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Epistemic Bootstrap for Fuzzy Data

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Abstract

Fuzzy data applied for modeling imprecise observations cause many problems in statistical reasoning and data analysis. To handle better such observations a new bootstrap technique designed for epistemic fuzzy data is proposed. Our new method is conceptually simple and is not hard computationally. Some simulation results reported in the paper show that the proposed new type of the bootstrap may increase the effectiveness of statistical inferential procedures used so far. Although these results are rather preliminary, they indicate that the epistemic bootstrap might be useful in different fields which is a good prognostic for further research.

Keywords: Bootstrap, Estimation, Fuzzy data, Fuzzy numbers, Fuzzy EM, Hypotheses testing.

1 Introduction

"The bootstrap is the most important new idea in statistics introduced in the last 20 years, and probably in the last 50 years" – these words, spoken by Jerome H. Friedman in 1998, have lost nothing of their relevance. A few years later George Casella [1] added that "the bootstrap has shown us how to use the power of the computer and iterated calculations to go where theoretical calculations cannot, which introduces a different way of thinking about all of statistics". It seems that the development of science and constantly opening new horizons confirm the truth of these words. Indeed, the bootstrap, invented by Bradley Efron [7], proved to be extremely useful and effective in various areas of research.

There are many reasons why the bootstrap is so successful. Firstly, it does not require any assumptions about the distribution of your data, such as normality, etc. Next, it can provide quite accurate inferences even if the sample size is small. Moreover, the bootstrap is relatively simple to apply for various sampling plans and data types. All in all, it can handle many difficult statistical problems, where standard methods fail or are unreliable. Therefore, it should come as no surprise that the bootstrap turned out to be useful in statistical reasoning with fuzzy data as well.

When discussing fuzzy data analysis one has to note that such data might be perceived from two different – ontic or epistemic – perspectives (see [17]). There are situations when the experimental data appear as essentially set-valued. Such data, called *ontic* represent an objective entity, so we can write, e.g., $X = A$, where a fuzzy set A is a value of a set-valued variable X. On the other hand, a fuzzy set A may represent an ill-known actual value of a point-valued quantity x, so we can write $x \in A$. Then A just represents the epistemic state of an agent and this is why it is called *epistemic*.

For the purposes of statistical inference, ontic fuzzy data are often modeled with fuzzy random variables defined by Puri and Ralescu [25]. Unfortunately, there are no suitable models for the distribution of fuzzy random variables. We also have no Central Limit Theorems for fuzzy random variables that can be straightforwardly applied for the actual data. That is why the bootstrap turned out to be very useful in statistical reasoning with ontic fuzzy data, especially. in hypotheses testing [2, 8, 9, 23, 28], classification [27], fuzzy rating in questionnaires [21, 22], quality control [26, 33], and so on.

At the first glance statistical reasoning with epistemic fuzzy data seems easier because the corresponding fuzzy random sample might be perceived as a perception of an unknown usual real-valued random sample (see [20]). Consequently, the probabilistic description of epistemic data is more straightforward to handle. Unfortunately, the nature of epistemic fuzzy data usually results in statistical procedures which come out too conservative and hence do not meet with much interest of practitioners. However, it seems that the bootstrap may be helpful also in this field.

The goal of this contribution is to suggest a novel bootstrap technique for epistemic fuzzy data. Although this work is preliminary, we show that applying our special type of bootstrap may increase the effectiveness of statistical inferential procedures used so far.

The paper is organized as follows: in Sec. 2 we recall the classical bootstrap technique as well as its variants designed for ontic fuzzy data. Then we suggest how to perform bootstrap in the framework of epistemic fuzzy data. Finally, in Sec. 3 we consider several examples showing the usefulness of the proposed approach both in estimation as in hypotheses testing. Sec. 4 contains the conclusions.

2 From classical to epistemic bootstrap

The key idea of the **bootstrap** proposed by Efron [7] is to create new samples by drawing randomly elements with replacement from the initial sample $x_1,\ldots,x_n \in \mathbb{R}$. This way one can produce any number (say B) of bootstrap samples, as it is shown in Figure 1, where $x^*_{ij} \in \{x_1,\ldots,x_n\}$ denotes the i-th element of the j-th sample.

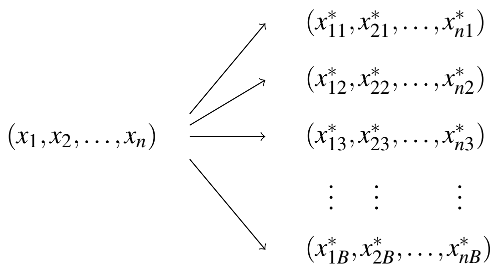

Figure 1: The classical bootstrap scheme.

Therefore, most of the bootstrap samples generated according to this classical scheme contain repeated values. Even worse, if the initial sample size is small all bootstrap samples consist of only a few distinct values. This is a highly unwanted effect, especially if the unknown population distribution is continuous. It is also undesirable if the bootstrap samples are generated to be used in procedures that do not accept ties. To overcome this disadvantage various improvements have been proposed, such as the balanced bootstrap [4, 10] or the smoothed bootstrap [5, 16, 32].

$$(\widetilde{x}_1,\widetilde{x}_2,\ldots,\widetilde{x}_n)$$
$$\downarrow$$
$$\big(\mathcal{O}(\widetilde{x}_1),\mathcal{O}(\widetilde{x}_2),\ldots,\mathcal{O}(\widetilde{x}_n)\big)$$
$$\downarrow$$
$$(\widetilde{x}^*_1,\widetilde{x}^*_2,\ldots,\widetilde{x}^*_n)$$

Figure 2: A flexible bootstrap scheme for fuzzy data.

The problem of excessive repetitions in bootstrap samples is also unwanted in statistics with fuzzy data. Further on let $(\widetilde{x}_1,\ldots,\widetilde{x}_n)$ denote a realization of a sample which consists of fuzzy observations, i.e. $\widetilde{x}_i \in \mathbb{F}(\mathbb{R})$, $i = 1,\ldots,n$, where $\mathbb{F}(\mathbb{R})$ stands for the family of all fuzzy numbers (let us recall that a fuzzy number is a fuzzy set in $\mathbb{R}$ which is normal, fuzzy-convex, which has upper-semicontinuous membership function and a bounded support).

To increase the variety of elements that appear in bootstrap samples a few flexible resampling methods were proposed. Without going into details they come down to drawing elements not directly from the initial sample $(\widetilde{x}_1,\ldots,\widetilde{x}_n)$ but from neighborhoods $(\mathcal{O}(\widetilde{x}_1),\ldots,\mathcal{O}(\widetilde{x}_n))$ of the original observations, as it is shown in Fig. 2, where $\mathcal{O}(\widetilde{x}_i)$ denotes a neighborhood of $\widetilde{x}_i$. Such neighborhoods might be generated from the primary sample by adding some incremental spreads on α-cuts [29, 30, 31] or by considering an extension of the initial sample which consists of all fuzzy numbers with the same canonical representation as those belonging to the primary sample [13, 14, 15]. However, all resampling methods for fuzzy data discussed above are oriented on the ontic data, where each fuzzy observation is considered as a realization of a set-valued random variable and hence it is treated as a whole.

On the other hand, when looking at data from the epistemic view each member of the initial fuzzy sample is just an imprecise perception of an unknown point-valued quantity. Hence the bootstrap technique designed for this type of data should take into account their specificity.

To be more exact, let us follow the process of generating this type of data. Suppose, our sample $X_1,\ldots,X_n$ consists of n independent and identically distributed random variables from the distribution F_θ, where $\theta \in \Theta$ usually denotes un unknown parameter. However, instead of observing a real-valued realization $(x_1,\ldots,x_n)$ of this sample we only have access to its imprecise perception modeled by a sample of fuzzy observations (or fuzzy sample, in brief), i.e. $(\widetilde{x}_1,\ldots,\widetilde{x}_n)$. This way a fuzzy set $\widetilde{x}_i$ contains somehow the actual real-valued realization of the i-th observation. A membership function of $\widetilde{x}_i$ attributes to each point of the real line the possibility that this very point is the true realization x_i of X_i.

The last remark suggests that a fuzzy perception $\widetilde{x}_i$ of the unknown value x_i might be used as a kind of fuzzy neighbor of x_i. Furthermore, by an appropriate selection of elements from the given fuzzy set which takes into account the degree of possibility that a generated candidate is the true outcome of the experiment, we obtain an innovative bootstrap technique oriented on epistemic fuzzy samples. Its basic idea can be clarified in a few words. Suppose we are provided with a fuzzy sample $\widetilde{x}_1,\dots,\widetilde{x}_n \in \mathbb{F}(\mathbb{R})$. To generate a single bootstrap sample we need just one loop with two steps only. Hence, for each $i = 1,\dots,n$ we generate randomly:

1) a real number α_i from the uniform distribution on the unit interval $[0,1]$, i.e. $\alpha_i \sim \mathrm{U}[0,1]$,

2) a real number x_i^* from the uniform distribution on the α-cut $(\widetilde{x}_i)_{\alpha_i}$, i.e. $x_i^* \sim \mathrm{U}\big[(\widetilde{x}_i)^L_{\alpha_i}, (\widetilde{x}_i)^U_{\alpha_i}\big]$, where $(\widetilde{x}_i)^L_{\alpha_i}$ and $(\widetilde{x}_i)^U_{\alpha_i}$ stand for the lower and upper bound of the α-cut $(\widetilde{x}_i)_{\alpha_i}$, respectively.

Proceeding in this way n times we complete the entire bootstrap sample $x_1^*,\dots,x_n^*$.

At this point someone might ask about the rationale for using the uniform distribution to generate both α-cuts and observations within an α-cut. The quickest form of response would be to invoke the tradition of using the uniform distribution to generate bootstrap samples. But that wouldn't be the answer, just a shifting responsibility to the bootstrap. However, for us, the main motivation for the use of a uniform distribution is, above all, the principle of maximum entropy. As Jaynes wrote in his famous work [18]: "...the maximization of entropy is not an application of a law of physics, but merely a method of reasoning which ensures that no unconscious arbitrary assumptions have been introduced". If we add to this - on the one hand - the relationship of the principle of maximum entropy with statistical mechanics, and on the other - the relationship with Bayesian statistics, in which we take the uniform distribution as a typical noninformative distribution, it seems that this very choice of this distribution is perfectly justified.

To settle for a single bootstrap sample seems not to be enough since according to Algorithm 1 we consider only one α-cut for each fuzzy observation. But the bootstrap world is generous: we can generate as many bootstrap samples as we want or have time for. Therefore, we may consider $B \geqslant 1$ α-cuts for each fuzzy set which provides a multiplicity of bootstrap samples, i.e. $(x^*_{1j},\dots,x^*_{nj})$, where $j = 1,\dots,B$. This general idea of our epistemic bootstrap, i.e. drawing bootstrap samples from epistemic fuzzy data is shown in Algorithm 1 and illustrated in Fig. 3.

Algorithm 1

Require: Initial fuzzy sample $\widetilde{x}_1,\dots,\widetilde{x}_n \in \mathbb{F}(\mathbb{R})$.
Ensure: B bootstrap samples.

```
for j = 1 to B do
    for i = 1 to n do
        Generate randomly a real number α_ij from
        the uniform distribution on the unit interval
        [0,1].
        Generate randomly a real number x*_ij from the
        uniform distribution on the α-cut (x̃_i)_α_ij.
    end for
end for
Bootstrap samples x*_1j,...,x*_nj, where j = 1,...,B.
```

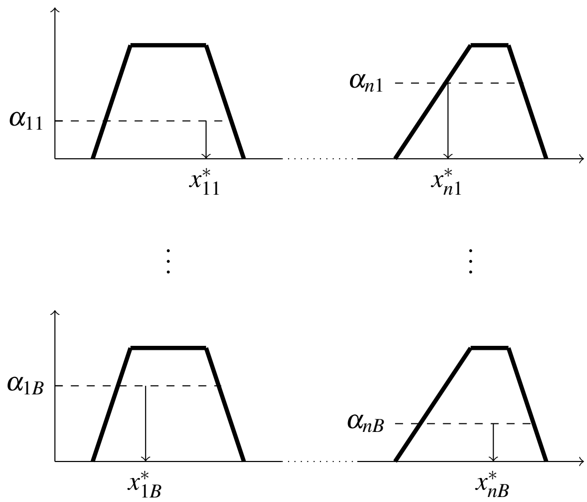

Figure 3: The idea of epistemic bootstrap: drawing bootstrap samples from epistemic fuzzy data.

In the real world where we are usually provided with a single random sample of outcomes $(x_1,\dots,x_n)$ which in turn gives us a single value of the desired statistics $T(x_1,\dots,x_n)$. Now, each bootstrap sample provides a bootstrap replication of the statistic of interest $T_j^*(x^*_{1j},\dots,x^*_{nj})$. Then, following Efron's idea the bootstrap statistic T^* is obtained by some aggregation of the bootstrap replications $T_1^*,\dots,T_B^*$. A typical solution is obtained by averaging the bootstrap replications, i.e. $T^* = \frac{1}{B}\sum_{j=1}^{B} T_j^*$.

Further on so determined bootstrap statistic can be used as a bootstrap estimator or a device for evaluating the standard error of an estimator, it may be applied for designing bootstrap confidence intervals or in hypotheses testing, etc. Some applications of the proposed bootstrap method in statistical reasoning with epistemic fuzzy data are discussed in Sec. 3.

3 Epistemic bootstrap in statistical reasoning

3.1 Empirical distribution function based on the epistemic bootstrap

Before applying our new technique in statistical reasoning we will check whether it really can approximate a population distribution.

Let $X_1, \ldots, X_n$ i.i.d. random variables from a distribution F. As it is known the empirical distribution function (e.d.f.) $\widehat{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{(-\infty, x_i)}(t)$, based on observations $x_1, \ldots, x_n$, estimates nicely the true c.d.f. F. Therefore our first goal is to check if its bootstrap counterpart $\widehat{F}_n^*$ determined for the bootstrap samples $(x_{1j}^*, \ldots, x_{nj}^*)$, $j = 1, \ldots, B$, obtained from initial fuzzy sample $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$, also estimates F.

In Fig. 4 and Fig. 5 one can find the theoretical c.d.f. (in black) of the normal and the exponential distribution, respectively, as well as the e.d.f. (in red) obtained from fuzzy samples of size $n = 200$ with the epistemic bootstrap with B α-cuts. In both cases our fuzzy samples consisted of trapezoidal fuzzy numbers simulated as follows: each fuzzy observation $\widetilde{x}_i$ is completely characterized by four real numbers corresponding to the endpoints of its support $[a_i, d_i]$ and core $[b_i, c_i]$, respectively. Obviously, $a_i \leqslant b_i \leqslant c_i \leqslant d_i$, for $i = 1, \ldots, n$. These endpoints are generated according to the following formulae

$$a_i = X - S^l - L, \quad b_i = X - S^l,$$
$$c_i = X + S^r, \quad d_i = X + S^l + R,$$

where X is a random variable corresponding to the "true" population distribution, while C^l, C^r, S^l and S^r stand for random variables used for fuzzification of the point-valued experimental result. In particular C^l and C^r are responsible for generating a core, while S^l and S^r are used to construct a support. In our simulation study X was generated from various distributions, like the normal, exponential and gamma, while C^l, C^r, S^l and S^r were generated mostly from the uniform distribution or from the gamma distribution. All random variables X, C^l, C^r, S^l and S^r were always independent. All simulations were conducted in R. Some simulation scenarios illustrated in this contribution is summarized in Table 1.

Fig. 4 and Fig. 5 show that if sample size is large enough then even drawing bootstrap samples from a single α-cut our method provides a satisfying approximation of the population distribution, which is also confirmed y the Q-Q plot given in Fig. 6.

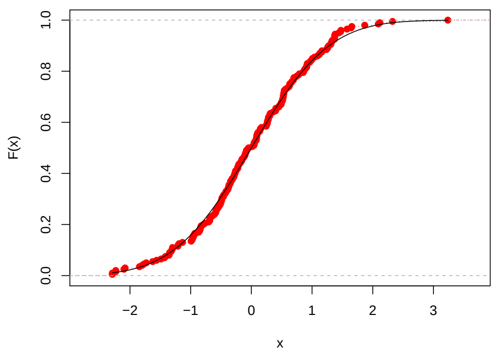

Figure 4: Empirical vs. theoretical c.d.f. for the normal distribution and the epistemic bootstrap from $\mathbb{F}_{(\text{N,U,U,1})}$ performed on a single α-cut, i.e. $B = 1$.

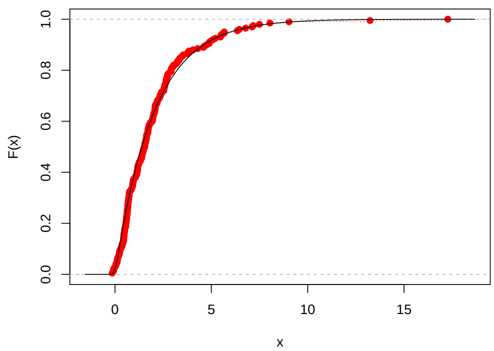

Figure 5: Empirical vs. theoretical c.d.f. for the exponential distribution and the epistemic bootstrap from $\mathbb{F}_{(\text{E,U,U,1})}$ performed on $B = 10$ α-cuts.

Table 1: Simulation scenarios for fuzzy samples.

Type	X	C^l, C^r	S^l, S^r
$\mathbb{F}_{(\text{N,U,U,1})}$	N(0,1)	U(0,0.5)	U(0,0.8)
$\mathbb{F}_{(\text{N,U,U,2})}$	N(0,2)	U(0,1)	U(0,2)
$\mathbb{F}_{(\text{E,U,U,1})}$	Exp(0.5)	U(0,0.4)	U(0,0.6)
$\mathbb{F}_{(\text{E,U,U,2})}$	Exp(1)	U(0,0.6)	U(0,1.2)
$\mathbb{F}_{(\text{N,E,U})}$	N(0,1)	Exp(1)	U(0,0.8)
$\mathbb{F}_{(\Gamma\text{,U,U})}$	$\Gamma(1,1)$	U(0,0.5)	U(0,0.8)

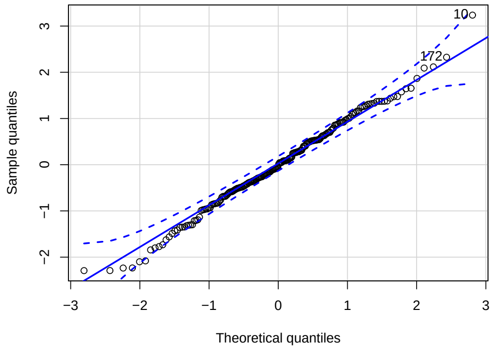

Figure 6: Q-Q plot for the normal distribution and the epistemic bootstrap from $\mathbb{F}_{(\mathrm{N,U,U,1})}$ performed on a single α-cut, i.e. $B = 1$.

3.2 Some remarks on the point estimation

In this section, we publish some results of the simulation study dedicated to the point estimation. In particular, we considered estimation of the mean μ and standard deviation σ of the normal distribution, i.e. for $X \sim \mathrm{N}(\mu,\sigma)$, and the parameter λ of the exponential distribution $X \sim \mathrm{Exp}(\lambda)$. We took into account different simulation scenarios (see Table 1), several sample sizes n and the number of α-cuts B.

Our simulations confirm that estimators obtained using the epistemic bootstrap are consistent, i.e. the quality of estimation improves with increasing sample size (and tending to the true value of the estimated parameter as $n \to \infty$). On the other hand, we noticed that increasing the number of α-cuts B used in the bootstrap has a rather slight effect on the estimation quality. Unfortunately, the standard error of the considered estimators was relatively large. It can be explained by the presence of two sources of variability: one, inherent to the estimator under study, and the second, connected with the effect of imprecision (fuzzification). Hence the resulting standard error of an estimator actually measures the overall variance instead of partitioning it and focusing on the inherent one.

The last remark is a hint for future research: we have to make use of some variance reduction method (see, e.g., [19]) or to learn how to separate both kinds of variability and then exclude the disturbing one when constructing the estimator. For instance, we implemented the RSS method (i.e. Ranked-Set Sampling) or the so-called antithetic variables. In the first case, we observe no significant improvement, while in the second case we recorded a decrease in the standard error of up to 30%. However, the problem seems to be open and still far from a definite solution.

Table 2: Empirical MSE comparison.

	MSE(EB)	MSE(FME)	MSE(FME*)
		$\mathbb{F}_{(\mathrm{N,U,U,1})}$	
μ	0.0983	0.8697	0.7758
σ	0.0439	15.6866	15.0551
		$\mathbb{F}_{(\mathrm{N,U,U,2})}$	
μ	0.3575	3.4967	2.3991
σ	0.2251	44.6723	45.2160
		$\mathbb{F}_{(\mathrm{E,U,U,1})}$	
λ	0.0492	8.8043	10.6394
		$\mathbb{F}_{(\mathrm{E,U,U,2})}$	
λ	0.2263	6.2483	1.2434
		$\mathbb{F}_{(\mathrm{N,E,U})}$	
μ	0.1493	0.64860	0.6743
σ	0.1982	13.6456	12.7209

Another goal was to compare the quality of some estimators produced with the proposed epistemic bootstrap and by the fuzzy version of the classical EM algorithm (known as the FEM method) introduced in [6] and available in EM.Fuzzy package (see [24]) implemented in R. Here we considered relatively small ($n = 10$) samples partially because the EM.Fuzzy package is intended for such samples. We set $B = 10$ for our epistemic bootstrap, and the default accuracy $\varepsilon = 0.001$ of the FEM algorithm (see [24] for necessary details). To compare both methods the empirical mean squared error of various estimators and distributions was determined. In particular, in Table 2 one can find some values of MSE for the epistemic bootstrap (MSE(EB)) and for two variants of the FEM method (MSE(FME) and MSE(FME*), where in the last case some additional information on the estimated parameter are required). It is seen that our method leads to estimators with significantly lower MSE – sometimes even more than 10 times than obtained with the FEM method implemented in the EM.Fuzzy package

3.3 Epistemic bootstrap in hypothesis testing

Finally, the power analysis of the suggested bootstrap method was conducted for the one-sided test for the mean. Results were compared with the outputs of a fuzzy test introduced in [11].

Suppose $X_1,\ldots,X_n$ are i.i.d. random variables from the normal distribution $\mathrm{N}(\mu,\sigma)$ with unknown mean μ and known standard deviation σ. Let us consider the null hypothesis $H_0 : \mu \leqslant \mu_0$ against the alternative

$H_1 : \mu > \mu_0$ on the significance level 0.05.

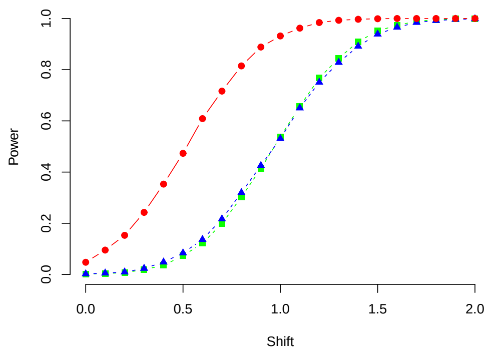

Figure 7: Power curves for the small sample size $n = 10$ and fuzzy data from $\mathbb{F}_{(\mathrm{N,U,U,1})}$.

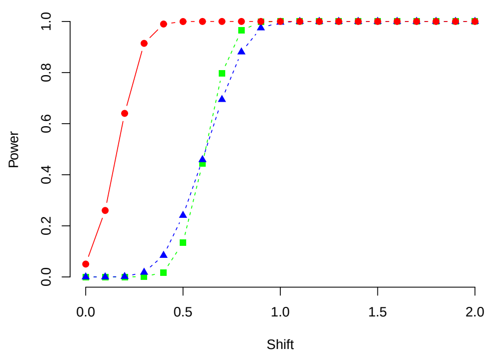

Figure 8: Power curves for the moderate sample size $n = 100$ and fuzzy data from $\mathbb{F}_{(\mathrm{N,U,U,1})}$.

Firstly, we combined our epistemic bootstrap with the classical Z-test for the mean. Simulated power curves obtained for this test are shown (red line) in Fig. 7 and Fig. 8 for the small $n = 10$ and moderate $n = 100$ sample size, respectively. Next, we compared these power curves with the others obtained for the test proposed by Grzegorzewski [11] supplemented by two defuzzification methods suggested in [12]. A defuzzification is applied since a fuzzy test leads not to a binary decision: either reject or accept H_0, but it provides a degree of rejection and acceptance of the null hypothesis.

In our study, we implemented the maximum value defuzzification operator (denoted further on as "max") and the randomized operator (denoted as "rand"). In Fig. 7 and Fig. 8 the corresponding power curves for the "max" operator are depicted in green, while for the "rand" operator in blue.

One can see immediately that the power of the classical Z-test combined with the epistemic bootstrap outmatches definitely the power of the fuzzy test whatever defuzzification method is used. Moreover, the test based on the epistemic bootstrap is uniformly better than the other ones, i.e. its power is bigger for any $\mu > \mu_0$. Although it is known that fuzzy tests are usually very conservative, the size of the power improvement obtained by applying the epistemic bootstrap is a good prognostic for further research.

4 Conclusions

Fuzzy data are encountered in various real-life situations where imprecise observations appear. Quite often they cause problems in statistical reasoning because of many restrictions involved by a specific data structure. To improve statistical inference based on such observations a new bootstrap technique was proposed. Contrary to other bootstrap approaches applied in fuzzy environment our method has been designed directly for epistemic fuzzy data.

Simulation results shown in this contribution give hope that the proposed bootstrap method may be useful in different fields of statistical inference. Obviously, the conclusions of so limited study cannot be treated as definite and require further research.

Finally, let us quote the words of Davison and Hinkley [3] that are both a warning and a clue for the users of any bootstrap method: "Despite its scope and usefulness, resampling must be carefully applied. Unless certain basic ideas are understood, it is all too easy to produce a solution to the wrong problem, or a bad solution to the right one. /.../ Bootstrap methods are intended to help avoid tedious calculations based on questionable assumptions, and this they do. But they can not replace clear critical thought about the problem, appropriate design of the investigation and data analysis, and incisive presentation of conclusions".

References

[1] G. Casella, Introduction to the silver anniversary of the bootstrap, Statistical Science 18 (2003) 133–134.

[2] A. Colubi, C. Fernández-García, M. Gil, Simulation of random fuzzy variables: An empirical approach to statistical/probabilistic studies with fuzzy experimental data, IEEE Transactions on Fuzzy Systems 10 (2002) 384–390.

[3] A. C. Davison, D. V. Hinkley, Bootstrap Methods and Their Application, Cambridge University Press, 1997.

[4] A. C. Davison, D. V. Hinkley, E. Schechtman, Effient bootstrap simulation, Biometrika 73 (1986) 555–566.

[5] D. De Angelis, G. A. Young, Smoothing the bootstrap, International Statistical Review 60 (1992) 45–56.

[6] T. Denœux, Maximum likelihood estimation from fuzzy data using the em algorithm, Fuzzy Sets and Systems 183 (2011) 72 – 91.

[7] B. Efron, Bootstrap methods: Another look at the jackknife, Annals of Statistics 7 (1979) 1–26.

[8] M. Gil, M. Montenegro, G. González-Rodríguez, A. Colubi, M. Casals, Bootstrap approach to the multi-sample test of means with imprecise data, Computational Statistics and Data Analysis 51 (2006) 148–162.

[9] G. González-Rodríguez, M. Montenegro, A. Colubi, M. Gil, Bootstrap techniques and fuzzy random variables: Synergy in hypothesis testing with fuzzy data, Fuzzy Sets and Systems 157 (2006) 2608–2613.

[10] R. Graham, D. V. Hinkley, P. W. M. John, S. Shi, Balanced design of bootstrap simulations, Journal of the Royal Statistical Socciety, Series B 52 (1990) 185–202.

[11] P. Grzegorzewski, Testing statistical hypotheses with vague data, Fuzzy Sets and Systems 112 (2000) 501–510.

[12] P. Grzegorzewski, Fuzzy tests – defuzzification and randomization, Fuzzy Sets and Systems 118 (2001) 437–446.

[13] P. Grzegorzewski, O. Hryniewicz, M. Romaniuk, Flexible bootstrap based on the canonical representation of fuzzy numbers, in: Proceedings of EUSFLAT 2019, Atlantis Press, 2019.

[14] P. Grzegorzewski, O. Hryniewicz, M. Romaniuk, Flexible bootstrap for fuzzy data based on the canonical representation, International Journal of Computational Intelligence Systems 13 (2020) 1650–1662.

[15] P. Grzegorzewski, O. Hryniewicz, M. Romaniuk, Flexible resampling for fuzzy data, International Journal of Applied Mathematics and Computer Science 30 (2020) 281–297.

[16] P. Hall, T. DiCiccio, J. Romano, On smoothing and the bootstrap, Annals of Statistics 17 (1989) 692–704.

[17] C. I., D. D., Statistical reasoning with set-valued information: Ontic vs. epistemic views, International Journal of Approximate Reasoning 55 (2014) 1502–1518.

[18] E. T. Jaynes, Information theory and statistical mechanics, Physical Review. Series II 106 (1957) 620–630.

[19] D. P. Kroese, T. Taimre, Z. I. Botev, Handbook of Monte Carlo Methods, John Wiley & Sons, 2011.

[20] R. Kruse, K. Meyer, Statistics with Vague Data, Riedel, Dordrecht, 1987.

[21] M. A. Lubiano, M. Montenegro, B. Sinova, S. de la Rosa de Sáa, M. A. Gil, Hypothesis testing for means in connection with fuzzy rating scale-based data: algorithms and applications, European Journal of Operational Research 251 (2016) 918–929.

[22] M. A. Lubiano, A. Salas, C. Carleos, M. A. de la Rosa de Sáa, S.and Gil, Hypothesis testing-based comparative analysis between rating scales for intrinsically imprecise data, International Journal of Approximate Reasoning 88 (2017) 128–147.

[23] M. Montenegro, A. Colubi, M. Casals, M. Gil, Asymptotic and bootstrap techniques for testing the expected value of a fuzzy random variable, Metrika 59 (2004) 31–49.

[24] A. Parchami, `https://cran.r-project.org/package=EM.Fuzzy`.

[25] M. Puri, R. D.A., Fuzzy random variables, Journal of the Mathematical Analysis and Applications 114 (1986) 409–422.

[26] A. Ramos-Guajardo, A. Blanco-Fernández, G. González-Rodríguez, Applying statistical methods with imprecise data to quality control in cheese manufacturing, in: P. Grzegorzewski, A. Kochanski, J. Kacprzyk (Eds.), Soft Modeling in Industrial Manufacturing, Springer, 2019, pp. 127–147.

[27] A. Ramos-Guajardo, P. Grzegorzewski, Distance-based linear discriminant analysis for interval-valued data, Information Sciences 372 (2016) 591–607.

[28] A. Ramos-Guajardo, M. Lubiano, *k*-sample tests for equality of variances of random fuzzy sets, Computational Statistics and Data Analysis 56 (2012) 956–966.

[29] M. Romaniuk, On some applications of simulations in estimation of maintenance costs and in statistical tests for fuzzy settings, in: A. Steland, E. Rafajłowicz, O. Okhrin (Eds.), Stochastic Models, Statistics and Their Applications, Springer International Publishing, 2019, pp. 437–448.

[30] M. Romaniuk, O. Hryniewicz, Interval-based, nonparametric approach for resampling of fuzzy numbers, Soft Computing 23 (2019) 5883–5903.

[31] M. Romaniuk, O. Hryniewicz, Discrete and smoothed resampling methods for interval-valued fuzzy numbers, IEEE Transactions on Fuzzy Systems 29 (2021) 599–611.

[32] B. W. Silverman, G. A. Young, The bootstrap: To smooth or not to smooth?, Biometrika 74 (1987) 469–479.

[33] D. Wang, O. Hryniewicz, A fuzzy nonparametric Shewhart chart based on the bootstrap approach, International Journal of Applied Mathematics and Computer Science 25 (2015) 389–401.