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Aggregation Operators for Comparative Possibility Distributions and Their Role in Group Decision Making

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Abstract

In this paper, we study an application of qualitative possibility theory to decision making under uncertainty. The word "qualitative" means that uncertainty is modeled using comparative possibility distributions on a universal set Ω. Such a possibility distribution defines how likely each elementary event $\omega \in \Omega$ is compared to others: more likely, less likely, equally likely, absolutely unlikely.

Based on the specificity relation introduced in our previous works, we define two operations on comparative possibility distributions: supremum and infimum. In group decision making, each of them can be used to combine possibility distributions representing opinions of different experts on the same subject into a "consensus" possibility distribution which produces the decisions acceptable (in different senses for supremum and infimum) by all the experts involved.

Keywords: Qualitative possibility theory, Comparative possibility distribution, Specificity relation, Aggregation operators, Group decision making.

1 Introduction

Let Ω be a universal set. For simplicity, let Ω be finite. In this paper, any element $\omega \in \Omega$ is called an *elementary event*. One and only one elementary event occurs, but it is unknown which one. Let some person called the *decision maker* have to make a decision the correctness of which depends on the elementary event ω that occurs. This uncertainty (unknown ω) prevents a decision maker from making a decision.

In some cases, the decision maker can obtain an expert opinion on ω. Most often this opinion is not precise, its nature is qualitative. To express expert opinions mathematically, comparative possibility orderings on Ω can be used [4, 3]. Such an ordering is a complete non-strict preorder on Ω, for any two elementary events it defines which one of them is more likely (non-strictly) to occur.

However, such an approach is not applicable when an expert thinks that some elementary events are absolutely unlikely. That is why we use a different concept of qualitative possibility theory suggested by Pyt'ev [5, 6, 7, 9, 8, 10, 13]. In Pyt'ev possibility theory, a possibility distribution is a function $\pi : \Omega \to [0,1]$ such that $\max\{\pi(\omega) \mid \omega \in \Omega\} = 1$ (all distributions are normalized). Only zero value of π is meaningful: $\pi(\omega) = 0$ means that ω is absolutely unlikely. All other values can be used only to compare them. That is, two possibility distributions π_1 and π_2 are equivalent iff $\operatorname{supp}\pi_1 = \operatorname{supp}\pi_2$, and for any two elementary events $\omega_1,\ \omega_2 \in \Omega, \pi_1(\omega_1) \leqslant \pi_1(\omega_2) \Leftrightarrow \pi_2(\omega_1) \leqslant \pi_2(\omega_2)$. In this paper, we denote this equivalence by "$\sim$": $\pi_1 \sim \pi_2$.

Difference between comparative possibility orderings [4, 3] and Pyt'ev possibility distributions can be demonstrated using the following simple example. Let $\Omega = \{1,\ 2\}$, and $\pi_1(1) = 0,\ \pi_1(2) = 1,\ \pi_2(1) = 0.5,\ \pi_2(2) = 1$. The comparative possibility orderings induced by the distributions π_1 and π_2 are the same: $\pi_1(1) < \pi_1(2)$, and $\pi_2(1) < \pi_2(2)$. However, in Pyt'ev possibility theory, π_1 and π_2 are not equivalent. According to π_1, elementary event 1 is absolutely unlikely. According to π_2, it is not so: 1 is less likely than 2, but 1 can occur.

In [13], the specificity relation "$\preceq$" on a set ϖ of all possibility distributions was defined:

Definition 1. $\pi_1 \preceq \pi_2$ *iff*

SpRel-1. $\operatorname{supp}\pi_1 \subset \operatorname{supp}\pi_2$, *SpRel-2.* $\forall \omega_1, \omega_2 \in \operatorname{supp} \pi_1$, $\pi_1(\omega_1) \le \pi_1(\omega_2)$ *implies* $\pi_2(\omega_1) \le \pi_2(\omega_2)$,

SpRel-3. $\forall \omega_1 \in \operatorname{supp} \pi_1, \forall \omega_2 \in \operatorname{supp} \pi_2 \setminus \operatorname{supp} \pi_1$, $\pi_2(\omega_1) \ge \pi_2(\omega_2)$.

This specificity relation is consistent with equivalence "$\sim$": $\pi_1 \sim \pi_2 \Leftrightarrow \pi_1 \preceq \pi_2$ and $\pi_2 \preceq \pi_1$.

For classification and decision-making problems, it was proved that the more specific a possibility distribution given by an expert is, the narrower (less ambiguous) a set of the optimal class numbers or decision consequence utilities is [13, 12]. Those results make it possible to compare expert opinions represented by possibility distributions.

In this paper, we introduce the corresponding aggregation operators that make it possible to combine different expert opinions represented by non-equivalent possibility distributions and produce "consensus" (in some sense) distributions.

2 Preliminary notes on the role of the specificity relation in classification and decision-making problems [13, 12]

In this paper, any function ζ defined on the universal set Ω is called an *ill-known element*. A value $\zeta(\omega)$ taken by ζ is unknown because it is unknown which elementary event $\omega \in \Omega$ occurs.

2.1 Classification

The classification problem formulated in [13] is to estimate an unknown value $\xi(\omega)$ of a latent (unobservable) ill-known element ξ using a value $\eta(\omega)$ (known to the decision maker) taken by an observable ill-known element η. The element η models a feature vector of an object to be classified, ξ is a class number (label). The expert opinion is represented by a joint possibility distribution of ξ and η: $\pi(x,y) \in [0,1]$ is possibility of the event $\{\xi(\omega) = x$ and $\eta(\omega) = y$ simultaneously$\} \subset \Omega$. Mathematically, the classification problem is formulated in [13] as the following optimization problem:

$$\max \left\{ \pi(x,y) \,\middle|\, \delta(y) \neq x \right\} \sim \min_{\delta}. \tag{1}$$

This is a problem of finding the classifiers δ (the mappings from the range of η to the range of ξ) minimizing possibility of incorrect classification. In Pyt'ev possibility theory this possibility is calculated as a maximum of possibilities of the elementary events resulting in incorrect classification (maxitivity of possibility).

Let $\Delta_*(\pi)$ be a set of all classifiers being optimal (according to (1)) under the possibilistic model represented by a joint possibility distribution π. Let D^{π}_{y} be a set of class labels produced by all optimal classifiers if a value $y \in Y$ of the ill-known element η is observed:

$$D^{\pi}_{y} = \{\delta_*(y) \mid \delta_* \in \Delta_*(\pi)\}.$$

The main thesis of [13] consists in the following: the more specific possibility distribution π is, the narrower the set D^{π}_{y} is. This is mathematically represented by the following theorem:

Theorem 1. *If* $\pi_1 \preceq \pi_2$, *then the event* $\left\{D^{\pi_1}_{\eta} \not\subset D^{\pi_2}_{\eta}\right\}$ *is* π_1*-impossible.*

2.2 Savage-style decision making

A conventional mathematical model of decision making is the Savage-style one [11]. The role of the specificity relation "$\preceq$" in Savage-style decision making was studied in [12]. According to Savage, the decision maker has to choose optimal act(s) from a set F of all feasible acts. Each act $f \in F$ is a mapping from the universal set Ω to a set X of potential consequences of decisions that are ranked according to their utility. That is, a real-valued utility function u is defined on X.

Therefore, if the decision maker knows for sure that the elementary event $\omega \in \Omega$ occurs, then it knows the consequence $f(\omega)$ of any act $f \in F$ and its utility $u(f(\omega))$ as well. In this case, a set of optimal acts F_ω is as follows:

$$F_\omega = \left\{ f \in F : \; u(f(\omega)) = u_{\max}(\omega) \right\}, \quad \text{where} \quad u_{\max}(\omega) \triangleq \max_{g \in F} u(g(\omega)).$$

Under qualitative possibilistic uncertainty which model is represented by a possibility distribution π defined on Ω, the most used formulation of the decision-making problem is based on the likely dominance rule (LDR) which establishes the preference between acts as follows [2]:

$$f >_\pi g \Leftrightarrow \max\left\{ \pi(\omega) \,\middle|\, u(f(\omega)) > u(g(\omega)) \right\} > \max\left\{ \pi(\omega) \,\middle|\, u(g(\omega)) > u(f(\omega)) \right\},$$

$$f \geqslant_\pi g \Leftrightarrow \text{not } g >_\pi f,$$

where $\max \varnothing$ is treated as zero. Let $F_{\mathrm{LDR}}(\pi)$ be a set of the most preferable acts according to the LDR:

$$F_{\mathrm{LDR}}(\pi) = \{ f \in F : \; \forall g \in F, \; f \geqslant_\pi g \}.$$

Let $U^\pi(\omega)$ be a set of all utilities that can be obtained by acting according to the LDR for the distribution π:

$$U^\pi(\omega) = \left\{ u = u(f(\omega)) \,\middle|\, f \in F_{\mathrm{LDR}}(\pi) \right\}. \tag{2}$$

Similarly to theorem 1, the following theorem is true[1] [12]:

Theorem 2. *If $\pi_1 \preceq \pi_2$, then the event $\{U^{\pi_1}(\omega) \not\subset U^{\pi_2}(\omega)\}$ is π_1-impossible.*

According to the theorem, minimum guaranteed utility of acting in accordance with LDR for distribution π_1 is not less than that for π_2 for any π_1-possible elementary event:

$$\min U^{\pi_1}(\omega) \geq \min U^{\pi_2}(\omega) \quad \forall \omega \in \operatorname{supp} \pi_1. \qquad (3)$$

I. e., in π_1-possible cases, acting according to LDR for π_1 is no worse than acting according to LDR for π_2 if minimum guaranteed utility is being considered.

Similarly,

$$\max U^{\pi_1}(\omega) \leq \max U^{\pi_2}(\omega) \quad \forall \omega \in \operatorname{supp} \pi_1. \qquad (4)$$

I. e., in π_1-possible cases, acting according to LDR for π_2 is no worse than acting according to LDR for π_1 if maximum achievable utility is being considered.

3 Defining aggregation operators for possibility distributions

In [13] it was proved that "$\preceq$" is a reflexive and transitive relation on the set ϖ of all possibility distributions $\Omega \to [0,1]$. I. e., "$\preceq$" is a preorder. It is consistent with equivalence "$\sim$", see the introduction.

This paper studies algebraic properties of the preordered set $(\varpi, \preceq)$ in more detail. Specifically, we prove bellow that in the case of finite Ω two binary operations can be introduced for distributions $\pi_1, \pi_2 \in \varpi$:

1. a supremum $\pi_1 \vee \pi_2$ defined by conditions
   - $\pi_1 \preceq \pi_1 \vee \pi_2$, $\pi_2 \preceq \pi_1 \vee \pi_2$,
   - $\forall \pi'$ such that $\pi_1 \preceq \pi'$ and $\pi_2 \preceq \pi'$, $\pi_1 \vee \pi_2 \preceq \pi'$,
2. an infimum $\pi_1 \wedge \pi_2$ defined by conditions
   - $\pi_1 \wedge \pi_2 \preceq \pi_1$, $\pi_1 \wedge \pi_2 \preceq \pi_2$,
   - $\forall \pi'$ such that $\pi' \preceq \pi_1$ and $\pi' \preceq \pi_2$, $\pi' \preceq \pi_1 \wedge \pi_2$.

3.1 Supremum of possibility distributions

In this section, the iterative algorithm for calculating $\pi_1 \vee \pi_2$ is introduced for any two possibility distributions $\pi_1, \pi_2 \in \varpi$, see algorithm 1.

[1] In fact, the statement of the theorem proved in [12] is slightly different. Theorem 2 is its consequence and proved in section 6 below.

At each step of the algorithm, the distributions π_1 and π_2 are transformed to strictly less specific ones. The first step of the algorithm is based on lemmas 1, 2 and makes the supports $\operatorname{supp} \pi_1$ and $\operatorname{supp} \pi_2$ equal. Then for the distributions with the same supports, the iterative step of the algorithm based on lemma 3 eliminates the following contradictions: for some ω', $\omega'' \in \operatorname{supp} \pi_1 = \operatorname{supp} \pi_2$, $\pi_1(\omega') \leq \pi_1(\omega'')$ and $\pi_2(\omega') \geq \pi_2(\omega'')$ with at least one strict inequality.

Theorem 4 states that the output distributions π_1 and π_2 are equivalent, and each of them is a supremum $\pi_1 \vee \pi_2$ according to the definition given in the beginning of section 3.

Algorithm 1: The iterative algorithm for calculating supremum $\pi_1 \vee \pi_2$.

```
if supp π1 ≠ supp π2 then
    if supp π1 ⊂ supp π2 then
        apply lemma 1: π1 ← π1';
    if supp π2 ⊂ supp π1 then
        swap π1 and π2: π1 ↔ π2;
        apply lemma 1: π1 ← π1';
    if supp π1 \ supp π2 ≠ ∅ and
      supp π2 \ supp π1 ≠ ∅ then
        apply lemma 2: π1 ← π1', π2 ← π2';
while for some ω', ω'' ∈ supp π1 = supp π2,
  π1(ω') ≤ π1(ω'') and π2(ω') ≥ π2(ω'') with at
  least one strict inequality do
    apply lemma 3: π1 ← π1', π2 ← π2';
```

Lemmas 1–3 and theorem 4 are formulated and proved in section 6.

3.2 Infimum of possibility distributions

Let $\widetilde{\varpi} = \varpi \cup \{\mathbf{0}\}$, where the function $\mathbf{0} : \Omega \to [0,1]$ is defined by $\mathbf{0}(\omega) = 0$ for all $\omega \in \Omega$. The function $\mathbf{0}$ is not a possibility distribution because $\sup\{\mathbf{0}(\omega) \mid \omega \in \Omega\} = 0 \neq 1$. Nevertheless, we formally append it to the set of all possibility distributions to form a mathematically convenient set $\widetilde{\varpi}$ (it is discovered below why $\widetilde{\varpi}$ is called "mathematically convenient").

In this section, the iterative algorithm for calculating $\pi_1 \wedge \pi_2 \in \widetilde{\varpi}$ is introduced for any two possibility distributions $\pi_1, \pi_2 \in \widetilde{\varpi}$, see algorithm 2.

At each step of the algorithm, the distributions π_1 and π_2 are transformed to strictly more specific ones. The iterative step of the algorithm based on lemma 4 eliminates the following contradictions: for some ω', $\omega'' \in \Omega$, $\pi_1(\omega') < \pi_1(\omega'')$ and $\pi_2(\omega') > \pi_2(\omega'')$. Lemma 5 makes the supports $\operatorname{supp} \pi_1$ and $\operatorname{supp} \pi_2$ of the resulting noncontradictory distributions equal. The final step of the algorithm based on lemma 6 chooses strict inequalities for all elementary events $\omega', \omega'' \in \operatorname{supp}\pi_1 = \operatorname{supp}\pi_2$ such that $\pi_1(\omega') < \pi_1(\omega'')$ and $\pi_2(\omega') = \pi_2(\omega'')$ or vise versa.

Theorem 5 states that the output distribution is an infimum $\pi_1 \wedge \pi_2$ according to the definition given in the beginning of section 3.

Algorithm 2: The iterative algorithm for calculating infimum $\pi_1 \wedge \pi_2$.

```
while for some ω′, ω″ ∈ Ω, π1(ω′) < π1(ω″) and
  π2(ω′) > π2(ω″) do
  apply lemma 4: π1 ← π1′, π2 ← π2′;
if supp π1 ≠ supp π2 then
  if supp π1 ⊂ supp π2 then
    apply lemma 5: π2 ← π2′;
  if supp π2 ⊂ supp π1 then
    swap π1 and π2: π1 ↔ π2;
    apply lemma 5: π2 ← π2′;
apply lemma 6
```

Lemmas 4–6 and theorem 5 are formulated and proved in section 6.

4 The aggregation operators "$\vee$" and "$\wedge$" as a basis for consensus group decision making

4.1 A set of optimal outcomes

Given a possibility distribution $\pi \in \varpi$ and an observation η, one can not decide in favor of any class label from D^{π}_{η} based on the possibilistic model π, see section 2.1. Similarly, decision maker can not decide in favor of any act from $F_{\mathrm{LDR}}(\pi)$, see section 2.2. Therefore, any utility from $U^{\pi}(\omega)$ (2) can be achieved if the elementary event ω occurs.

Let $D^{\pi}(\omega) = D^{\pi}_{\eta(\omega)}$ in the case of classification or $D^{\pi}(\omega) = U^{\pi}(\omega)$ in the case of Savage-style decision making. We refer to $D^{\pi}(\omega)$ as the *set of optimal outcomes*. Thus, the meaning of both theorems 1 and 2 can be summarized as follows: if $\pi_1 \preceq \pi_2$ then the set of optimal outcomes according to π_1 is a subset of the set of optimal outcomes according to π_2 for all π_1-possible elementary events ω.

4.2 "Positive" consensus — a consensus on optimal outcomes

Let π_1 and π_2 be two possibility distributions that represent the opinions of two different experts on the same subject. So π_1 and π_2 are, in general, defined using distinct scales of possibility values.

According to theorems 1 and 3:

$$D^{\pi_1 \wedge \pi_2}(\omega) \subset D^{\pi_i}(\omega) \quad \forall \omega \in \operatorname{supp}\pi_1 \wedge \pi_2,\ i = 1, 2.$$

That is, the possibility distribution $\pi_1 \wedge \pi_2$ produces a "positive" consensus between the experts: it identifies the outcomes that are optimal according to both expert opinions.

Note that this is true only for elementary events from $\operatorname{supp}\pi_1 \wedge \pi_2$. However, if ω' is possible according to one expert and absolutely unlikely according to $\pi_1 \wedge \pi_2$, then ω' is absolutely unlikely according to the other expert (provided that both expert opinions are correct, it means that ω' is absolutely unlikely at all), or there is ω'' such that the pair ω', ω'' satisfy lemma 4, i. e., there is a contradiction between the expert opinions that can not be resolved mathematically (in this case, consensus can not be achieved with mathematical methods alone).

4.3 "Negative" consensus — a consensus on non-optimal outcomes

Similarly, according to theorems 1 and 2, any outcome which is non-optimal under $\pi_1 \vee \pi_2$ is non-optimal under π_1 and under π_2 as well. Thus, the possibility distribution $\pi_1 \vee \pi_2$ produces a "negative" consensus between the experts: it identifies the outcomes that are non-optimal according to both expert opinions.

4.4 A set of arguable outcomes

Let $\overline{D}^{\pi}(\omega)$ be a set of non-optimal outcomes if an elementary event $\omega \in \Omega$ occurs.

We have discovered above that $D^{\pi_1 \wedge \pi_2}(\omega)$ and $\overline{D}^{\pi_1 \vee \pi_2}(\omega)$ are the sets of outcomes considered to be optimal and non-optimal correspondingly by both experts. Note also that according to theorems 1 and 2, $D^{\pi_1 \wedge \pi_2}(\omega) \subset D^{\pi_1 \vee \pi_2}(\omega)$ for any elementary event $\omega \in \Omega$ such that $\pi_1 \wedge \pi_2(\omega) \neq 0$ because $\pi_1 \wedge \pi_2 \preceq \pi_1 \vee \pi_2$. Hence, $D^{\pi_1 \vee \pi_2}(\omega) \setminus D^{\pi_1 \wedge \pi_2}(\omega)$ accumulates all outcomes $d \notin D^{\pi_1 \wedge \pi_2}(\omega)$ and $d \notin \overline{D}^{\pi_1 \vee \pi_2}(\omega)$ and can be non-empty.

Neither "positive" nor "negative" consensus can be achieved on the outcomes $d \in D^{\pi_1 \vee \pi_2}(\omega) \setminus D^{\pi_1 \wedge \pi_2}(\omega)$: d is optimal according to one expert opinion and is non-optimal according to the other one. I. e., $D^{\pi_1 \vee \pi_2} \setminus D^{\pi_1 \wedge \pi_2}$ is a set of arguable outcomes — a consensus on them can not be achieved using the mathematical methods described above and requires other mathematical approaches or non-mathematical methods of resolving contradictions such as discussion, voting, and so on.

Table 1: The possibility distributions π_1 and π_2 representing the opinions of two experts, an infimum $\pi_1 \wedge \pi_2$ and a supremum $\pi_1 \vee \pi_2$ of them, and the corresponding sets $D^{\pi_1}, \ldots, D^{\pi_1 \vee \pi_2}$ of the optimal act utilities.

	ω_1	ω_2	ω_3
π_1	1	1	1/2
D^{π_1}	$\{0, \alpha, 2\alpha\}$	$\{0, \alpha, 2\alpha\}$	$\{0\}$
π_2	1	1/2	1
D^{π_2}	$\{0, \alpha, 2\alpha\}$	$\{0\}$	$\{0, \alpha, 2\alpha\}$
$\pi_1 \wedge \pi_2$	1	0	0
$D^{\pi_1 \wedge \pi_2}$	$\{2\alpha\}$	$\{0\}$	$\{0\}$
$\pi_1 \vee \pi_2$	1	1	1
$D^{\pi_1 \vee \pi_2}$	$\{0, \alpha, 2\alpha\}$	$\{0, \alpha, 2\alpha\}$	$\{0, \alpha, 2\alpha\}$

4.5 An illustrative example

Let us consider betting as an example of a decision making problem. Suppose, there are three alternatives $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and the decision maker is up to either bet all on one of those or to split his betting stake equally between two of them. Let f_{ij} be the stake, one half of which is bet on ω_i, and the other half is bet on ω_j (if $i = j$, it is an all on ω_i bet). Suppose, potential returns are proportional to the stake on the winning alternative, i. e. $u(f_{ij}(\omega_k)) = \alpha(\delta_{ik} + \delta_{jk})$, where α is some coefficient, δ_{ij} is Kronecker delta.

Suppose, one of the decision maker friends thinks that ω_1 and ω_2 are more likely to win than ω_3, whereas another friend thinks that ω_1 and ω_3 are more likely to win than ω_2. These opinions can be expressed via possibility distributions π_1 and π_2, see table 1. According to the first friend, the most preferable acts are $F_{\mathrm{LDR}}(\pi_1) = \{f_{11}, f_{12}, f_{22}\}$. According to the second friend, the most preferable acts are $F_{\mathrm{LDR}}(\pi_2) = \{f_{11}, f_{13}, f_{33}\}$. The corresponding sets of optimal outcomes can be seen in table 1.

According to $\pi_1 \wedge \pi_2$, ω_2 and ω_3 are impossible because the friend's opinions about those alternatives contradict each other. $F_{\mathrm{LDR}}(\pi_1 \wedge \pi_2) = \{f_{11}\}$, and the act f_{11} is no worse than any other act $f \in F_{\mathrm{LDR}}(\pi_1) \cup F_{\mathrm{LDR}}(\pi_2)$ for the alternative ω_1: $u(f_{11}(\omega_1)) = 2\alpha \geq u(f(\omega_1))$, see (3). $F_{\mathrm{LDR}}(\pi_1 \vee \pi_2) = \{f_{ij} \,|\, i, j = 1, 2, 3\}$, for any alternative $\omega_k \in \Omega$ there is an act $f' \in F_{\mathrm{LDR}}(\pi_1 \vee \pi_2)$ which is no worse than any act $f'' \in F_{\mathrm{LDR}}(\pi_1) \cup F_{\mathrm{LDR}}(\pi_2)$: $u(f'(\omega_k)) \geq u(f''(\omega_k))$, see (4).

5 The algebra of subjective judgments with algebraic operations "$\vee$" and "$\wedge$"

We have introduced two operations on the extended set $\widetilde{\varpi}$ of possibility distributions above: supremum "$\vee$" and infimum "$\wedge$".

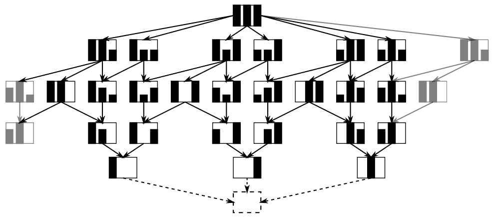

Figure 1: The black histograms represent all non-equivalent possibility distributions in the case of $\Omega = \{\omega_1, \omega_2, \omega_3\}$ — bar i of each histogram is $\pi(\omega_i)$ in height. Any other distribution is equivalent to one of the distributions shown in the figure. The arrows demonstrate the specificity relation "$\preceq$". They point from less specific distributions to more specific ones. Note that the specificity relation is transitive. The histograms shown in gray are the duplicates of the black histograms, they are added to the figure in order to avoid arrows crossing it from left to right and vise versa. The dashed rectangle corresponds to the quasi-distribution $\mathbf{0}(\cdot) = 0$.

Remark 1. *Note that formally "$\vee$" and "$\wedge$" are point-to-set mappings: if $\pi = \pi_1 \circ \pi_2$, then any possibility distribution $\pi' \sim \pi$ satisfies $\pi' = \pi_1 \circ \pi_2$ as well, where "$\circ$" is a placeholder for "$\vee$" or "$\wedge$". This results from the fact that "$\preceq$" is not reflexive. Nevertheless, "$\preceq$" is consistent with equivalence "$\sim$": $\pi_1 \preceq \pi_2$ and $\pi_2 \preceq \pi_1$ imply $\pi_1 \sim \pi_2$ and vise versa. So, it would be mathematically correct to consider "$\vee$" and "$\wedge$" the operations on a quotient space of a set of all possibilities by "$\sim$". However, we avoid consideration in terms of classes of equivalence and the quotient space below because it would result in introducing redundant notations and cluttering the reasoning.*

Let us consider the quartet $(\widetilde{\varpi}, \preceq, \vee, \wedge)$ which consists of the following objects: the extended set $\widetilde{\varpi}$ of the possibility distributions, the specificity relation "$\preceq$" on $\widetilde{\varpi}$, the operations "$\vee$" and "$\wedge$" on $\widetilde{\varpi}$. $\mathbf{0} \preceq \pi \preceq \mathbf{1}$ for any $\pi \in \widetilde{\varpi}$, where $\mathbf{1} \in \widetilde{\varpi}$ is defined by $\mathbf{1}(\omega) = 1$ for all $\omega \in \Omega$. Hence, supremum $\pi_1 \vee \pi_2$ and infimum $\pi_1 \wedge \pi_2$ are defined for any $\pi_1, \pi_2 \in \widetilde{\varpi}$. In other words, $(\widetilde{\varpi}, \preceq, \vee, \wedge)$ is a lattice. Its diagram for $\Omega = \{\omega_1, \omega_2, \omega_3\}$ is shown in figure 1.

The lattice $(\widetilde{\varpi}, \preceq, \vee, \wedge)$ can be used as a basis for an algebra of subjective judgments expressed via possibility distributions. That algebra can be used in group decision making to achieve "positive" and "negative" consensus between the experts as described above.

The operations "$\vee$" and "$\wedge$" extend the usual set operations: union "$\cup$" and intersection "$\cap$". Indeed, if $\pi_i = \mathbf{1}_{A_i}$ is an indicator function of a subset $A_i \subset \Omega$, $i = 1, 2$, then $\pi_1 \vee \pi_2 = \mathbf{1}_{A_1 \cup A_2}$ and $\pi_1 \wedge \pi_2 = \mathbf{1}_{A_1 \cap A_2}$ are the indicator functions of $A_1 \cup A_2$ and $A_1 \cap A_2$ correspondingly.

6 Lemmas, theorems, and their proofs

In [12], the following theorem was proved:

Theorem 3. *If* $\pi_1 \preceq \pi_2$*, then* $f \in F_{\text{LDR}}(\pi_1)$ *implies either* $f \in F_{\text{LDR}}(\pi_2)$ *or* $u(f(\omega)) = u(g(\omega))$ *for any* $g >_{\pi_2} f$ *and all* π_1*-possible elementary events* $\omega \in \Omega$*.*

Let us prove that theorem 2 formulated above at page 3 is its consequence.

Proof. The following is to be proved: $\pi_1 \preceq \pi_2$ implies $U^{\pi_1}(\omega) \subset U^{\pi_2}(\omega)$ for all π_1-possible elementary events $\omega \in \operatorname{supp}\pi_1$.

Suppose $\omega^* \in \operatorname{supp}\pi_1$ and $u^* \in U^{\pi_1}(\omega^*)$. Let us prove that $u^* \in U^{\pi_2}(\omega^*)$.

There is $f^* \in F_{\text{LDR}}(\pi_1)$ such that $u^* = u(f^*(\omega^*))$ by definition of $U^{\pi_1}(\omega^*)$. According to theorem 3, there are two options. First, $f^* \in F_{\text{LDR}}(\pi_2)$. Therefore, $u^* = u(f^*(\omega^*)) \in U^{\pi_2}(\omega^*)$ by definition of $U^{\pi_2}(\omega^*)$. Second, $f^* \notin F_{\text{LDR}}(\pi_2)$ and $u(f^*(\omega)) = u(g(\omega))$ for any $g >_{\pi_2} f^*$ and all π_1-possible elementary events $\omega \in \Omega$. Let g be any act from $F_{\text{LDR}}(\pi_2)$, therefore, $g >_{\pi_2} f^*$ as $f^* \notin F_{\text{LDR}}(\pi_2)$. I. e., $u^* = u(f^*(\omega^*)) = u(g(\omega^*))$ which implies $u^* \in U^{\pi_2}(\omega^*)$. $\square$

Lemma 1. *Let* $\operatorname{supp}\pi_1 \subset \operatorname{supp}\pi_2$*, and* π_1' *be defined as follows:*

$$\pi_1'(\omega) = \begin{cases} \pi_1(\omega), & \omega \in \operatorname{supp}\pi_1, \\ \alpha\pi_2, & \omega \in \Omega \setminus \operatorname{supp}\pi_1, \end{cases}$$

where a constant $\alpha > 0$ *satisfies the condition*

$$\min\{\pi_1(\omega) | \omega \in \operatorname{supp}\pi_1\} > \\ > \max\{\alpha\pi_2(\omega) | \omega \in \Omega \setminus \operatorname{supp}\pi_1\}. \quad (5)$$

Then $\pi_1 \preceq \pi$ *and* $\pi_2 \preceq \pi \Leftrightarrow \pi_1' \preceq \pi$ *and* $\pi_2 \preceq \pi$*.*

Proof. The "$\Rightarrow$" statement follows from the fact that $\pi_1 \preceq \pi_1'$ and transitivity of "$\preceq$". Conversely, assume that $\pi_1 \preceq \pi$ and $\pi_2 \preceq \pi$. Let us prove that $\pi_1' \preceq \pi$.

As $\pi_2 \preceq \pi$, $\operatorname{supp}\pi_1' = \operatorname{supp}\pi_2 \subset \operatorname{supp}\pi$. That is, SpRel-1 in definition 1 is proved.

Let us consider two points $\omega_1, \omega_2 \in \operatorname{supp}\pi_1' = \operatorname{supp}\pi_2$. There are three options. First, $\omega_1, \omega_2 \in \operatorname{supp}\pi_1$. In this case $\pi_1'(\omega_1) = \pi_1(\omega_1)$, and $\pi_1'(\omega_2) = \pi_1(\omega_2)$. Therefore, if $\pi_1'(\omega_1) \le \pi_1'(\omega_2)$, then $\pi_1(\omega_1) \le \pi_1(\omega_2)$. Recalling $\pi_1 \preceq \pi$, it implies $\pi(\omega_1) \le \pi(\omega_2)$. Second, $\omega_1, \omega_2 \in \operatorname{supp}\pi_2 \setminus \operatorname{supp}\pi_1$. In this case $\pi_1'(\omega_1) = \alpha\pi_2(\omega_1)$, $\pi_1'(\omega_2) = \alpha\pi_2(\omega_2)$, $\alpha > 0$. Therefore, if $\pi_1'(\omega_1) \le \pi_1'(\omega_2)$, then $\pi_2(\omega_1) \le \pi_2(\omega_2)$. Recalling $\pi_2 \preceq \pi_1$, it implies $\pi(\omega_1) \le \pi(\omega_2)$. Third, $\omega_1 \in \operatorname{supp}\pi_2 \setminus \operatorname{supp}\pi_1$, $\omega_2 \in \operatorname{supp}\pi_1$. In this case $\pi_1'(\omega_1) = \alpha\pi_2(\omega_1) \le \pi_1(\omega_2) \le \pi_1'(\omega_2)$, see (5). On the other hand, $\pi_1 \preceq \pi$ implies $\pi(\omega_1) \le \pi(\omega_2)$. That is, SpRel-2 in definition 1 is proved.

Finally, let us consider two points $\omega_1 \in \operatorname{supp}\pi_1' = \operatorname{supp}\pi_2$ and $\omega_2 \in \operatorname{supp}\pi \setminus \operatorname{supp}\pi_1' = \operatorname{supp}\pi \setminus \operatorname{supp}\pi_2$. As $\pi_2 \preceq \pi$, $\pi(\omega_1) \ge \pi(\omega_2)$. That is, SpRel-3 is proved. $\square$

Lemma 2. *Let* $\operatorname{supp}\pi_1 \setminus \operatorname{supp}\pi_2 \neq \varnothing$*,* $\operatorname{supp}\pi_2 \setminus \operatorname{supp}\pi_1 \neq \varnothing$*, and distributions* π_1'*,* π_2' *be defined as follows:*

$$\pi_i'(\omega) = \begin{cases} \max\{\pi_i(\omega), \alpha_i\}, & \omega \in \operatorname{supp}\pi_1 \cup \operatorname{supp}\pi_2, \\ 0, & \textit{otherwise}, \end{cases}$$

where

$$\alpha_1 = \max\{\pi_1(\omega) | \omega \in \operatorname{supp}\pi_1 \setminus \operatorname{supp}\pi_2\},$$
$$\alpha_2 = \max\{\pi_2(\omega) | \omega \in \operatorname{supp}\pi_2 \setminus \operatorname{supp}\pi_1\}.$$

Then $\pi_1 \preceq \pi$ *and* $\pi_2 \preceq \pi \Leftrightarrow \pi_1' \preceq \pi$ *and* $\pi_2' \preceq \pi$*.*

Proof. The "$\Rightarrow$" statement follows from the fact that $\pi_i \preceq \pi_i'$, $i = 1, 2$ and transitivity of "$\preceq$". Conversely, assume that $\pi_1 \preceq \pi$ and $\pi_2 \preceq \pi$. Let us prove that $\pi_i' \preceq \pi$, $i = 1, 2$.

As $\pi_i \preceq \pi$, $\operatorname{supp}\pi_1 \subset \operatorname{supp}\pi$ and $\operatorname{supp}\pi_2 \subset \pi$. Therefore, $\operatorname{supp}\pi_i' = \operatorname{supp}\pi_1 \cup \operatorname{supp}\pi_2 \subset \operatorname{supp}\pi$. That is, SpRel-1 is proved.

Let us consider two points $\omega_1, \omega_2 \in \operatorname{supp}\pi_i' = \operatorname{supp}\pi_1 \cup \operatorname{supp}\pi_2$. There are three options. First, $\pi_i(\omega_1), \pi_i(\omega_2) \ge \alpha_i$. In this case, $\pi_i'(\omega_1) = \pi_i(\omega_1)$ and $\pi_i'(\omega_2) = \pi_i(\omega_2)$. Therefore, if $\pi_i'(\omega_1) \le \pi_i'(\omega_2)$, then $\pi_i(\omega_1) \le \pi_i(\omega_2)$. Recalling $\pi_i \preceq \pi$, it implies $\pi(\omega_1) \le \pi(\omega_2)$. Second, $\pi_i(\omega_1) \le \alpha_i \le \pi_i(\omega_2)$. In this case, $\pi_i'(\omega_1) = \alpha_i \le \pi_i(\omega_2) = \pi_i'(\omega_2)$, and $\omega_2 \in \operatorname{supp}\pi_i$. Recalling $\pi_i \preceq \pi$, it implies $\pi(\omega_1) \le \pi(\omega_2)$. Third, $\pi_i(\omega_1), \pi_i(\omega_2) \le \alpha_i$. In this case, $\pi_i'(\omega_1) = \pi_i'(\omega_2) = \alpha_i$. Let ω' be a point of $\operatorname{supp}\pi_1 \setminus \operatorname{supp}\pi_2$ such that $\pi_1(\omega') = \alpha_1$, i. e., $\omega' = \arg\max\{\pi_1(\omega) | \omega \in \operatorname{supp}\pi_1 \setminus \operatorname{supp}\pi_2\}$. Similarly, $\omega'' = \arg\max\{\pi_2(\omega) | \omega \in \operatorname{supp}\pi_2 \setminus \operatorname{supp}\pi_1\}$. As $\omega'' \in \operatorname{supp}\pi_2$ and $\omega' \notin \operatorname{supp}\pi_2$, $\pi_2 \preceq \pi$ implies $\pi(\omega') \le \pi(\omega'')$. As $\omega' \in \operatorname{supp}\pi_1$ and $\omega'' \notin \operatorname{supp}\pi_1$, $\pi_1 \preceq \pi$ implies $\pi(\omega'') \le \pi(\omega')$. I. e., $\pi(\omega') = \pi(\omega'')$. As $0 = \pi_2(\omega') \le \pi_2(\omega_j) \le \alpha_2 = \pi_2(\omega'')$, and $\pi(\omega') = \pi(\omega'')$, $\pi_2 \preceq \pi$ implies $\pi(\omega') = \pi(\omega_j) = \pi(\omega'')$, $j = 1, 2$. I. e., $\pi_i'(\omega_1) = \pi_i'(\omega_2)$, and $\pi(\omega_1) = \pi(\omega_2)$. That is, SpRel-2 is proved.

Finally, let us consider two points $\omega_1 \in \operatorname{supp}\pi_i' = \operatorname{supp}\pi_1 \cup \operatorname{supp}\pi_2$ and $\omega_2 \in \operatorname{supp}\pi \setminus \operatorname{supp}\pi_i' =$

$\operatorname{supp}\pi\backslash(\operatorname{supp}\pi_1 \cup \operatorname{supp}\pi_2)$. As $\omega_1 \in \operatorname{supp}\pi_1$ or $\omega_1 \in \operatorname{supp}\pi_2$ while $\omega_2 \notin \operatorname{supp}\pi_1$ and $\omega_2 \notin \operatorname{supp}\pi_1$, at least one of relations $\pi_1 \preceq \pi$ and $\pi_2 \preceq \pi$ implies $\pi(\omega_1) \geq \pi(\omega_2)$. That is, SpRel-3 is proved. □

Lemma 3. *Let* $\operatorname{supp}\pi_1 = \operatorname{supp}\pi_2$*, and there be two points* ω', ω'' *such that* $\pi_1(\omega') \leq \pi_1(\omega'')$ *and* $\pi_2(\omega') \geq \pi_2(\omega'')$ *with at least one strict inequality. Let distributions* π_1' *and* π_2' *be defined as follows:*

$$\pi_i'(\omega) = \begin{cases} \hat{\alpha}_i, & \check{\alpha}_i \leq \pi_i(\omega) \leq \hat{\alpha}_i, \\ \pi_i(\omega), & \textit{otherwise}, \end{cases}$$

where

$$\check{\alpha}_i = \min\{\pi_i(\omega'), \pi_i(\omega'')\},\ \hat{\alpha}_i = \max\{\pi_i(\omega'), \pi_i(\omega'')\}.$$

Then $\pi_1 \preceq \pi$ *and* $\pi_2 \preceq \pi \Leftrightarrow \pi_1' \preceq \pi$ *and* $\pi_2' \preceq \pi$.

Proof. The "$\Rightarrow$" statement follows from the fact that $\pi_i \preceq \pi_i'$, $i = 1,2$ and transitivity of "$\preceq$". Conversely, assume that $\pi_1 \preceq \pi$ and $\pi_2 \preceq \pi$. Let us prove that $\pi_1' \preceq \pi$ and $\pi_2' \preceq \pi$

As $\pi_i \preceq \pi$, $\operatorname{supp}\pi_i' = \operatorname{supp}\pi_i \subset \operatorname{supp}\pi$, $i = 1,2$. That is, SpRel-1 is proved.

Let us consider two points $\omega_1, \omega_2 \in \operatorname{supp}\pi_i' = \operatorname{supp}\pi_1 = \operatorname{supp}\pi_2$. Similarly to lemma 2, in order to prove SpRel-2, three options are to be considered: 1) $\pi_i(\omega_1), \pi_i(\omega_2) \notin [\check{\alpha}_i, \hat{\alpha}_i]$; 2) $\pi_i(\omega_1) \notin [\check{\alpha}_i, \hat{\alpha}_i]$, $\pi_i(\omega_2) \in [\check{\alpha}_i, \hat{\alpha}_i]$.; 3) $\pi_i(\omega_1), \pi_i(\omega_2) \in [\check{\alpha}_i, \hat{\alpha}_i]$. The proof is analogous. SpRel-3 is obvious because $\operatorname{supp}\pi_1' = \operatorname{supp}\pi_2' = \operatorname{supp}\pi_1 = \operatorname{supp}\pi_2$. □

Theorem 4. *In the case of finite* Ω*, algorithm 1 produces equivalent possibility distributions, any of them is supremum* $\pi_1 \vee \pi_2$ *of possibility distributions* $\pi_1, \pi_2 \in \varpi$.

Proof. It follows from lemmas 1-3. □

Lemma 4. *Let there be two points* $\omega', \omega'' \in \Omega$ *such that* $\pi_1(\omega') < \pi_1(\omega'')$ *and* $\pi_2(\omega') > \pi_2(\omega'')$*, and distributions* π_1' *and* π_2' *be defined as follows:*

$$\pi_i'(\omega) = \begin{cases} \pi_i(\omega), & \pi_i(\omega) \geq \alpha_i, \textit{ and } \omega \neq \omega', \\ & \textit{and } \omega \neq \omega'', \\ 0, & \textit{otherwise}, \end{cases}$$

where $\alpha_i = \max\{\pi_i(\omega'), \pi_i(\omega'')\}$, $i = 1,2$.

Then $\pi \preceq \pi_1$ *and* $\pi \preceq \pi_2 \Leftrightarrow \pi \preceq \pi_1'$ *and* $\pi \preceq \pi_2'$.

Proof. The "$\Rightarrow$" statement follows from the fact that $\pi_i \preceq \pi$, $i = 1,2$ and transitivity of "$\preceq$". Conversely, assume that $\pi \preceq \pi_1$ and $\pi \preceq \pi_2$. Let us prove that $\pi \preceq \pi_1'$ and $\pi \preceq \pi_2'$.

First, let us prove that $\pi(\omega') = \pi(\omega'') = 0$. Assume that $\pi(\omega'), \pi(\omega'') > 0$. Recalling $\pi \preceq \pi_1$, $\pi \preceq \pi_2$, $\pi_1(\omega') < \pi_1(\omega'')$, and $\pi_2(\omega') > \pi_2(\omega'')$, it implies $\pi(\omega') < \pi(\omega'')$ and $\pi(\omega') > \pi(\omega'')$ simultaneously. However, these inequalities contradict each other. Assume that $\pi(\omega') = 0$, $\pi(\omega'') > 0$. In this case, $\pi \preceq \pi_2$ implies $\pi_2(\omega') \leq \pi_2(\omega'')$ which contradicts to $\pi_2(\omega') > \pi_2(\omega'')$. Similarly, $\pi(\omega') > 0$ and $\pi(\omega'') = 0$ contradicts to $\pi \preceq \pi_1$ and $\pi_1(\omega') < \pi_1(\omega'')$. That is, $\pi(\omega') = \pi(\omega'') = 0$.

Let us consider any point ω''' such that $\pi_i(\omega''') < \alpha_i = \max\{\pi_i(\omega'), \pi_i(\omega'')\}$. Assume that $\pi(\omega''') > 0 = \pi(\omega') = \pi(\omega'')$. As $\pi \preceq \pi_i$, it implies $\pi_i(\omega') \leq \pi_i(\omega''')$ and $\pi_i(\omega'') \leq \pi_i(\omega''')$ which contradicts to $\max\{\pi_i(\omega'), \pi_i(\omega'')\} > \pi_i(\omega''')$. That is, $\pi_i(\omega''') = 0$. Recalling $\pi \preceq \pi_i$, it implies $\pi(\omega''') = 0$. As the result, we have proved that $\pi_i(\omega) < \alpha_i$ for some $i = 1,2$ implies $\pi(\omega) = 0$. Also $\pi(\omega') = \pi(\omega'') = 0$. That is, $\pi_i'(\omega) = 0$ implies $\pi(\omega) = 0$, i.e., $\operatorname{supp}\pi \subset \operatorname{supp}\pi_i'$. SpRel-1 is proved.

Let us consider two points $\omega_1, \omega_2 \in \operatorname{supp}\pi \subset \operatorname{supp}\pi_i'$. If $\pi(\omega_1) \leq \pi(\omega_2)$, then $\pi_i'(\omega_1) = \pi_i(\omega_1) \leq \pi_i(\omega_2) = \pi_i'(\omega_2)$ cause $\pi \preceq \pi_i$. That is, SpRel-2 is proved.

Let us consider two points $\omega_1 \in \operatorname{supp}\pi \subset \operatorname{supp}\pi_i' \subset \operatorname{supp}\pi_i$ and $\omega_2 \notin \operatorname{supp}\pi$. As $\pi \preceq \pi_i$ and by definition of π_i', $\pi_i'(\omega_1) = \pi_i(\omega_1) \geq \pi_i(\omega_2) \geq \pi_i'(\omega_2)$. That is, SpRel-3 is proved. □

Lemma 5. *Let* $\operatorname{supp}\pi_1 \subset \operatorname{supp}\pi_2$*, and* π_2' *be defined as follows:*

$$\pi_2'(\omega) = \begin{cases} \pi_2(\omega), & \omega \in \operatorname{supp}\pi_1, \\ 0, & \textit{otherwise}. \end{cases}$$

Then $\pi \preceq \pi_1$ *and* $\pi \preceq \pi_2 \Leftrightarrow \pi \preceq \pi_1$ *and* $\pi \preceq \pi_2'$.

Proof. It is obvious because of transitivity of relation "$\preceq$", definitions of "$\preceq$" and π_2'. □

Beside the representation of comparative possibility distribution in terms of a real-valued function π, a mathematically equivalent representation in terms of a well-ordered partition $\operatorname{WOP}(\pi) = \left(\Omega_1^\pi,\ \ldots,\ \Omega_n^\pi;\ Z^\pi\right)$ of Ω [1] is used below:

$$\pi(\omega) = 0 \Leftrightarrow \omega \in Z^\pi, \quad \pi(\omega_1) > \pi(\omega_2) > 0 \Leftrightarrow \\ \Leftrightarrow \omega_1 \in \Omega_i^\pi,\ \omega_2 \in \Omega_j^\pi,\ i < j. \quad (6)$$

Lemma 6. *Let there be no points* ω', ω'' *from lemma 4 for possibility distributions* π_1 *and* π_2*, and* $\operatorname{WOP}(\pi_i) = \left(\Omega_1^{\pi_i},\ \ldots,\ \Omega_{n_i}^{\pi_i};\ Z\right)$, $i = 1,2$. *Then there is a possibility distribution* π *which* $\operatorname{WOP}(\pi) = \left(\Omega_1^\pi,\ \ldots,\ \Omega_n^\pi;\ Z\right)$ *is defined by conditions* (7) *and* (8)*:*

$$\forall k = 1, \ldots, n,\ \exists i, j:\ \Omega_k^\pi = \Omega_i^{\pi_1} \cap \Omega_j^{\pi_2}, \quad (7)$$

for any $\Omega_i^\pi = \Omega_{i_1}^{\pi_1} \cap \Omega_{i_2}^{\pi_2}$ *and* $\Omega_j^\pi = \Omega_{j_1}^{\pi_1} \cap \Omega_{j_2}^{\pi_2}$,

$$i \leq j \Leftrightarrow i_1 \leq j_1 \text{ and } i_2 \leq j_2, \qquad (8)$$

Such a distribution π *is an infimum* $\pi = \pi_1 \wedge \pi_2$.

Proof. Existence of WOP(π) satisfying (8) is guaranteed since there are no points ω', ω'' from lemma 4.

Let us prove that $\pi \preceq \pi_1$. SpRel-1 and SpRel-3 are obviously true: $\operatorname{supp}\pi_1 = \operatorname{supp}\pi_2 = \operatorname{supp}\pi = \Omega \setminus Z$. Let us consider two points $\omega_1 \in \Omega_i^\pi = \Omega_{i_1}^{\pi_1} \cap \Omega_{i_2}^{\pi_2}$ and $\omega_2 \in \Omega_j^\pi = \Omega_{j_1}^{\pi_1} \cap \Omega_{j_2}^{\pi_2}$ such that $\pi(\omega_1) \geq \pi(\omega_2)$. According to (6), $i \leq j$. Therefore, $i_1 \leq i_2$, see (8), and $\pi_1(\omega_1) \geq \pi_1(\omega_2)$. That is, SpRel-2 is true. Similarly, $\pi \preceq \pi_2$. Let us prove that $\pi' \preceq \pi$ for any π' such that $\pi' \preceq \pi_k$, $k = 1, 2$. SpRel-1 is true since $\operatorname{supp}\pi' \subset \operatorname{supp}\pi_k = \operatorname{supp}\pi$.

Let us consider two points $\omega_1 \in \Omega_i^\pi = \Omega_{i_1}^{\pi_1} \cap \Omega_{i_2}^{\pi_2}$ and $\omega_2 \in \Omega_j^\pi = \Omega_{j_1}^{\pi_1} \cap \Omega_{j_2}^{\pi_2}$. If $\pi'(\omega_1) \geq \pi'(\omega_2) > 0$ or $\pi'(\omega_1) > \pi'(\omega_2) = 0$, then $\pi_k(\omega_1) \geq \pi_k(\omega_2)$, see SpRel-2 or SpRel-3 correspondingly. Therefore, $i_k \leq j_k \Rightarrow i \leq j \Rightarrow \pi(\omega_1) \geq \pi(\omega_2)$, see (6), (8). That is, SpRel-2 and SpRel-3 are true. □

Theorem 5. *In the case of finite* Ω*, algorithm 2 produces infimum* $\pi_1 \wedge \pi_2 \in \widetilde{\varpi}$ *of possibility distributions* $\pi_1, \pi_2 \in \widetilde{\varpi}$.

Proof. It follows from lemmas 4-6. □

7 Conclusion

Two aggregation operators on a set of comparative possibility distributions (understood according to Pyt'ev possibility theory) are defined. Their role in classification and decision-making problems is studied. It is demonstrated that the operators can be used as a basis for an algebra of subjective judgments expressed via possibility distributions. That algebra can be used in group decision making to achieve "positive" and "negative" consensus between the experts. "Positive" consensus between the experts identifies the outcomes that are optimal according to all expert opinions. "Negative" consensus identifies the outcomes that are non-optimal according to all expert opinions.

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