $a = \star$, $b = 1$ and $c \notin \{0,\star,1\}$. Then, we may check that for $\theta \in \{\mathrm{K},\mathrm{Mc},\mathrm{N},\mathrm{L}\}$, $1 \rightarrow_\theta c = c$ which is incomparable with $\star \rightarrow_\theta c = \star$. □

Proposition 23. (Monotonicity of the multiplication) *For any $a,b,c \in L^\star$:*

$$a \leq b \Rightarrow a \otimes_\theta c \leq b \otimes_\theta c \tag{17}$$

where $\theta \in \{\mathrm{B},\mathrm{Be},\mathrm{S}\}$.

Sketch of the proof: Let us consider $\theta = \mathrm{S}$. Let $a,b \neq \star$ and $a \leq b$ then for $c = \star$ we get $a \otimes_\mathrm{S} \star = a$ which is smaller than or equal to $b \otimes_\mathrm{S} \star = b$. Let $a = b = \star$ then the right-hand side inequality holds trivially. □

Proposition 24. *The monotonicity of the multiplication $\otimes$ expressed in (17) cannot be guaranteed for arbitrary $a,b,c \in L^\star$ whenever $\theta \in \{\mathrm{K},\mathrm{Mc},\mathrm{N},\mathrm{L},\mathrm{Le},\mathrm{D}\}$ is considered.*

Sketch of the proof: By finding the counterexamples. Let $a = \star$, $b = 1$ and $c \notin \{0,\star,1\}$. Then it holds for all $\theta \in \{\mathrm{K},\mathrm{Mc},\mathrm{N},\mathrm{L},\mathrm{Le},\mathrm{D}\}$ that $\star \otimes_\theta c = \star$ which is incomparable with $1 \otimes_\theta c = c$. □

Proposition 25. (Residuation) *For any $a,b,c \in L^\star$:*

$$a \rightarrow_\theta (b \rightarrow_\theta c) = b \rightarrow_\theta (a \rightarrow_\theta c) \tag{18}$$

where $\theta \in \{\mathrm{B},\mathrm{S},\mathrm{K},\mathrm{Mc},\mathrm{L},\mathrm{Le}\}$.

Sketch of the proof: Let $\theta = \mathrm{Le}$. Let $a = \star$. Then case of $b \in \{0,\star,1\}$ leads to the equality (18). Let $b \notin \{0,\star,1\}$. Then, both sides of (18) result in $b \rightarrow_\mathrm{Le} 0$ if $c = 0$, in $\star$ if $c = \star$, in 1 if $c = 1$, and in $b \rightarrow_\mathrm{Le} c$ if $c \notin \{0,\star,1\}$. The case of $a \neq \star$ can be verified easily. The proofs for the remaining algebras are analogous. □

Proposition 26. *The residuation property expressed in (18) cannot be guaranteed for arbitrary $a,b,c \in L^\star$ whenever $\theta \in \{\mathrm{Be},\mathrm{N},\mathrm{D}\}$ is considered.*

Sketch of the proof: By finding the counterexamples. Let $\theta = \mathrm{D}$ and let $a = \star$, $b \notin \{0,\star,1\}$, $c = 0$ and it can be checked that $\star \rightarrow_\mathrm{D} \neg b \neq b \rightarrow_\mathrm{D} \star$. The proofs for the remaining algebras are analogous. □

Proposition 27. (1st De Morgan law) *For any $a,b \in L^\star$:*

$$\neg_\theta (a \vee_\theta b) = \neg_\theta a \wedge_\theta \neg_\theta b \tag{19}$$

where $\theta \in \{\mathrm{B},\mathrm{Be},\mathrm{K},\mathrm{Mc},\mathrm{L}\}$.

Sketch of the proof: Consider $\theta = \mathrm{B}$. Let $a = \star$ or $b = \star$. Then both sides of (19) equal to $\star$. Consider $\theta = \mathrm{Be}$. Let e.g. $a = \star$, then for any $b \in L^\star$, both sides of (19) equal to $\neg_\theta b$. Consider $\theta \in \{\mathrm{K}, \mathrm{Mc}, \mathrm{L}\}$. Let $a = \star$ and $b = 1$. Then we may see that both sides of (19) result in 0. Let $a = \star$ and $b \neq 1$. Then, it can be verified that both sides of (19) equal to $\star$. The proof for the case of $b = \star$ is similar. □

Proposition 28. *The first De Morgan law expressed in (19) cannot be guaranteed for arbitrary $a, b \in L^\star$ whenever $\theta \in \{\mathrm{S}, \mathrm{N}\}$ is considered.*

Sketch of the proof: By finding the counterexamples. Consider $\theta = \mathrm{S}$ and $a = \star$, $b = 0$ then it can be verified that $\neg_\mathrm{S}(\star \vee_\mathrm{S} 0) = 1$ while $\neg_\mathrm{S} \star \wedge_\mathrm{S} \neg_\mathrm{S} 0 = 0$. Consider $\theta = \mathrm{N}$, $a = \star$, $b \notin \{0, 1, \star\}$. Then $\neg_\mathrm{N}(\star \vee_\mathrm{N} b) = 1$ while $\neg_\mathrm{N} \star \wedge_\mathrm{N} \neg_\mathrm{N} b = \neg b$. □

Proposition 29. (1st De Morgan law – restricted) *Let $\mathscr{L}$ be a residuated lattice such that $\neg$ is strict. Then (19) holds for any $a, b \in L^\star$ and for $\theta \in \{\mathrm{Le}, \mathrm{D}\}$.*

Sketch of the proof: Let us demonstrate the proof for $\theta = \mathrm{D}$. Let $a = \star$, $b \notin \{0, \star, 1\}$. Then one may observe that (19) is equivalent to $\neg b = \star \wedge_\mathrm{D} \neg b$. This equality is preserved only for $\neg b = 0$. □

4 Summary

In order to summarize the results into a comprehensible form, we introduce the following notations that will be used in Table 5 and Table 6 gathering the results:

(i) "✓" and "×" – denotation of the preservation and non-preservation of the considered property in the given partial algebra, respectively;

(ii) "•" – denotation of the preservation of a given property under the assumption that the underlying algebra is without zero divisors;

(iii) "!" - warning denotation of the change in the preservation of a given property for switched positions of the arguments.

Let us once more summarize all the investigated axioms and properties in a comprehensible form in Table 4.

The summary of the preservation and the non-preservation of the investigated properties is then gathered in Table 5 using the denotation fixed in this Section.

Furthermore, all studied partial algebras with their lists of the preserved axioms (R_1)-(R_9) are presented in a comprehensible form in Table 6 where one can easily check up to which extent the "quality" of the residuated lattice is preserved for the particular partial extensions.

It shows, that (R_7) is not preserved by any of the algebras however, Lower estimation algebra then preserves all the axioms, under the assumption that it is build based on an underlying residuated lattice without zero divisors. This additional property is required in order to meet (R_5) and (R_9). The same holds for the Dragonfly algebra with the difference of non-preserved axiom (R_8). The other algebras do not meet even some other properties, e.g., (R_3). Some of the algebras do not require the restriction on the underlying residuated lattice without zero divisors in order to preserve (R_5) and (R_9), namely Bochvar, Bochvar external, and Sobociński. However, this fact should not be misleadingly viewed as an automatic advantage as the reasons are

	Axioms and properties	
1		$a \wedge_\theta (b \wedge_\theta c) = (a \wedge_\theta b) \wedge_\theta c$
2	Associativity	$a \otimes_\theta (b \otimes_\theta c) = (a \otimes_\theta b) \otimes_\theta c$
3		$a \vee_\theta (b \vee_\theta c) = (a \vee_\theta b) \vee_\theta c$
4	Absorption	$a \vee_\theta (a \wedge_\theta b) = a$
5	Annihilator	$a \wedge_\theta 0 = 0$
6	Neutrality	$a \otimes_\theta 1 = a$
7		$a \wedge_\theta (b \rightarrow_\theta ((a \otimes_\theta b) \vee_\theta c)) = a$
8		$((a \rightarrow_\theta b) \otimes_\theta a) \vee_\theta b = b$
9	Distributivity	$a \otimes_\theta (b \vee_\theta c) = (a \otimes_\theta b) \vee_\theta (a \otimes_\theta c)$
10	Isotonicity	$a \leq b \Rightarrow c \rightarrow_\theta a \leq c \rightarrow_\theta b$
11	Antitonicity	$a \leq b \Rightarrow b \rightarrow_\theta c \leq a \rightarrow_\theta c$
12	Monotonicity	$a \leq b \Rightarrow a \otimes_\theta c \leq b \otimes_\theta c$
13	Residuation	$a \rightarrow_\theta (b \rightarrow_\theta c) = b \rightarrow_\theta (a \rightarrow_\theta c)$
14	1st De Morgan	$\neg_\theta (a \vee_\theta b) = \neg_\theta a \wedge_\theta \neg_\theta b$

Table 4: Investigated axioms and properties.

		B	Be	S	K	Mc	N	L	Le	D
1	(R_1)	✓	✓	✓	✓	✓	✓	✓	✓	✓
2	(R_5)	✓	✓	✓	•	•	•	•	•	•
3	(R_2)	✓	✓	✓	✓	✓	✓	✓	✓	✓
4	(R_3)	×	×	×	×	×	×	×	✓	✓
5	(R_4)	×	✓	✓	✓	×(!)	✓	✓	✓	✓
6	(R_6)	✓	×	×	✓	✓	✓	✓	✓	✓
7	(R_7)	×	×	×	×	×	×	×	×	×
8	(R_8)	×	×	×	×	×	×	×	✓	×
9	(R_9)	✓	✓	✓	•	•	•	•	•	•
10		✓	✓	✓	•	•	×	•	•	×
11		✓	✓	✓	×	×	×	×	✓	✓
12		✓	✓	✓	×	×	×	×	×	×
13		✓	×	✓	✓	✓	×	✓	✓	×
14		✓	✓	×	✓	✓	×	✓	•	•

Table 5: Preservation of the axioms and properties provided in Table 4 in the partial algebras.

B	$(R_1),(R_2),(R_5),(R_6),(R_9)$
Be	$(R_1),(R_2),(R_4),(R_5),(R_9)$
S	$(R_1),(R_2),(R_4),(R_5),(R_9)$
K	$(R_1),(R_2),(R_4),(R_5)^\bullet,(R_6),(R_9)^\bullet$
Mc	$(R_1),(R_2),(R_4)^!,(R_5)^\bullet,(R_6),(R_9)^\bullet$
N	$(R_1),(R_2),(R_4),(R_5)^\bullet,(R_6),(R_9)^\bullet$
L	$(R_1),(R_2),(R_4),(R_5)^\bullet,(R_6),(R_9)^\bullet$
Le	$(R_1),(R_2),(R_3),(R_4),(R_5)^\bullet,(R_6),(R_8),(R_9)^\bullet$
D	$(R_1),(R_2),(R_3),(R_4),(R_5)^\bullet,(R_6),(R_9)^\bullet$

Table 6: Particular partial algebras and the lists of the preserved axioms of residuated lattices.

rather lying in specific treatment of the dummy value than by some application advantages. Indeed, e.g., in the case of the Bochvar algebra, the preservation of, e.g., the associativity (R_5) is simply the direct conclusion of the anihilating effect of $\star$.

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