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Qualitative Capacities and Their Informational Comparison

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Abstract

Capacities are monotonically increasing set functions that generalize probability and possibility measures. They are qualitative when they range on a finite linearly ordered scale. This paper pursues a parallel between qualitative capacities and the quantitative setting of belief functions. The qualitative setting appears to be as rich as the quantitative one, but qualitative counterparts of belief function notions do not always have the same meaning and/or behavior as in the quantitative case. The paper especially focuses on the comparison of qualitative capacities in terms of information content. Various information orderings are studied, including one based on a counterpart of Dempster rule of combination.

Keywords: Qualitative capacity, belief function, possibility distribution, Möbius transform.

1 Introduction

Set functions that are monotonically increasing in the wide sense under inclusion and that generalize probability measures have appeared independently in several works, especially by G. Choquet [3] in 1953, who uses the name "capacity", and by M. Sugeno, in his 1974 Ph.D thesis [20] who calls them "fuzzy measures" in reference to Zadeh's fuzzy sets [22] introduced in 1965. In fact, Sugeno proposed a counterpart of Lebesgue integral, where the set-function is not additive and where sum and product are respectively replaced by max and min, the original union and intersection connectives for fuzzy sets. Capacities have been a key tool especially in decision-related topics [1]. In particular, numerical capacities may represent coalition weights in cooperative games, or yet uncertainty measures, such as upper or lower probabilities, and belief functions. See [16] for a recent monograph on capacities.

Interestingly an important special case of fuzzy measure is the *possibility measure* proposed by Zadeh later in 1978, where the addition in the basic axiom of probability measures is replaced by the maximum. In [23] Zadeh highlights the fact that when defining a possibility measure, the probability distribution is replaced by a fuzzy set, giving birth to possibility theory [11, 12].

In this paper, we consider normalized possibility distributions over a power set as a qualitative counterpart of the basic probability assignment in Shafer evidence theory [19]. Any qualitative capacity can be defined from such a "possibility assignment" [13]. Interpreting qualitative capacities is not easy. In [7], three different ways of using them are presented. They can be interpreted either as bounds on ill-known possibility or necessity measures, or as a tool to model the decision maker attitude (pessimism or optimism) in qualitative criteria under uncertainty, or yet as qualitative counterparts of belief functions that handle both incompleteness and inconsistency of pieces of information stemming from several sources. In contrast to the quantitative case, a qualitative capacity can be associated with several possibility assignments [5, 6], and we use inner and outer Möbius transforms for representing qualitative capacities. This enables us to define a qualitative counterpart of Dempster rule of combination for qualitative capacities. Then six information orderings between qualitative capacities are investigated, which are respectively based on natural dominance between set functions, on contour functions, on so-called outer qualitative capacities, on counterparts of commonality, and specialization, including Dempster-like specialization. The study shows that these orderings do not always behave with respect to each other like their quantitative counterparts.

2 Qualitative capacities

After basic definitions, we recall qualitative Möbius transforms, introduce inner and outer set functions and discuss commonality in the qualitative setting.

Let $W = \{w_1, \ldots, w_m\}$ be a finite set of possible states. A capacity (or fuzzy measure) is a set function $g : 2^W \to [0,1]$ such that [16]:
$g(\emptyset) = 0; \quad g(W) = 1;$
$A \subseteq B \Rightarrow g(A) \le g(B)$ (increasing monotonicity).

The conjugate g^c of a capacity g is defined by $g^c(A) = 1 - g(A^c)$ where A^c is the complement of A. The capacity is said to be additive if $g(A \cup B) = g(A) + g(B)$, whenever $A \cap B = \emptyset$. It is then a probability measure. The following inequalities hold for a capacity g: $g(A \cup B) \ge \max(g(A), g(B))$ and $g(A \cap B) \le \min(g(A), g(B))$. When the first inequality is an equality for all pairs of events (maxitivity axiom), g is called a possibility measure and is denoted by Π, such that $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$. When the second inequality is an equality for all pairs of events (minitivity axiom), g is called a necessity measure, denoted by N and such that $N(A \cap B) = \min(N(A), N(B))$.

A *qualitative* capacity γ is defined similarly, except that $[0,1]$ is replaced by a finite totally ordered set.

Definition 1 *Let L be a finite totally ordered set with a bottom and a top denoted by* 0 *and* 1 *respectively, i.e.,* $L = \{0 = \lambda_0 < \lambda_1 < \ldots < \lambda_l = 1\}$.

A qualitative capacity *(q-capacity, for short) γ is is an L-valued capacity, i.e.,* $\gamma(\emptyset) = 0; \quad \gamma(W) = 1; \quad A \subseteq B \Rightarrow \gamma(A) \le \gamma(B)$.

When $L = \{0,1\}$, we speak of a Boolean capacity. L can be equipped with an order-reversing map, i.e., a mapping: $\nu : L \to L$ such that $\nu(\lambda_i) = \lambda_{l-i}$ [13]. The conjugate of γ can then be defined and it is the capacity γ^c defined as $\gamma^c(A) = \nu(\gamma(A^c)), \forall A \subseteq W$.

A probability measure is not a special case of a q-capacity since addition is not defined on the scale L. This absence in the qualitative setting creates a difficulty when trying to understand the meaning of q-capacities. Indeed, in the numerical framework, some capacities capture a convex family of probability measures [16], which is very helpful to grasp their meaning. Nevertheless, even if the framework of bounded chains is less expressive than the reals, many existing concepts defined for numerical capacities have a qualitative counterpart, such as Möbius transforms, contour functions, and so on.

2.1 Möbius transforms

The notion of Möbius transform is instrumental for studying capacities [16], especially in the theory of evidence, where this transform is interpreted in terms of a probability mass function over a family of subsets [19]. The Mobius transform m_g of a capacity g on a finite set is the set function $m_g : 2^W \to \mathbb{R}$, called *basic probability assignment (BPA)* and given by: $m_g(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} g(B)$.

It can be checked that $\sum_{B \subseteq W} m_g(B) = 1$. Sets A such that $m_g(A) \neq 0$ are said to be focal for g. The function m_g is the unique solution to the linear system of equations: $g(A) = \sum_{B \subseteq A} m_g(B), \forall A \subseteq W$.

A capacity is convex if for all $A, B \in W, g(A \cap B) + g(A \cup B) \ge g(A) + g(B)$. Convexity is generalized by k-monotonicy property. The capacity g is called k-monotone for some $k \ge 2$, if for all families of k subsets $A_1, \ldots, A_k$, it holds that:

$$g(\cup_{i=1}^k A_i) \ge \sum_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} g(\cap_{i \in I} A_i).$$

A capacity is called totally monotone if it is k-monotone for all $k \ge 2$. For example, in the framework of the theory of evidence, a plausibility and a belief function are capacities and a capacity is totally monotone if and only if it is a belief function [19].

An important result for capacities is the following: The Möbius transform m_g of a capacity g is non-negative if and only if g is a belief function, i.e., a totally monotone capacity [19]. The focal sets A are then such that $m_g(A) > 0$.

By analogy, in the qualitative case, m_g is replaced by a normalized *possibility* distribution over subsets [9]:

Definition 2 *A* basic possibility assignment *(BΠA) is a mapping* $\rho : 2^W \to L$, *such that* $\max_{A \subseteq W} \rho(A) = 1$ *(top normalization) and* $\rho(\emptyset) = 0$.

In contrast to the quantitative case, any q-capacity γ can be expressed by means of a BΠA, in the form:

$$\gamma(A) = \max_{B \subseteq A} \rho(B), \forall A \subseteq W, \text{ for some } \rho. \quad (1)$$

This form is really similar to the definition of a capacity in terms of its Möbius transform, replacing sum by maximum. In contrast to the quantitative case, there exists a whole family of BΠAs ρ generating the same q-capacity γ. The least one is the qualitative Möbius transform:

Definition 3 ([16]) *The* qualitative Möbius transform *(QMT) $\gamma_\#$ of a q-capacity γ is the least solution of Eq. (1), namely:*

$$\gamma_\#(A) = \begin{cases} \gamma(A) \text{ if } \gamma(A) > \gamma(A \setminus \{w\}), \forall w \in A; \\ 0 \text{ otherwise} \end{cases} \quad (2)$$

It is such that, as expected:

$$\gamma(A) = \max_{B \subseteq A} \gamma_{\#}(B). \quad (3)$$

An equivalence relation $\sim$ on BΠAs can be defined as follows: $\rho_1 \sim \rho_2$ iff ρ_1 and ρ_2 induce the same capacity by (1). The set of BΠAs inducing γ is $\{\rho : \gamma_{\#} \leq \rho \leq \gamma\}$.

The QMT $\gamma_{\#}$ represents the minimal information necessary to reconstruct γ. The sets A such that $\gamma_{\#}(A) > 0$ are again called *focal sets* of γ and form the set $\mathscr{F}(\gamma)$. For any two focal sets A and B such that $B \subset A$, we have $\gamma_{\#}(B) < \gamma_{\#}(A)$, a restricted monotony condition that is specific to the qualitative setting. In particular if W is focal, we must have $\gamma_{\#}(W) = 1$, a property that characterizes *non-dogmatic* q-capacities. Indeed, $0 < \gamma_{\#}(W) < 1$ is forbidden, since then $\gamma_{\#}(A) < 1, \forall A \subseteq W$, which would imply $\gamma(W) < 1$.

Example 1 *Let* $W = \{w_1, w_2, w_3\}$ *and* $\gamma(\{w_1\}) = \gamma(\{w_1, w_3\}) = 0.3$, $\gamma(\{w_1, w_2\}) = 0.7$, $\gamma(\{w_2, w_3\}) = \gamma(\{w_1, w_2, w_3\}) = 1$ *(here, L is chosen arbitrarily as* $\{0, 0.3, 0.7, 1\}$*), and* $\gamma(A) = 0$ *otherwise. Then* $\mathscr{F}(\gamma) = \{\{w_1\}, \{w_1, w_2\}, \{w_2, w_3\}\}$, *with* $\gamma_{\#}(\{w_1\}) = 0.3$, $\gamma_{\#}(\{w_1, w_2\}) = 0.7$ *and* $\gamma_{\#}(\{w_2, w_3\}) = 1$. □

All focal sets of a q-capacity γ are singletons (i.e., $\gamma_{\#}(A) > 0$ if and only if $\exists w \in W : A = \{w\}$), if and only if it is a possibility measure. A q-capacity γ has focal sets forming a nested sequence if and only if it is a necessity measure.

As an important example of q-capacity, we can adapt the notion of simple support functions present in the Dempster Shafer theory to the qualitative setting.

Definition 4 *A (qualitative)* simple support function (SSF) *focused on a set* $E \neq W$ *is a non-dogmatic necessity measure, denoted by* N_E, *with focal sets E and W. Its qualitative Möbius transform is thus of the form*

$$N_{E\#}(A) = \begin{cases} s < 1 \text{ if } A = E \\ 1 \text{ if } A = W \\ 0 \text{ otherwise} \end{cases}.$$

Clearly $N_E(A) = s$ whenever $E \subseteq A \neq W$ and 0 if $E \not\subseteq A$. In this definition, we restrict to $E \neq W$, since the set-function N_W corresponds to the vacuous capacity γ^0 such that $\gamma^0(A) = \gamma^0_{\#}(A) = 0$ for $A \neq W$.

2.2 Outer capacities

The capacity value $\gamma(A)$ is recovered from its QMT $\gamma_{\#}$ via weights assigned to subsets of set A, which reminds of inner measures. Hence the QMT $\gamma_{\#}$ is actually an *inner* qualitative Möbius transform.

Using supersets instead, another qualitative counterpart of Möbius transforms, we can call *outer qualitative Möbius transform*, can be used to represent a capacity γ. It is a mapping $\gamma^{\#} : 2^W \to L$ defined by

$$\gamma^{\#}(A) = \begin{cases} \gamma(A) & \text{if } \gamma(A) < \min_{A \subset F} \gamma(F) \\ 1 & \text{otherwise.} \end{cases} \quad (4)$$

The set of outer focal sets of γ is $\mathscr{F}^o_\gamma = \{A \subseteq W : \gamma^{\#}(A) < 1\}$. The original capacity is retrieved as [4]:

$$\gamma(A) = \min_{A \subseteq F} \gamma^{\#}(F). \quad (5)$$

An outer QMT is a set function that is monotonic on the set of outer focal sets, with some set $E \in \mathscr{F}^o_\gamma$ such that $\gamma^{\#}(E) = 0$ (in order to ensure that $\gamma(\emptyset) = 0$). In particular, the outer focal sets of a Boolean capacity have zero weights.

Example 2 *In the previous example, it can be checked that* $\gamma^{\#}(\{w_2\}) = \gamma^{\#}(\{w_3\}) = 0$, $\gamma^{\#}(\{w_1, w_2\}) = 0.7$, $\gamma^{\#}(\{w_1, w_3\}) = 0.3$ *and* $\gamma^{\#}(E) = 1$ *otherwise.* □

It is tempting to consider a kind of duality between q-capacities by exchanging inner and outer QMTs in the expressions (3) and (5) of γ. For instance, consider $\min_{A \subseteq F} \gamma_{\#}(F)$. Unfortunately, it is easy to see that this set function is not monotonic. To fix this difficulty one may consider a kind of generalized QMT. Let $\mathscr{M} \subseteq 2^W$ and $\delta : \mathscr{M} \to L$ be a strictly increasing map (with respect to $\subseteq$), for which there are $E, F \in \mathscr{M}$ such that $\delta(E) = 0$ (E can be $\emptyset$) and $\delta(F) = 1$ (F can be W).

Two q-capacities can be induced by δ:

- an inner q-capacity $\gamma_*(A) = \max_{E \in \mathscr{M}, E \subseteq A} \delta(E)$

- an outer q-capacity $\gamma^*(A) = \min_{F \in \mathscr{M}, F \supseteq A} \delta(F)$.

Proposition 1 *q-capacities* γ_* *and* γ^* *are such that:*

- $\gamma^*(A) \geq \gamma_*(A)$ *for all* $A \subseteq W$
- *if* $1 > \delta(A) > 0$, *then* $\gamma^*(A) = \gamma_*(A) = \delta(A)$
- *if* $A \neq \emptyset, W$ *and* $A \notin \mathscr{M}$, *then* $\gamma^*(A) > \gamma_*(A)$

These definition and result remind of inner and outer measures in probability theory [15]. Any q-capacity γ expressed from its QMT can be viewed as an inner q-capacity, and has an outer counterpart letting $\mathscr{M} = \mathscr{F} \cup \{\emptyset\}$. For instance, suppose $\gamma_* = N$ is a necessity measure with nested focal sets $E_p \subset E_{p-1} \cdots \subset E_1$ with $\alpha_i = \delta(E_i)$, $\alpha_1 = 1$, $\delta(\emptyset) = 0$. Here $\alpha_1 > \alpha_2 > \cdots > \alpha_p$. It is easy to check that N^* is a possibility measure with distribution π such that $\pi(w) = \alpha_i$ if $w \in E_i \setminus E_{i+1}, i = 2, \ldots p$ and $\pi(w) = \alpha_p$ if $w \in E_p$. Note that the possibility measure N^* is obtained from N without duality.

However this construction is not interesting for Boolean capacities, since for them, when $A \neq \emptyset$, $\gamma^*(A) = 1$ whether A is contained in a focal set of γ or not (then it is 1 by default). So if γ is Boolean, then $\gamma^*(A) = \gamma^{0*}(A) = 1$ except if $A = \emptyset$.

The capacity γ^* can be viewed as a counterpart of a plausibility function or outer measure. However, the upper capacity γ^* differs from the conjugate of γ. The latter is not intrinsic to q-capacities, unlike in the quantitative case, since it needs the order-reversing map ν. Both also differ from the *upper q-capacity* defined by

$$Pl_\gamma(A) = \max_{B \cap A \neq \emptyset} \gamma_\#(B), \quad (6)$$

similar to the upper probability or plausibility. Clearly, $Pl_\gamma(A) \geq \gamma(A)$. However, in contrast to the numerical case, the function Pl_γ is always maxitive. It is the possibility measure based on the qualitative counterpart, for q-capacities, of the contour function:

$$\pi_\gamma(w) = \max_{w \in B} \gamma_\#(B) \neq \gamma^c_\nu(\{w\}), \quad (7)$$

namely, $Pl_\gamma(A) = \max_{w \in A} \pi_\gamma(w)$ [13].

2.3 Commonality in the qualitative setting

In the numerical setting, there is a third set-function that is useful for belief functions, i.e., the commonality function $Q_g(A) = \sum_{A \subseteq B} m_g(B)$. This function has values in [0, 1] and is anti-monotonic ($A \subseteq B \Rightarrow Q_g(A) \geq Q_g(B)$) only for BPAs m_g.

It is tempting to define its qualitative counterpart as done in [18]: $Q_\gamma(A) = \max_{A \subseteq B} \gamma_\#(B)$. This is clearly an anti-monotonic set function. Note that $Q_\gamma(\emptyset) = 1$, but we do not have that $Q_\gamma(W) = 0$ in general, since $Q_\gamma(W) = \gamma_\#(W)$. It is clear that a large part of the information contained in γ may be lost by Q_γ. More generally we can prove the following result.

Proposition 2 *Given a q-capacity γ with set of focals $\mathscr{F}_\gamma$, let $\hat{\mathscr{F}}_\gamma = \{E \in \mathscr{F}_\gamma : \nexists F \in \mathscr{F}_\gamma, E \subset F\}$ be the maximal elements for inclusion in $\mathscr{F}_\gamma$. Let $\hat{\gamma}$ be the q-capacity whose set of focals is $\hat{\mathscr{F}}_\gamma$ and such that $\forall E \in \hat{\mathscr{F}}_\gamma, \hat{\gamma}_\#(E) = \gamma_\#(E)$. Then $Q_\gamma = Q_{\hat{\gamma}}$.*

Note that the same result is valid for the contour function, i.e., $\pi_\gamma = \pi_{\hat{\gamma}}$, since it is clear that $Q_\gamma(\{w\}) = \pi_\gamma(w)$. Moreover, it is easy to check that

Proposition 3 *$Q_\gamma(E) = \gamma^*(E)$ when $E \in \hat{\mathscr{F}}_\gamma$.*

However, $Q_\gamma(A) = \min_{w \in A} \pi_\gamma(w)$ does not hold. It is easy to see that for any possibility measure Π, $Q_\Pi(A) \neq \min_{w \in A} \pi(w)$ since $Q_\Pi(A) = 0$ as soon as A is not a singleton. In short, the qualitative counterpart of the quantitative commonality is not very attractive. However we could define the commonality of γ as

$$\Delta_\gamma(A) = \min_{w \in A} \pi_\gamma(w).$$

It would behave as a standard commonality function.

3 Merging rules for capacities

The problem of merging capacities is important for the purpose of information fusion. The most popular combination rule for belief functions is Dempster rule of combination [19] (here given without the normalization step), namely $m_{12} = m_1 \odot m_2$ such that

$$m_{12}(A) = \sum_{B,C : A = B \cap C} m_1(B) m_2(C), \forall A \subseteq W. \quad (8)$$

3.1 Qualitative Dempster-like combination

In this section we study a qualitative counterpart of this combination rule for BΠAs, first suggested in [9].

Definition 5 *Let ρ_1 and ρ_2 be two BΠAs. Their* conjunctive combination *is defined by*

$$\forall A \subseteq W, (\rho_1 \otimes \rho_2)(A) = \max_{B \cap C = A} \min(\rho_1(B), \rho_2(C))$$

The combination rule is commutative, associative and possesses an identity: the vacuous BΠA $\rho_0(A) = 0$ for $A \neq W$ and $\rho_0(W) = 1$.

However, the set function $\rho_1 \otimes \rho_2$ is generally not a BΠA because possibly $\rho_1 \otimes \rho_2(\emptyset) \neq 0$.

Moreover, this definition may fail to preserve top normalization via combination, when there are no B and C such that $\rho_1(B) = \rho_2(C) = 1$ with $B \cap C \neq \emptyset$.

In [2], the bottom normalization condition $(\rho_1 \otimes \rho_2)(\emptyset) = 0$ is enforced and added to Def. 5. In the following, we use a bottom- and top-normalized conjunctive rule denoted by $\hat{\otimes}$ and defined by:

$$(\rho_1 \hat{\otimes} \rho_2)(A) = (\rho_1 \otimes \rho_2)(A) \text{ if } A \neq \emptyset, W$$
$$(\rho_1 \hat{\otimes} \rho_2)(\emptyset) = 0.$$
$$(\rho_1 \hat{\otimes} \rho_2)(W) = \begin{cases} 1 \text{ if } \nexists \text{A: } (\rho_1 \otimes \rho_2)(A) = 1 \\ (\rho_1 \otimes \rho_2)(W) \text{ otherwise.} \end{cases}.$$

It can be checked that, omitting the top normalization step (the third condition above), this definition preserves the associativity of the combination rule for BΠAs. But top normalization leads to losing associativity. However, we can make the top normalization step after combining all the items of information ρ_i, that is, compute $(\rho_1 \hat{\otimes} \rho_2 \hat{\otimes} \rho_3)(A), A \neq W$ first (which is associative).

Alternatively, we could renormalize $\rho_1 \otimes \rho_2$ in the style of qualitative possibility theory, i.e., let the weight $(\rho_1 \otimes \rho_2)(A)$ of the focal set A with maximal weight be set to 1.

3.2 Conjunctive combination of q-capacities

Extending the Dempster-like rule from BΠAs to capacities is not trivial because, choosing different BΠAs generating the capacities may lead to different results after combination.

Example 3 *For instance consider* A,B,C *with* $\rho_1(W) = 1, \rho_1(A) = a > \rho_1(A\cap B) = b$ *where* $A\cap B \neq \emptyset$, *and* $\rho_1(E) = 0$ *otherwise. Let* $\tau_1 = \rho_1$ *but for* $\tau_1(B) = b$. *Clearly,* $\rho_1 \sim \tau_1$. *Lastly let* $\rho_2(C) = c$, *with* $B\cap C \neq \emptyset, \rho_2(W) = 1$ *and* $\rho_2(E) = 0$ *otherwise. Suppose* $A\cap C = \emptyset$. *Note that* $(\rho_1 \otimes \rho_2)(B\cap C) = 0$ *and it yields a capacity* γ_{12} *such that* $\gamma_{12}(B\cap C) = 0$. *However* $(\tau_1 \otimes \rho_2)(B\cap C) = \min(b,c)$ *yielding a capacity* γ'_{12} *such that* $\gamma'_{12}(B\cap C) = \min(b,c)$. □

An option is to combine the QMTs of the capacities.

Definition 6 *The* conjunctive combination *of a k-tuple of capacities* γ_i *consists in first computing the BΠA*

$$\rho_{\hat{\otimes}}(A) = \begin{cases} \max_{A_1\ldots A_k} \ \min(\gamma_{1\#}(A_1),\ldots,\gamma_{k\#}(A_k)) \\ \qquad\qquad \text{if } \cap_{i=1}^{k} A_i = A \neq \emptyset \\ 0 \qquad \text{if } A = \emptyset. \\ 1 \qquad \text{if } A = W. \end{cases}$$

and the resulting capacity is $\gamma(A) = \max_{E\subseteq A} \rho_{\hat{\otimes}}(A)$. *This combination is denoted by* $\gamma = \hat{\otimes}_{i=1}^{k} \gamma_i$.

This combination rule is commutative.

The three cases appearing in this definition are motivated by possible inconsistencies between the capacities to be combined. There are various forms of mutually (in)consistent capacities:

Definition 7 *Two capacities* γ_1 *and* γ_2 *are said to be*

- top mutually consistent

if $\exists E,F : E\cap F \neq \emptyset, \gamma_{1\#}(E) = \gamma_{2\#}(F) = 1$

- mutually consistent *if they are top consistent and*

\- *all focal sets of* γ_1 *intersect at least one focal set of* γ_2

\- *all focal sets of* γ_2 *intersect at least one focal set of* γ_1

- strongly mutually consistent *if*

\- *all focal sets of* γ_1 *intersect at least one focal set of* γ_2 *with weight 1*

\- *all focal sets of* γ_2 *intersect at least one focal set of* γ_1 *with weight 1*

- fully mutually consistent *if*

\- *all focal sets of* γ_1 *intersect all focal sets of* γ_2

\- *all focal sets of* γ_2 *intersect all focal sets of* γ_1

If the QMTs are not top consistent (in the sense that if $\gamma_{1\#}(A_i) = 1, \forall i = 1,\ldots k$ then $\cap_{i=1}^{k} A_i = \emptyset$), we still get a capacity as the result since we enforce $\gamma(W) = 1$.

4 Comparing Q-capacities

In this section we propose different methods to compare q-capacities from the standpoint of their informational content. These methods are formally similar to comparison methods existing in the quantitative context. Let m_1 and m_2 two BPAs; we denote by $m_1 \sqsubseteq_\iota m_2$ the fact that a BPA m_1 is more informative than a BPA m_2, for a generalized inclusion ι. The main quantitative definitions are as follows:

1. *cf-ordering* [14]: $m_1 \sqsubseteq_{cf} m_2$ iff $cf_1(\{w\}) \leq cf_2(\{w\}), \forall \omega \in W$ where $cf_i(\{w\}) = Pl_i(\{w\})$ is the contour function of $m_i, i = 1,2$. This is precisely the informational comparison according to relative specificity in possibility theory.

2. *bel-ordering* [10]: $m_1 \sqsubseteq_{bel} m_2$ iff $Bel_1(A) \geq Bel_2(A), \forall A \subseteq W$. It suggests that making information more precise leads to stronger beliefs.

3. *pl-ordering* [10]: $m_1 \sqsubseteq_{pl} m_2$ iff $Pl_1(A) \leq Pl_2(A), \forall A \subseteq W$. It is equivalent to the bel-ordering by duality.

4. *q-ordering* [10]: $m_1 \sqsubseteq_q m_2$ iff $Q_1(A) \leq Q_2(A), \forall A \subseteq W$. The idea is that the larger the focal sets, the less informative belief function and then the greater is the commonality function.

5. *s-ordering* [21]: $m_1 \sqsubseteq_s m_2$ iff there exists a stochastic matrix $S(A,B)$ where A is focal for m_1 and B is focal for m_2 such that $\sum_{A:A\subseteq B} S(A,B) = 1$ (so $S(A,B) = 0$ if $A \not\subseteq B$), and $m_1 = S \cdot m_2$ (short for $m_1(A) = \sum_{B:A\subseteq B} S(A,B) m_2(B)$). Then, m_1 is called a *specialization* of m_2. Formally, it corresponds to a random set inclusion of m_1 in m_2. When the focal elements are nested it coincides with the possibilistic specificity ordering.

6. *d-ordering* [17]: $m_1 \sqsubseteq_d m_2$ iff there exists a BPA m such that $m_1 = m \odot m_2$. Then, m_1 is said to be a *Dempster-like specialization* of m_2. The idea is that if a mass function results from combining information coming from two sources, the former is more informed than each source individually. These information comparison relations are more or less strong. It has been proved in [10, 17, 14] that

$$m_1 \sqsubseteq_d m_2 \Rightarrow m_1 \sqsubseteq_s m_2 \Rightarrow \left\{ \begin{array}{c} m_1 \sqsubseteq_q m_2 \\ m_1 \sqsubseteq_{bel} m_2 \\ \Updownarrow \\ m_1 \sqsubseteq_{pl} m_2 \end{array} \right\} \Rightarrow m_1 \sqsubseteq_{cf} m_2$$

We are going to study counterparts of such information orderings for q-capacities.

4.1 Natural dominance between capacities

Let us consider two q-capacities γ_1 and γ_2. A simple idea to compare γ_1 and γ_2 is to consider the natural order $\gamma_1 \geq \gamma_2$ as a counterpart of the bel-ordering $Bel_1 \geq Bel_2$, we name *natural dominance* between q-capacities. It makes sense if we see q-capacities as similar to support functions in the sense of Shafer [19].

Proposition 4 $\gamma_1 \geq \gamma_2$ *if and only if* $\forall F \in \mathscr{F}_{\gamma_2}$ $\exists E \in \mathscr{F}_{\gamma_1}$ *such that* $E \subseteq F$ *and* $\gamma_{1\#}(E) \geq \gamma_{2\#}(F)$.

Intuitively, $\gamma_1 \geq \gamma_2$ means that γ_1 has smaller focal sets than γ_2, and with greater weights, hence is more informative.

4.2 Contour function ordering

Definition 8 *The* contour function ordering, *denoted by* $\gamma_1 \geq_{cf} \gamma_2$, *stands for* $\pi_{\gamma_1}(w) \geq \pi_{\gamma_2}(w)\ \forall w \in W$ *where* $\pi_\gamma(w) = \max_{w\in B} \gamma_\#(B)$.

We note that $\gamma_1 \geq_{cf} \gamma_2$ intuitively means that the contour function of the former is less narrow (hence less specific, i.e., less informative, in the sense of possibility theory) than the one of the latter.

Note that, contrary to the quantitative case, the cf-ordering $\pi_{\gamma_1} \geq \pi_{\gamma_2}$ is the same as comparing the qualitative counterpart of the plausibility functions ($Pl_{\gamma_1} \geq Pl_{\gamma_2}$) since the plausibility function Pl_γ induced by γ is the possibility measure associated to the contour function. Moreover if $\gamma_i = \Pi_i, i = 1,2$ are possibility measures, the natural dominance and the cf-dominance coincide. However if they are necessity measures they are at odds with each other, since $N_1 \geq N_2$ implies $\pi_{N_1} \leq \pi_{N_2}$. So, we cannot relate the cf-ordering to the natural dominance of capacities, since $\gamma_1 > \gamma_2$ neither implies $\pi_{\gamma_1} \geq \pi_{\gamma_2}$ nor the converse inequality.

Any dogmatic q-capacity is cf-dominated by any non-dogmatic capacity, since the contour function of the latter is uniform. A weak counterpart to Proposition 4 for cf-dominance is as follows.

Proposition 5 $\gamma_1 \geq_{cf} \gamma_2$ *implies for each* $E \in \mathscr{F}_{\gamma_1}$, $\exists F \in \mathscr{F}_{\gamma_2}$ *such that* $E \cap F \neq \emptyset$ *and* $\gamma_{1\#}(E) \geq \gamma_{2\#}(F)$.

The converse is not true.

4.3 Ordering outer q-capacities

In addition to the plausibility ordering, which coincides with the cf-ordering, we can compare outer q-capacities induced by γ_1 and γ_2:

Definition 9 γ_1 outer-dominates γ_2 *(*$\gamma_1 \geq_{outer} \gamma_2$*) if* $\gamma_1^* \geq \gamma_2^*$, *where* $\gamma^*(A) = \min_{B\in\mathscr{M}:A\subseteq B} \gamma_\#(B)$ *and* $\mathscr{M} = \mathscr{F}_\gamma \cup \{\emptyset\}$ *with* $\gamma_\#(\emptyset) = 0$.

For instance, the vacuous (non-informative) capacity γ^0 is such that $\mathscr{F}_{\gamma^0} = \{W\}$ with weight 1. It is clear that $\gamma^{0*}(A) = 1, \forall A \neq \emptyset$ is the vacuous possibility measure. Hence, $\gamma^{0*} \geq \gamma^*$ for all γ. Even if Pl_γ, and γ^* can both be viewed as counterparts of numerical plausibility functions in the qualitative setting, the cf-dominance $\geq_{cf}$ is not related to the outer dominance as counterexamples can show. The counterpart to Prop. 4 for outer-dominance is

Proposition 6 $\gamma_1^* \geq \gamma_2^*$ *iff* $\forall E \in \mathscr{F}_{\gamma_1}$ *such that* $\gamma_{1\#}(E) < 1$, $\exists F \in \mathscr{F}_{\gamma_2}$ *s.t.* $E \subseteq F$ *and* $\gamma_{2\#}(F) \leq \gamma_{1\#}(E)$.

Compared to Proposition 4, it exchanges E and F. The orderings and $\gamma_1 \geq \gamma_2$ and $\gamma_1^* \geq \gamma_2^*$ are thus not related. Moreover, note that for Boolean capacities, the proposition is always valid since the second part is always trivially true (no focal sets with weights less than 1). And indeed it can be checked that if γ_1 is Boolean then $\gamma_1^* = \gamma^{0*}$. The merit of this ordering is thus limited.

4.4 Qualitative specialisation

We can try to define a qualitative counterpart of the specialization ordering as a generalization of inclusion between focal sets. In the numerical case, masses flow from focal sets of one belief function to larger focal sets of the other, while preserving the total mass. This is equivalent to letting the masses of the focal sets of the second belief function flow down to smaller focal sets of the first one [10]. However here we need two conditions, one from the first capacity to the other, and one from the latter to the former. A big difference with the quantitative case is that we cannot split the masses.

One shall thus view specialisation as a generalized form of set inclusion. In the case of Boolean capacities, we say that γ_1 is more specific than γ_2 ($\gamma_1 \geq_s \gamma_2$) if each focal set E_1 of γ_1 is contained in some focal set F_2 of γ_2 and each focal set F_2 of γ_2 contains some focal set E_1 of γ_1. In the general case we may add that $\gamma_{1\#}(E) \geq \gamma_{2\#}(F)$.

The first inclusion condition, with this inequality, already implies natural dominance $\gamma_1 \geq \gamma_2$, in view of Proposition 4.

However, this proposal is not satisfactory. For instance suppose γ_1 is a simple support function focused on $E \subset W$ with weight λ and γ_2 is the vacuous capacity with a single weight 1 on W. Then it is clear that γ_1 should be considered as included in γ_2 as W contains both focal sets of γ_1. However the inequality is violated since the weight λ of E can be assigned no focal set of γ_2, that contains it with smaller or equal weight. Worse, if we take another simple support function γ_1' focused on $E' \supset E$ with the same or smaller weight. It is clear that $\gamma_1 \geq_s \gamma_1'$,while they are not specialisations of the vacuous capacity.

One way out is the use of cuts of capacities, that is $\gamma_\alpha(A) = 1$ if $\gamma(A) \geq \alpha$, and 0 otherwise. Then we can say that $\gamma_1 \geq_s \gamma_2$ if and only if $\gamma_{1\alpha} \geq_s \gamma_{2\alpha}, \forall \alpha > 0$. In the previous counterexample, at levels $\alpha > \lambda$, $W \subseteq W$, at levels $\alpha \leq \lambda$, focal sets are E and W, respectively.

4.5 Commonality-based orderings

Now we discuss the qualitative counterpart of the q-ordering for commonalities $Q_\gamma(A) = \max_{A \subseteq E} \gamma_\#(E)$: $\gamma_1 \geq_q \gamma_2$ stands for $Q_{\gamma_1} \geq Q_{\gamma_2}$. We can prove:

Proposition 7 *$Q_{\gamma_1} \geq Q_{\gamma_2}$ iff $\forall F \in \hat{\mathscr{F}}_{\gamma_2}$ $\exists E \in \hat{\mathscr{F}}_{\gamma_1}$ such that $F \subseteq E$ and $\gamma_{1\#}(E) \geq \gamma_{2\#}(F)$ where $\hat{\mathscr{F}}$ is the set of maximal subsets in $\mathscr{F}$.*

It follows that: $\gamma_1 \geq_q \gamma_2$ implies $\gamma_1 \geq_{cf} \gamma_2$. Indeed, Q_γ and π_γ coincide on singletons. However, it is easy to find examples showing the converse is false. It can also be shown that the natural dominance of capacities neither implies nor is implied by the commonality ordering. The same holds for outer-dominance.

So far we get the following links between q-capacity comparison methods:
$\gamma_1 \geq_s \gamma_2 \Rightarrow \gamma_1 \geq \gamma_2$ and $\gamma_1 \geq_q \gamma_2 \Rightarrow \gamma_1 \geq_{cf} \gamma_2$.

4.6 Dempster-like qualitative specialisation

Another idea is to compare q-capacities using the Dempster-like combination rule.

Definition 10 *The* Dempster specialization ordering *$\gamma_1 \geq_d \gamma_2$ is defined by $\exists \gamma$ such that $\gamma_1 = \gamma \hat{\otimes} \gamma_2$.*

It is easy to check that:

Proposition 8 *A non-empty set $E_1 \neq W$ is a focal set of $\gamma \hat{\otimes} \gamma_2$ if and only if $\exists E \in \mathscr{F}_\gamma$ and $\exists E_2 \in \mathscr{F}_{\gamma_2}$ such that $E_1 = E \cap E_2$ and $(\gamma \hat{\otimes} \gamma_2)_\#(E_1) = \min(\gamma_\#(E), \gamma_{2\#}(E_2))$.*

As a consequence, note that a focal set of $\gamma \hat{\otimes} \gamma_2$ different from W is always included in a focal set of γ and in another of γ_2 (possibly W).

Intuitively, merging two pieces of information should increase informativeness, at least when these pieces of information are mutually consistent [8]. And indeed the combination rule produces smaller focal sets (i.e. more informative) since it performs the intersection of original ones. But,

it possesses idempotent elements, namely capacities γ such that $\gamma \hat{\otimes} \gamma = \gamma$. They are such that their focal sets are nested or disjoint. The combination yields no additional information in that case. Moreover, if we omit a consistency condition between γ_1 and γ_2, as per Def. 7, $\gamma_1 = \gamma \hat{\otimes} \gamma_2$ is comparable neither to γ nor to γ_2. Under mutual consistency we get:

Proposition 9 *If γ_1 and γ_2 are mutually consistent q-capacities, then $\gamma_1 \hat{\otimes} \gamma_2 \geq \min(\gamma_1, \gamma_2)$.*

This form of mutual consistency is not enough to ensure that $\gamma_1 = \gamma \hat{\otimes} \gamma_2$ dominates *any of* γ and γ_2:

Example 4 *Suppose γ with focal sets $\gamma_\#(E) = \alpha < 1$ $\gamma_\#(F) = \beta > \alpha$, $\gamma_\#(G) = 1$ where E, F and G are disjoint. Then γ_2 has the same focal sets with $\gamma_{2\#}(E) = \beta$ and $\gamma_{2\#}(F) = \alpha$, $\gamma_{2\#}(G) = 1$. So $\gamma \hat{\otimes} \gamma_2 = \min(\gamma_1, \gamma_2)$, but $\gamma \hat{\otimes} \gamma_2 < \gamma$ and $\gamma \hat{\otimes} \gamma_2 < \gamma_2$.*

A stronger form of mutual consistency is needed to ensure that the qualitative Dempster-like combination rule improves informativeness in the sense that $\gamma \hat{\otimes} \gamma_2$ dominates both γ and γ_2.

Proposition 10 *If γ and γ_2 are strongly mutually consistent q-capacities, then $\gamma \hat{\otimes} \gamma_2 \geq \max(\gamma, \gamma_2)$.*

So in case of strong mutual consistency between γ and γ_2, it is clear that $\gamma_1 \geq_d \gamma_2$, i.e., $\gamma_1 = \gamma \hat{\otimes} \gamma_2$, implies natural dominance $\gamma_1 \geq \gamma_2$.

In fact, only one of the strong consistency conditions is needed for the d-ordering to imply the natural dominance of capacities.

Proposition 11 *If there is a capacity γ such that $\gamma_1 = \gamma \hat{\otimes} \gamma_2$, where all focal sets of γ_2 intersect a focal set of weight 1 of γ, then $\gamma_1 \geq \gamma_2$.*

Indeed it is clear that $\gamma \hat{\otimes} \gamma_2$ possesses at least focal sets of the form $F \cap G$, where F is focal of weight 1 for γ and G is focal for γ_2. The weight of $F \cap G$ is not less than $\gamma_{2\#}(G)$. Generally, $\gamma \hat{\otimes} \gamma_2$ will possess more focal sets, of the form $F \cap G$ where F is some other focal set of γ and G a focal set of γ_2. Then, $\gamma \hat{\otimes} \gamma_2 > \gamma_2$, i.e., the combination improves informativeness.

For instance, a non-dogmatic q-capacity γ satisfies the condition of the previous proposition. So we have as a special case of the above result:

Corollary 12 *If γ is non-dogmatic then: $\gamma \hat{\otimes} \gamma_2 \geq \gamma_2$.*

However, there are cases when natural dominance holds but the d-ordering does not hold. For instance, if γ_1 is an SSF focused on E with weight a and γ_2 is an SSF with weight $b < a$ focused on E as well, there is no way of expressing γ_1 as the combination of γ_2 and some other γ, since $(\gamma_2 \hat{\otimes} \gamma)_\#(E)$ is of the form $\min(\gamma_{2\#}(E), \gamma_\#(F)) \leq b$ for $E \subseteq F$. There is also a link between the d-ordering and the cf-ordering.

Proposition 13 *If $\gamma_1 = \gamma \hat{\otimes} \gamma_2$ where γ is top-consistent with γ_2, then $\gamma_1 \leq_{cf} \gamma_2$. Moreover, if γ is non-dogmatic, γ_1 and γ_2 have the same contour functions.*

In the first case, it can be seen that $\pi_{\gamma_1} = \min(\pi_\gamma, \pi_{\gamma_2}) \leq \pi_{\gamma_2}$. If γ is non-dogmatic, π_γ has value 1 everywhere, and cf_1 and cf_2 are the same functions. We can then compare the Dempsterian specialisation and the ordering of qualitative commonalities:

Proposition 14 *Under the same assumption as Prop. 13, $\gamma_1 \geq_d \gamma_2 \Rightarrow \gamma_1 \leq_q \gamma_2$ where $Q_\gamma(A) = \max_{A \subseteq B} \gamma_\#(B)$.*

In summary, we have shown the following:

$$\gamma_1 \geq_d \gamma_2 \Rightarrow \gamma_1 \leq_q \gamma_2 \Rightarrow \gamma_1 \leq_{cf} \gamma_2$$

The entailments between $\geq_d$, $\geq_q$ and $\geq_{cf}$ are the exact counterparts of the situation in the quantitative case. This is not true for $\geq$ and $\geq_{cf}$.

5 Concluding remarks

This paper highlights a similarity between belief functions and qualitative capacities. The latter can be seen as support functions, where the QMT represent more or less imprecise pieces of evidence with various strengths coming from several sources. Here we started the investigation of information comparison methods for q-capacities. These orderings can be useful to discuss results of qualitative information fusion methods as outlined in [5]. They can also be useful under other interpretations of qualitative capacities described in [7].

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