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Graded Fuzzy Preconcept Lattices: Theoretical Basis

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Abstract

Noticing certain limitations of concept lattices in the fuzzy context, especially in view of their practical applications, in this paper we propose a more general approach based on what we call *graded fuzzy preconcept lattices*. We believe that this approach is more adequate for dealing with fuzzy information than the one based on fuzzy concept lattices. In this paper we develop theoretical basis for graded fuzzy preconcept lattices. Some possible applications of graded fuzzy lattices are discussed in the Conclusion section.

Keywords: Fuzzy context, Fuzzy concept lattice, Fuzzy preconcept, Fuzzy preconcept lattice, Gradation of fuzzy preconcept lattices.

1 Introduction

Formal concept analysis or just concept analysis for short was developed mainly in eighties of the previous century. The principles and fundamental results of concept analysis were exposed in the paper [31] and further expanded in the book [13]. Concept analysis starts with the notion of a (formal) context that is a triple (X,Y,R) where X and Y are sets and $R \subseteq X \times Y$ is a relation between the elements of these sets. The elements of X are interpreted as some abstract objects, the elements of Y are interpreted as some abstract properties or attributes, and the entry $(x,y) \in R$ means that an object x has attribute y. The idea of the concept analysis is to reveal all pairs (A,B) of sets $A \subseteq X$ and $B \subseteq Y$ (called concepts) such that every object $x \in A$ has all properties $y \in B$ and every property $y \in B$ holds for all objects $x \in A$.

In the first decade of the 21$^{\text{th}}$ century different fuzzy counterparts of the formal concept analysis were introduced. In the fuzzy case a context is a tuple (X,Y,L,R) where X and Y are sets, L is a lattice, and $R: X \times Y \to L$ is an L-fuzzy relation. Fuzzy concepts in this fuzzy context are pairs (A,B), where A and B are L-fuzzy subsets of the sets X and Y respectively, which are interrelated in a way, regarding the relation in the crisp case (see Definition 4.8). The most important work in the first decade of the 21$^{\text{th}}$ century in the field of fuzzy concept analysis was carried out by R. Bělohlávek, see, e.g. [3, 4], see also [5, 21, 7, 25], etc.

Concept analysis and concept lattices, crisp as well as fuzzy, aroused great interest both among theorists in mathematics and among researchers working in applied fields. The theoretical interest in concept lattices can be explained, in particular, by the fact that they form interesting connections with other mathematical structures, specifically with (fuzzy) rough sets [24, 12] and (fuzzy) mathematical morphology [10, 11, 23], see, e.g. [2, 21, 32, 28].

Since its inception, crisp concept analysis has found important applications in the study of “real-world" problems. On the other hand, we found only a few works, where fuzzy concept analysis is used in the research of any practical-type problems. In our opinion, the problem to use fuzzy concept lattices in practice is that the request in the concept analysis of the *precise correspondence* between the fuzzy set A of objects and the fuzzy set B of attributes in "real-world" situations is (almost) impracticable. In this case one sooner has to deal with the weaker request asking *that the correspondence between A and B must hold up to a certain degree*. In order to develop a theoretical basis for a solution of this problem, in the paper [29] we first replaced the notion of a fuzzy concept by a much weaker notion of a fuzzy preconcept, and then proposed two methods, allowing to evaluate “how far a fuzzy preconcept is from the nearest fuzzy concept". In its turn, this lead to two kinds of graded fuzzy preconcept lattices, called $\mathscr{D}$-graded and $\mathscr{M}$-graded. The study of these lattices was initiated in [29]. In this paper we go further in the study of the $\mathscr{D}$-graded preconcept lattices. The main goal of this paper is to develop the theoretical basis of $\mathscr{D}$-graded fuzzy preconcept lattices in order that they could be used for some practical applications. The perspective fields for application of graded fuzzy preconcept lattices are discussed in the Conclusion section.

The paper is organized as follows. In the second, preliminary, section we briefly recall the notions related to lattices, residuated lattices or quantales, fuzzy sets and fuzzy relations. Further in this section we remind the reader the concept of the measure of inclusion of fuzzy sets; just the measure of such inclusion lies in the base of the grade of fuzzy preconcepts - the main concept considered further in this paper.

In the third section we define fuzzy preconcepts, introduce partial order relation $\preceq$ on the family of all fuzzy preconcepts of a given fuzzy context (X,Y,L,R) and show that the resulting structure $(\mathbb{P}(X,Y,L,R),\preceq)$ is a lattice. In the fourth section we consider operators $R^{\uparrow}$ and $R^{\downarrow}$ on fuzzy preconcept lattices; these operators play fundamental role in our work and, in particular, they are used in order to distinguish "real" fuzzy concepts from arbitrary fuzzy preconcepts and to define a lattice $(\mathbb{C}(X,Y,L,R),\preceq)$ of fuzzy concepts. However this lattice structure is different from the lattice induced from the lattice $(\mathbb{P}(X,Y,L,R),\preceq)$ of fuzzy preconcepts. Some of the results of this section are known (the corresponding references are given); we reproduce them here in the form appropriate for this work and in order to make the paper self contained.

The following, fifth section is the central one in the work. Here we propose a method allowing to determine the grade showing the distinction of a fuzzy preconcept from "being a real fuzzy concept" and study the corresponding graded preconcept lattices. The definition of a grade of a fuzzy preconcept is based on the evaluation of mutual "contentment" of the fuzzy set of objects and the fuzzy set of attributes. The corresponding lattice of fuzzy preconcepts endowed with gradated evaluation defined in such way is called a $\mathscr{D}$-graded preconcept lattice. Some examples of $\mathscr{D}$-graded fuzzy preconcept lattices are given.

In the last, conclusion section, we briefly summarize main results obtained and survey some directions for the future work.

2 Preliminaries

2.1 Lattices, quantales and residuated lattices

We use the standard terminology accepted in theory of lattices, see, e.g. [1, 14, 22, 9] for details. For the reader convenience, we make clarification of some, possibly less known, terms.

A complete lattice $\mathbb{L}=(\mathbb{L},\leq,\wedge,\vee)$ is called infinitely distributive if $a\wedge(\bigvee_{i\in I} b_i)=\bigvee_{i\in I}(a\wedge b_i)$ for every $a\in\mathbb{L}$ and every $\{b_i \mid i\in I\}\subseteq\mathbb{L}$. A complete lattice $\mathbb{L}$ is called infinitely co-distributive if $a\vee(\bigwedge_{i\in I} b_i)=\bigwedge_{i\in I}(a\vee b_i)$ for every $a\in\mathbb{L}$ and every $\{b_i \mid i\in I\}\subseteq\mathbb{L}$. A complete lattice is called infinitely bi-distributive if it is infinitely and co-infinitely distributive. It is known that every completely distributive lattice is infinitely bi-distributive. Every MV-algebra which is join-distributive is also meet-distributive.

Let $\mathbb{L}$ be a complete lattice and $*:\mathbb{L}\times\mathbb{L}\to\mathbb{L}$ be a binary associative commutative monotone operation. Then the tuple $(\mathbb{L},\leq,\wedge,\vee,*)$ is called *a (commutative) quantale* [26] if $*$ distributes over arbitrary joins: $a*(\bigvee_{i\in I}b_i)=\bigvee_{i\in I}(a*b_i)$. Operation $*$ will be referred to as the product in $\mathbb{L}$. A quantale is called integral if the top element $\top_{\mathbb{L}}$ of the lattice $\mathbb{L}$ acts as the unit, that is $\top_{\mathbb{L}}*a=a*\top_{\mathbb{L}}=a$ for every $a\in\mathbb{L}$; in this case we write $\top_{\mathbb{L}}=1_{\mathbb{L}}$ and $\perp_{\mathbb{L}}=0_{\mathbb{L}}$ were $\mathbb{L}_0$ is the least element of $\mathbb{L}$. In what follows saying *a quantale* we mean *a commutative integral quantale*. A typical example of a quantale is the unit interval endowed with a lower semi-continuous t-norm, see, e.g. [20].

In a quantale a further binary operation $\mapsto:\mathbb{L}\times\mathbb{L}\to\mathbb{L}$, the residuum, can be introduced as associated with operation $*$ of the quantale $(\mathbb{L},\leq,\wedge,\vee,*)$ via the Galois connection, that is $a*b\leq c \iff a\leq b\mapsto c$ for all $a,b,c\in\mathbb{L}$. A quantale $(\mathbb{L},\leq,\wedge,\vee,*)$ provided with the derived operation $\mapsto$, that is the tuple $(\mathbb{L},\leq,\wedge,\vee,*,\mapsto)$, is known also as a (complete) residuated lattice [22]. In the following proposition we collect well-known properties of the residuum:

Proposition 2.1 (see, e.g. [17, 18])
(1) $(\bigvee_i a_i)\mapsto b=\bigwedge_i(a_i\mapsto b)$ *for all* $\{a_i \mid i\in I\}\subseteq\mathbb{L}$, *for all* $b\in\mathbb{L}$;
(2) $a\mapsto(\bigwedge_i b_i)=\bigwedge_i(a\mapsto b_i)$ *for all* $a\in\mathbb{L}$ *and for all* $\{b_i \mid i\in I\}\subseteq\mathbb{L}$,;
(3) $1_{\mathbb{L}}\mapsto a=a$ *for all* $a\in\mathbb{L}$;
(4) $a\mapsto b=1_{\mathbb{L}}$ *whenever* $a\leq b$;
(5) $a*(a\mapsto b)\leq b$ *for all* $a,b\in\mathbb{L}$;
(6) $(a\mapsto b)*(b\mapsto c)\leq a\mapsto c$ *for all* $a,b,c\in\mathbb{L}$;
(7) $a\mapsto b\leq(a*c\mapsto b*c)$ *for all* $a,b,c\in\mathbb{L}$;
(8) $a*b\leq a\wedge b$ *for all* $a,b\in\mathbb{L}$;
(9) $(a*b)\mapsto c=a\mapsto(b\mapsto c)$ *for all* $a,b,c\in\mathbb{L}$.

2.2 Fuzzy sets and fuzzy relations

Given a set X, an L-fuzzy subset is a mapping $A:X\to L$. The lattice and the quantale structure of L is extended point-wise to the L-exponent of X, that is to the set $\mathbb{L}=L^X$ of all L-fuzzy subsets of X. Specifically, the union and intersection of a family of L-fuzzy sets $\{A_i | i \in I\} \subseteq L$ are defined by their join $\bigvee_{i\in I} A_i$ and meet $\bigwedge_{i\in I} A_i$ respectively. An L-fuzzy relation between two sets X and Y is an L-fuzzy subset of the product $X \times Y$, that is a mapping $R : X \times Y \to L$, see, e.g. [30, 33].

In this work we will deal with several different lattices and quantales. The symbol $\mathbb{L}$ is used as the common notation for any lattice (quantale). On the other hand, notation L is used when we speak about L-fuzzy sets. Besides, since L, as the lattice of values for L-fuzzy subsets and L-fuzzy relations, in the paper is an arbitrary but a fixed lattice, we shall omit the prefix L and speak just of fuzzy sets and fuzzy relations.

2.3 Measure of inclusion of L-fuzzy sets

The gradation of a preconcept lattice introduced below is based on the fuzzy inclusion between fuzzy sets. We present here a brief introduction into this field.

In order to fuzzify the inclusion relation $A \subseteq B$ "a fuzzy set A is a subset of a fuzzy set B", we have to interpret it as a certain fuzzy relation $\hookrightarrow$ based on "if - then" rule, that is on some implication $\Rightarrow$ defined on the lattice L. In the result we come to the formula $A \hookrightarrow B = \inf_{x\in X}(A(x) \Rightarrow B(x))$. As far as we know, for the first time this approach was applied in [27] where it was based on the Kleene-Dienes implication $\Rightarrow$. Later the fuzzified relation of inclusion between fuzzy sets was studied and used by many authors, see, e.g. [6], [8], [19], et al. In most of the papers the implication $\Rightarrow$ was defined by means of residuum $\mapsto$ of the underlying quantale $(L, \wedge, \vee, *)$. This implication behaves in this situation "much better" than the Kleene-Dienes or some other implication on $(L, \wedge, \vee, *)$. In our paper we stick to the residuum based measure of inclusion specified in the following definition:

Definition 2.2 *By setting $A \hookrightarrow B = \bigwedge_{x\in X}(A(x) \mapsto B(x))$ for all $A, B \in L^X$, we obtain a mapping $\hookrightarrow: L^X \times L^X \to L$. We call $A \hookrightarrow B$ by the (L-valued) measure of inclusion of the L-fuzzy set A into the L-fuzzy set B. We denote $A \cong B =_{def} (A \hookrightarrow B) \wedge (B \hookrightarrow A)$ and view it as the degree of equality of L-fuzzy sets A and B.*

Proposition 2.3 (see, e.g. [15, 16])
Mapping $\hookrightarrow: L^X \times L^X \to L$ satisfies the following properties:
(1) $(\bigvee_i A_i) \hookrightarrow B = \bigwedge_i (A_i \hookrightarrow B)$ *for all* $\{A_i \mid i \in I\} \subseteq L^X$ *and for all* $B \in L^X$;
(2) $A \hookrightarrow (\bigwedge_i B_i) = \bigwedge_i (A \hookrightarrow B_i)$ *for all* $A \in L^X$ *and for all* $\{B_i \mid i \in I\} \subseteq L^X$;
(3) $A \hookrightarrow B = 1_L$ *whenever* $A \le B$;
(4) $1_X \hookrightarrow A = \bigwedge_x A(x)$ *for all* $A \in L^X$ *where* $1_X : X \to L$ *is a constant function with the value* $1_L \in L$;
(5) $(A \hookrightarrow B) \le (A * C \hookrightarrow B * C)$ *for all* $A, B, C \in L^X$;
(6) $(A \hookrightarrow B) * (B \hookrightarrow C) \le (A \hookrightarrow C)$ *for all* $A, B, C \in L^X$;
(7) $(\bigwedge_i A_i) \hookrightarrow (\bigwedge_i B_i) \ge \bigwedge_i (A_i \hookrightarrow B_i)$ *for all* $\{A_i : i \in I\}$, $\{B_i : i \in I\} \subseteq L^X$;
(8)$(\bigvee_i A_i) \hookrightarrow (\bigvee_i B_i) \ge \bigwedge_i (A_i \hookrightarrow B_i)$ *for all* $\{A_i : i \in I\}$, $\{B_i : i \in I\} \subseteq L^X$.

3 Preconcepts and preconcept lattices

Let L be a complete lattice. Further, let X, Y be sets and $R : X \times Y \to L$ be a fuzzy relation. Following terminology accepted in the theory of (fuzzy) concept lattices, see, e.g. [31], [3], [4], we refer to the tuple (X, Y, L, R) as a fuzzy context.

Definition 3.1 *Given a fuzzy context (X, Y, L, R), a pair $P = (A, B) \in L^X \times L^Y$ is called a fuzzy preconcept*[1].

On the set $L^X \times L^Y$ of all fuzzy preconcepts we introduce a partial order $\preceq$ as follows. Given $P_1 = (A_1, B_1)$ and $P_2 = (A_2, B_2)$, we set $P_1 \preceq P_2$ if and only if $A_1 \le A_2$ and $B_1 \ge B_2$. Let $(\mathbb{P}, \preceq)$ be the set $L^X \times L^Y$ endowed with this partial order. Further, given a family of fuzzy preconcepts $\{P_i = (A_i, B_i) : i \in I\} \subseteq L^X \times L^Y$, we define its join (supremum) by $\veebar_{i\in I} P_i = (\bigvee_{i\in I} A_i, \bigwedge_{i\in I} B_i)$ and its meet (infimum) as $\barwedge_{i\in I} P_i = (\bigwedge_{i\in I} A_i, \bigvee_{i\in I} B_i)$.

Theorem 3.2 *$\mathbb{P}$ is a complete lattice. Besides, if L is an infinitely bi-distributive lattice, then $(\mathbb{P}, \preceq, \barwedge, \veebar)$ is also an infinitely bi-distributive lattice.*

Proof Let $\mathscr{P} = \{P_i = (A_i, B_i) \mid i \in I\} \subseteq \mathbb{P}$. Then $\barwedge_{i\in I} P_i = (\bigwedge_{i\in I} A_i, \bigvee_{i\in I} B_i)$ and $\veebar_{i\in I} P_i = (\bigvee_{i\in I} A_i, \bigwedge_{i\in I} B_i) \in \mathbb{P}$. So $(\mathbb{P}, \preceq, \barwedge, \veebar)$ is a complete lattice. Its top and bottom elements are $\top_{\mathbb{P}} = (1_X, 0_Y)$ and $\bot_{\mathbb{P}} = (0_X, 1_Y)$ where a_X and a_Y are respectively constant fuzzy subsets of X and Y with values $a \in L$.

Further, assume that L is infinitely bi-distributive. To show that $(\mathbb{P}, \preceq, \barwedge, \veebar)$ is infinitely distributive, let $\mathscr{P} = \{P_i = (A_i, B_i) \mid i \in I\} \subseteq \mathbb{P}$ and $P = (A, B) \in \mathbb{P}$. Then $(\veebar_{i\in I} P_i) \barwedge P = ((\vee_{i\in I} A_i), (\wedge_{i\in I} B_i)) \barwedge (A, B) = ((\vee_{i\in I} A_i) \wedge A, (\wedge_{i\in I} B_i) \vee B) = ((\vee_{i\in I}(A_i \wedge A)), (\wedge_{i\in I}(B_i \vee B))) = \veebar_{i\in I}(P_i \barwedge P)$.
In the same manner we prove that $(\mathbb{P}, \preceq, \barwedge, \veebar)$ is an infinitely co-distributive lattice. □

In the sequel we write just $\mathbb{P}$ or $(\mathbb{P}, \preceq)$ instead of $(\mathbb{P}, \preceq, \barwedge, \veebar)$ if no misunderstanding is possible, or $(\mathbb{P}, X, Y, L, R)$ in case when we need to specify the fuzzy context we are working in.

[1]This notion of a fuzzy preconcept is not related with the notion of a preconcept as it is defined in [13, p. 59].

4 Operators $R^{\uparrow}$ and $R^{\downarrow}$ on L-powersets and fuzzy concept lattices

Let X and Y be sets and let $R : X \times Y \to L$ be a fuzzy relation, where L is a fixed *quantale*. Given a fuzzy context (X,Y,L,R), we define operators $R^{\uparrow} : L^X \to L^Y$ and $R^{\downarrow} : L^Y \to L^X$ as follows:

Definition 4.1 (see, e.g. [4]) *Given $A \in L^X$ and $B \in L^Y$, we define $A^{\uparrow} \in L^Y$ and $B^{\downarrow} \in L^X$ by setting*
$A^{\uparrow}(y) = \bigwedge_{x\in X}(A(x) \mapsto R(x,y))\ \forall y \in Y,$
$B^{\downarrow}(x) = \bigwedge_{y\in Y}(B(y) \mapsto R(x,y))\ \forall x \in X.$
By setting $R^{\uparrow}(A)$ for all $A \in L^X$ and $R^{\downarrow}(B) = B^{\downarrow}$ for all $B \in L^Y$, we get operators $R^{\uparrow} : L^X \to L^Y$ and $R^{\downarrow} : L^Y \to L^X$.

In the crisp case, that is when $A \subseteq X, B \subseteq Y$ and $R : X \times Y \to \{0,1\}$, this definition is obviously equivalent to the original definition of operators $A \longrightarrow A'$ and $B \longrightarrow B'$ in [31]. From the properties of the residuum one can easily justify the following:

Proposition 4.2 *Operators $R^{\uparrow} : L^X \to L^Y$ and $R^{\downarrow} : L^Y \to L^X$ are decreasing:*
$A_1 \le A_2 \Rightarrow A_1^{\uparrow} \ge A_2^{\uparrow}$, $B_1 \le B_2 \Rightarrow B_1^{\downarrow} \ge B_2^{\downarrow}$.

In the sequel we write $A^{\uparrow\downarrow}$ instead of $(A^{\uparrow})^{\downarrow}$ and $B^{\downarrow\uparrow}$ instead of $(B^{\downarrow})^{\uparrow}$. We write also $R^{\uparrow\downarrow}$ for the composition $R^{\downarrow} \circ R^{\uparrow} : L^X \to L^X$ and $R^{\downarrow\uparrow}$ for the composition $R^{\uparrow} \circ R^{\downarrow} : L^Y \to L^Y$.

Proposition 4.3 (cf, e.g. [31] in crisp case, [4]) *$A^{\uparrow\downarrow} \ge A$ for every $A \in L^X$ and $B^{\downarrow\uparrow} \ge B$ for every $B \in L^Y$.*

Proposition 4.4 (cf, e.g. [31] in crisp case, [4])
$A^{\uparrow} = A^{\uparrow\downarrow\uparrow}$ for every $A \in L^X$ and $B^{\downarrow} = B^{\downarrow\uparrow\downarrow}$ for every $B \in L^Y$.

From propositions 4.3 and 4.4 one can easily get the following fundamental fact:

Theorem 4.5 (cf, e.g. [31] in crisp case, [4]) *Operators $R^{\uparrow} : L^X \to L^Y$ and $R^{\downarrow} : L^Y \to L^X$ satisfy an antitone Galois connection:*

$$A \le B^{\downarrow} \iff B \le A^{\uparrow} \quad \forall A \in L^X, B \in L^Y.$$

Example 4.6 Let a fuzzy context (X,Y,L,R) be given and let $A \subseteq X$.[2] Then for every $y \in Y$ $A^{\uparrow}(y) = \bigwedge_{x\in X} A(x) \mapsto R(x,y) = \bigwedge_{x\in A} R(x,y)$. In the same way we prove that if $B \subseteq Y$, then $B^{\downarrow}(x) = \bigwedge_{y\in B} R(x,y)$. Hence, in case when $A \subseteq X$, $B \subseteq Y$ a pair (A,B) can be a concept (either crisp or fuzzy) only in case when R is also crisp, that is $R : X \times Y \to \{0,1\}$. This shows the limitation for the use of concept lattices in the case of a fuzzy context and gives an additional evidence in favour of the graded approach to fuzzy preconcept lattices.

Continuing the previous example we calculate $A^{\uparrow\downarrow}$ and $B^{\downarrow\uparrow}$ in case of crisp sets A and B:
$A^{\uparrow\downarrow}(x) = \bigwedge_{y\in Y}\left(\bigwedge_{x'\in A}(R(x',y) \mapsto R(x,y))\right),$
$B^{\downarrow\uparrow}(y) = \bigwedge_{x\in X}\left(\bigwedge_{y'\in B}(R(x,y') \mapsto R(x,y))\right).$ □

Proposition 4.7 (cf, e.g. [31] for the crisp case, [4]) *Given a family $\{A_i \mid i \in I\} \subseteq L^X$, we have $(\bigvee_{i\in I} A_i)^{\uparrow} = \bigwedge_{i\in I} A_i^{\uparrow}$. Given a family $\{B_i \mid i \in I\} \subseteq L^Y$, we have $(\bigvee_{i\in I} B_i)^{\downarrow} = \bigwedge_{i\in I} B_i^{\downarrow}$.*

Referring to the definition of a (fuzzy) concept given in [31, 4], we reformulate it as follows:

Definition 4.8 *A fuzzy preconcept (A,B) is called a (fuzzy) concept if $A^{\uparrow} = B$ and $B^{\downarrow} = A$.*

Let $\mathbb{C} = \mathbb{C}(X,Y,L,R)$ be the subset of $\mathbb{P} = \mathbb{P}(X,Y,L,R)$ consisting of fuzzy concepts (A,B) and let $\preceq$ be the partial order on $\mathbb{C}$ induced by the partial order $\preceq$ from the lattice $(\mathbb{P},\preceq)$. Then $(\mathbb{C},\preceq)$ is a partially ordered subset of the lattice $(\mathbb{P},\preceq)$. However, generally $(\mathbb{C},\preceq)$ is not a sublattice of the lattice $(\mathbb{P},\preceq,\barwedge,\veebar)$ and we need to define joins and meets in $(\mathbb{C},\preceq)$ differently from $\barwedge$ and $\veebar$. To do this, first we show the following simple lemma:

Lemma 4.9 *Let (A_1,B_1), (A_2,B_2) be fuzzy concepts. If $A_1 \le A_2$, then $B_1 \ge B_2$ and if $B_1 \ge B_2$ then $A_1 \le A_2$.*

Proof If $A_1 \le A_2$, then from Proposition 4.2 it follows that $A_1^{\uparrow} \ge A_2^{\uparrow}$ and hence $B_1 \ge B_2$. The proof of the second statement is similar. □

Corollary 4.10 *Let $(A_1,B_1),(A_2,B_2) \in \mathbb{C}$. Then $(A_1,B_1) \preceq (A_2,B_2)$ if and only if $A_1 \le A_2$ if and only if $B_1 \ge B_2$.*

Proposition 4.11 *If $A \in L^X$, then $(A^{\uparrow\downarrow},A^{\uparrow})$ is the smallest concept containing A as the fuzzy set of objects. If $B \in L^Y$, then $(B^{\downarrow},B^{\downarrow\uparrow})$ is the smallest concept containing B as the fuzzy set of attributes.*

Proof From Proposition 4.4 it follows that $(A^{\uparrow\downarrow},A^{\uparrow})$ is a fuzzy context. Further, from Proposition 4.3 we know that $A \le A^{\uparrow\downarrow}$. Assume that there is a fuzzy concept $(A_o,A_o^{\uparrow})$ such that $A \le A_o \le A^{\uparrow\downarrow}$. Then $A^{\uparrow} \ge A_o^{\uparrow} \ge A^{\uparrow\downarrow\uparrow} = A^{\uparrow}$ an hence $A_o^{\uparrow} = A^{\uparrow}$. Therefore $A_o = A^{\uparrow\downarrow}$ and $(A_o,A_o^{\uparrow}) = (A^{\uparrow\downarrow},A^{\uparrow})$. In a similar way we can prove that $(B^{\downarrow},B^{\downarrow\uparrow})$ is the smallest context containing B as the fuzzy set of attributes. □

[2] Here and in the sequel we do not distinguish between a crisp set $A \subseteq X$ and its characteristic function $\chi_A : X \to \{0,1\}$.

Theorem 4.12 *Let (X,Y,L,R) be a fuzzy context and let $\preceq$ be the partial order induced from the lattice $\mathbb{P}(X,Y,L,R,\preceq)$. Then $\mathbb{C}(X,Y,L,R \preceq)$ is a complete lattice.*

We know already that $\mathbb{C}(X,Y,L,R),\preceq)$ is a partially ordered set. So the proof will follow directly from the next proposition.

Proposition 4.13 (cf [31] for the crisp case, [4]) *Let $\{\mathscr{C} = \{C_i = (A_i,B_i)\} \subseteq \mathbb{C}(X,Y,L,R,\preceq)$ be a family of fuzzy concepts. Then*
$\curlywedge_{i\in I} C_i = \left(\bigwedge_{i\in I} A_i, (\bigvee_{i\in I} B_i)^{\downarrow\uparrow}\right)$ *is its infimum in the partially ordered set $(\mathbb{C},\preceq)$,*
$\curlyvee_{i\in I} C_i = \left((\bigvee_{i\in I} A_i)^{\uparrow\downarrow}, \bigwedge_{i\in I} B_i\right)$ *is its supremum in the partially ordered set $(\mathbb{C},\preceq)$.*

Proof To show the first statement we have to prove only that $\curlywedge_{i\in I} C_i$ is a fuzzy concept; its minimality will be clear from its definition since $(\bigwedge_{i\in I} A_i, \bigvee_{i\in I} B_i)$ is the meet of $\mathscr{C}$ in $\mathbb{P}(X,Y,L,R)$. Indeed, $(\bigvee_{i\in I} B_i)^{\downarrow\uparrow} = (\bigwedge_{i\in I} A_i)^{\uparrow\downarrow\uparrow} = (\bigwedge_{i\in I} A_i)^{\uparrow}$; $(\bigwedge_{i\in I} A_i)^{\uparrow} = (\bigvee_{i\in I} B_i)^{\downarrow\uparrow}$.
To show the second statement we have to prove only that $\curlyvee_{i\in I} C_i$ is a fuzzy concept; its maximality will be clear from its definition since $(\bigvee_{i\in I} A_i, \bigwedge_{i\in I} B_i))$ is the join of $\mathscr{C}$ in $\mathbb{P}(X,Y,L,R)$. Indeed $(\bigvee_{i\in I} A_i)^{\uparrow\downarrow} = (\bigwedge_{i\in I} B_i))^{\downarrow\uparrow\downarrow} = (\bigwedge_{i\in I} B_i)^{\downarrow}$; $(\bigwedge_{i\in I} B_i)^{\downarrow} = (\bigvee_{i\in I} A_i)^{\uparrow\downarrow}$. $\square$

Taking into account that in a fuzzy concept (A_i,B_i) it holds $A_i^{\uparrow} = B_i$ and $B_i^{\downarrow} = A_i$, we get the following corollary from the previous Proposition 4.13:

Corollary 4.14 *Let $\mathscr{C} = \{C_i = (A_i,B_i) \mid i \in I\} \subseteq \mathbb{C}$ be a family of fuzzy concepts. Then*
$\curlywedge_{i\in I} C_i = \left(\bigwedge_{i\in I} A_i, (\bigwedge_{i\in I} A_i)^{\uparrow}\right)$ *is its infimum in the lattice $(\mathbb{C},\preceq)$,*
$\curlyvee_{i\in I} C_i = ((\bigwedge_{i\in I} B_i)^{\downarrow}, \bigwedge_{i\in I} B_i)$ *is its supremum in the lattice $(\mathbb{C},\preceq)$.*

5 Conceptuality degree of a fuzzy preconcept and $\mathscr{D}$-graded preconcept lattices

5.1 Degrees of object and attribute based contentments of a fuzzy preconcept

Let (X,Y,L,R) be a fuzzy context and $(A,B) \in \mathbb{P}(X,Y,L,R)$.

Definition 5.1 *The degree of contentment of a fuzzy set A of objects by a fuzzy set B of attributes or the degree object based contentment of the preconcept (A,B) for short is defined by $\mathscr{D}^{\uparrow}(A,B) =_{def} A^{\uparrow} \cong B$.*

Definition 5.2 *The degree of contentment of a fuzzy set B of attributes by a fuzzy set A of objects or the attribute based contentment of the preconcept (A,B) is defined by $\mathscr{D}^{\downarrow}(A,B) =_{def} A \cong B^{\downarrow}$.*

Definition 5.3 *The degree of conceptuality of a preconcept (A,B) in the fuzzy preconcept lattice $\mathbb{P}$ is defined by $\mathscr{D}(A,B) = \mathscr{D}^{\uparrow}(A,B) \wedge \mathscr{D}^{\downarrow}(A,B)$.*

Changing pairs $(A,B) \in \mathbb{P}$, we obtain mappings
$\mathscr{D}^{\uparrow} : \mathbb{P} \to L$, $\mathscr{D}^{\downarrow} : \mathbb{P} \to L$ and $\mathscr{D} : \mathbb{P} \to L$.

We illustrate the evaluation of conceptuality degree in the fuzzy context (X,Y,L,R) in some simple situations. To simplify calculations we distinguish the following three special conditions. The first one concerns the properties of the product $*$ of the quantale while the other two have to do with fuzzy relation R. Besides the second and the third one is only applicable for pairs of crisp sets $(A,B) \in \mathbb{P}(X,Y,L,R)$. In this case we denote $A^c = X \setminus A$ and $B^c = Y \setminus B$, that is the complements of the sets A and B respectively.

$(\dagger_*)$ Operation $*$ has no zero divisors, that is $a * b = 0 \Rightarrow a = 0$ or $b = 0$ for any $a,b \in L$.

$(\dagger_{R_{BA}})$ $\bigvee_{y\in B^c} \bigwedge_{x\in A} R(x,y) = 0$. In particular, this relation holds if $B = Y$.

$(\dagger_{R_{AB}})$ $\bigvee_{y\in A^c} \bigwedge_{x\in B} R(x,y) = 0$. In particular, this relation holds if $A = X$.

Finally let $(\dagger_R)$ mean that both $(\dagger_{R_{BA}})$ and $(\dagger_{R_{AB}})$ hold.

Example 5.4 Let $A \subseteq X, B \subseteq Y$, let $(L,\leq,\wedge,\vee,*)$ be an arbitrary quantale, $\mapsto: L \times L \to L$ its residuum, and $R : X \times Y \to L$ a fuzzy relation. Then
$A^{\uparrow} \hookrightarrow B = \bigwedge_{y\in Y} (\bigwedge_{x\in X} (A(x) \mapsto R(x,y)) \mapsto B(y)) = \bigwedge_{y\in B^c} ((\bigwedge_{x\in X} (A(x) \mapsto R(x,y)) \mapsto 0) = \bigwedge_{y\in B^c} (\bigwedge_{x\in A} R(x,y) \mapsto 0)$;
$B \hookrightarrow A^{\uparrow} = \bigwedge_{y\in Y} (B(y) \mapsto (\bigwedge_{x\in X} (A(x) \mapsto R(x,y)))) = \bigwedge_{y\in Y} (B(y) \mapsto \bigwedge_{x\in A} R(x,y)) = \bigwedge_{y\in B} \bigwedge_{x\in A} R(x,y)$.

$\mathscr{D}^{\uparrow}(A,B) = (\bigwedge_{y\in B^c} (\bigwedge_{x\in A} (R(x,y) \mapsto 0))) \wedge (\bigwedge_{x\in A, y\in B} R(x,y))$.
From the above it easily follows that if either $(\dagger_*)$ or $(\dagger_{R_{BA}})$ holds, then $B \hookrightarrow A^{\uparrow} = 1$ and hence $\mathscr{D}^{\uparrow}(A,B) = \bigwedge_{x\in A, y\in B} R(x,y)$.

In a similar way we prove that
$A \hookrightarrow B^{\downarrow} = \bigwedge_{x\in A, y\in B} R(x,y)$,
$B^{\downarrow} \hookrightarrow A = \bigwedge_{x\in A^c} (\bigwedge_{y\in B} (R(x,y) \mapsto 0))$,
$\mathscr{D}^{\downarrow}(A,B) = (\bigwedge_{x\in A^c} \bigwedge_{y\in B} (R(x,y) \mapsto 0)) \wedge (\bigwedge_{x\in A, y\in B} R(x,y))$ and $\mathscr{D}^{\downarrow}(A,B) = \bigwedge_{x\in A, y\in B} R(x,y)$ in case when either $(\dagger_*)$ or $(\dagger_{R_{AB}})$ holds.

Thus in case of crisp object and attribute sets A, B and under assumption that either $(\dagger_*)$ or $(\dagger_R)$ holds, the degree of the conceptuality of the pair (A,B) is $\mathscr{D}(A,B) = \bigwedge_{x\in A, y\in B} R(x,y)$. $\square$

Example 5.5 Let now $(\mathbb{P}(X,Y,L,R)$ be a fuzzy context where $L=[0,1]$, $a\in(0,1)$, $X_a\subseteq X$, $B\subseteq Y$ and a fuzzy set $A: X\to[0,1]$ be defined by

$$A(x)=\begin{cases} a & \text{if } x\in X_a \\ 0, & \text{if } x\notin X_a.\end{cases}$$

Then $B\hookrightarrow A^{\uparrow}=\bigwedge_{y\in Y}(B(y)\mapsto A^{\uparrow}(y))=\bigwedge_{y\in B}A^{\uparrow}(y)=\bigwedge_{y\in B,x\in X_a}(a\mapsto R(x,y))$,
$A^{\uparrow}\hookrightarrow B=\bigwedge_{y\in B^c}\left(\bigwedge_{x\in X_a}(a\mapsto R(x,y))\right)\mapsto 0$ and hence $\mathscr{D}^{\uparrow}(A,B)=\left(\bigwedge_{y\in B,x\in X_a}(a\mapsto R(x,y))\right)\wedge\left(\bigwedge_{y\in B^c}(\bigwedge_{x\in X}(a\mapsto R(x,y)))\mapsto 0\right)$ and $\mathscr{D}^{\uparrow}(A,B)=\bigwedge_{y\in B,x\in X_a}(a\mapsto R(x,y))$ if condition $(\dagger_*)$ or condition $(\dagger_{R_{BA}})$ is satisfied.

In a similar way, in order to calculate $\mathscr{D}^{\downarrow}(A,B)$, we have $A\hookrightarrow B^{\downarrow}=\bigwedge_{x\in X}(A(x)\mapsto B^{\downarrow}(x))=\bigwedge_{x\in X_a}\left(a\mapsto\bigwedge_{y\in B}R(x,y)\right)$;
$B^{\downarrow}\hookrightarrow A=\left(\bigwedge_{x\in X_a}\left(\bigwedge_{y\in B}(R(x,y)\mapsto a)\right)\right)\wedge\left(\bigwedge_{x\in X_a^c}\left(\bigwedge_{y\in B}(R(x,y)\mapsto 0)\right)\right)$.

$\mathscr{D}^{\downarrow}(A,B)=\bigwedge_{x\in X_a^c}\left(\bigwedge_{y\in B}(R(x,y)\mapsto 0)\right)\wedge\bigwedge_{x\in X_a}\left(\bigwedge_{y\in B}(R(x,y)\mapsto a)\right)\bigwedge_{x\in X_a}\left(a\mapsto\bigwedge_{y\in B}R(x,y)\right)$.

Under assumption of $(\dagger_*)$ or $(\dagger_R)$ the formula can be simplified and we get
$\mathscr{D}(A,B)=\left(\bigwedge_{x\in X_a}(a\mapsto\bigwedge_{y\in B}R(x,y))\right)\wedge\left(\bigwedge_{x\in X_a}\left(\bigwedge_{y\in B}R(x,y)\mapsto a\right)\right)$.

Example 5.6 Let now $A\subseteq X$, $Y_b\subseteq Y$, $L=[0,1]$, $b\in(0,1)$ and $B: Y\to L=[0,1]$ be defined by

$$B(y)=\begin{cases} b & \text{if } y\in Y_b \\ 0, & \text{if } y\notin Y_b.\end{cases}$$

Then, under assumption of $(\dagger_*)$ or $(\dagger_R)$, calculating similar as in the previous example, we get:

$\mathscr{D}(A,B)=\left(\bigwedge_{y\in Y_b}(b\mapsto\bigwedge_{xx\in A}R(x,y))\right)\wedge\left(\bigwedge_{y\in Y_b}(\bigwedge_{x\in A}(R(x,y)\mapsto b))\right)$.

Example 5.7 We demonstrate the previously obtained formulas for calculating $\mathscr{D}(A,B)$ in case of the Example 5.5 for the three basic t-norms $*$ on $[0,1]$: $*_{\wedge}=\wedge$ - the minimum t-norm, $*_L$ - the Łukasiewicz t-norm and $*_P$ - the product t-norm, see, e.g. [20].

(1) *Łukasiewicz t-norm* has zero divisors. Therefore, to simplify situation we will consider the case when $X_a=X$, $B=Y$, that is in case when assumption $(\dagger_R)$ is satisfied. Then from the above formulas we have
$\mathscr{D}^{\uparrow}(A,B)=\left(\bigwedge_{x\in X,y\in Y}(1-a+R(x,y))\right)\wedge 1$;
$\mathscr{D}^{\downarrow}(A,B)=\left(1-\bigwedge_{y\in Y}R(x,y)+a\right)\wedge 1$ and hence

$$\mathscr{D}(A,B)=\bigwedge_{x\in X,y\in Y}(1-|a-R(x,y)|).$$

(2) *The product t-norm* has no zero-divisors, that is satisfies assumption $(\dagger_*)$. Hence, under this assumption we can apply formulas obtained in Example 5.5, that is
$\mathscr{D}(A,B)=\left(\bigwedge_{x\in X_a}(a\mapsto\bigwedge_{y\in B}R(x,y))\right)\wedge\left(\bigwedge_{x\in X_a}\left(\bigwedge_{y\in B}(R(x,y)\mapsto a)\right)\right)$.

To describe $\mathscr{D}(A,B)$ in this situation we denote
$X_1=\{x\in X\mid a\leq\bigwedge_{y\in B}R(x,y)\}$,
$X_2=\{x\in X\mid a\geq\bigwedge_{y\in B}R(x,y)\}$. Then $\mathscr{D}(A,B)=\left(\bigwedge_{x\in X_1}\frac{a}{\bigwedge_{y\in B}R(x,y)}\right)\wedge\left(\bigwedge_{x\in X_2}\frac{\bigwedge_{y\in B}R(x,y)}{a}\right)$ if $X_1\cup X_2\neq\emptyset$ and $\mathscr{D}(A,B)=1$ otherwise.

(3) *The minimum t-norm* has no zero-divisors, that is satisfies assumption $(\dagger_*)$. Therefore, using formulas obtained in Example 5.5, we have

$$\mathscr{D}(A,B)=\begin{cases}\bigwedge_{x\in X_2,y\in Y}R(x,y) & \text{if } X_2\neq\emptyset;\\ a & \text{otherwise}.\end{cases}\qquad\square$$

5.2 $\mathscr{D}$-graded preconcept lattices

Given a fuzzy context (X,Y,L,R), let $(\mathbb{P}(X,Y,L,R),\preceq)$ be the corresponding fuzzy preconcept lattice, and let $\mathscr{D}^{\uparrow}$, $\mathscr{D}^{\downarrow}$ be the operators defined in the previous subsection. Then the tuple $(\mathbb{P},\preceq,\mathscr{D}^{\uparrow},\mathscr{D}^{\downarrow})$ will be referred to as a $\mathscr{D}$-graded fuzzy preconcept lattice of the fuzzy context (X,Y,L,R). In the next two theorems we prove the most important properties of the operators of $\mathscr{D}$-gradation and $\mathscr{D}$-graded fuzzy preconcept lattices.

Theorem 5.8 *Let $\mathbb{P}=(\mathbb{P},\preceq\veebar,\barwedge)$ be a fuzzy preconcept lattice. Given a family of fuzzy preconcepts $\mathscr{P}=\{P_i=(A_i,B_i)\mid i\in I\}\subseteq\mathbb{P}$ it holds*

$$\mathscr{D}^{\uparrow}(\veebar_{i\in I}P_i)\geq\bigwedge_{i\in I}\mathscr{D}^{\uparrow}(P_i)$$

Proof The proof follows from the next series of (in)equalities:
$\mathscr{D}^{\uparrow}(\veebar_{i\in I}(A_i,B_i))=(\bigvee_{i\in I}A_i)^{\uparrow}\cong(\bigwedge_{i\in I}B_i)=(\bigwedge_{i\in I}A_i^{\uparrow})\cong(\bigwedge_{i\in I}B_i)=\left((\bigwedge_{i\in I}A_i^{\uparrow})\hookrightarrow(\bigwedge_{i\in I}B_i)\right)\wedge\left((\bigwedge_{i\in I}A_i^{\uparrow})\hookleftarrow(\bigwedge_{i\in I}B_i)\right)\geq\left(\bigwedge_{i\in I}(A_i^{\uparrow}\hookrightarrow B_i)\right)\wedge\left(\bigwedge_{i\in I}(B_i\hookrightarrow A_i^{\uparrow})\right)=\bigwedge_{i\in I}\mathscr{D}^{\uparrow}(A_i,B_i)$. $\square$

Theorem 5.9 *Given a family of fuzzy preconcepts $\mathscr{P}=\{P_i=(A_i,B_i)\mid i\in I\}\subseteq\mathbb{P}$ it holds*

$$\mathscr{D}^{\downarrow}(\barwedge_{i\in I}P_i)\geq\bigwedge_{i\in I}\mathscr{D}^{\downarrow}(P_i).$$

The proof follows from the next series of (in)equalities:
$\mathscr{D}^{\downarrow}(\barwedge_{i\in I}(A_i,B_i))=(\bigwedge_{i\in I}A_i)\cong(\bigvee_{i\in I}B_i)^{\downarrow}=(\bigvee_{i\in I}A_i)\cong(\bigwedge_{i\in I}B_i^{\downarrow})=\left((\bigwedge_{i\in I}A_i)\hookrightarrow(\bigwedge_{i\in I}B_i^{\downarrow})\right)\wedge\left((\bigwedge_{i\in I}A_i)\hookleftarrow(\bigwedge_{i\in I}B_i^{\downarrow})\right)\geq\left(\bigwedge_{i\in I}(A_i\hookrightarrow B_i^{\downarrow})\right)\wedge\left(\bigwedge_{i\in I}(A_i\hookleftarrow B_i^{\downarrow})\right)=\bigwedge_{i\in I}\mathscr{D}^{\downarrow}(A_i,B_i)$. $\square$

6 Conclusions

Noticing limitation of concept lattices in case of a fuzzy context in view of possible applications, especially for "real world" problems, we introduce a very general notion of a preconcept on one hand, and on the other hand resrtrict it by assigning to a preconcept a certain *degree of its conceptuality*. In the result we come to what we call *a graded fuzzy preconcept lattice.* In our paper [29] we proposed two approaches of gradation of preconcept lattices. Here we go further with the study of the first one of these approaches, namely the one which is based on the evaluation of a certain mutual contentment of a fuzzy set of potential objects and a fuzzy set of its potential attributes. We call it *an inner approach* and the graded fuzzy preconcept lattice obtained in this way by $\mathscr{D}$*-graded*.

Concerning the immediate plans for the future work, we consider theoretical as well as practical issues. With regard to the perspectives in the research of theoretical issues, we see both a deeper study of the fuzzy $\mathscr{D}$-graded lattices themselves (in particular the research of categorical properties and constructions of fuzzy $\mathscr{D}$-graded preconcept lattices - products, coproducts, quotients et al. seems to us very challenging) and the study of the relations between fuzzy $\mathscr{D}$-graded lattices and other fuzzy mathematical structures. Specifically, we see challenging to reveal the relations between fuzzy graded, in particular, $\mathscr{D}$-graded, preconcept lattices on one side and fuzzy rough structures, fuzzy topology and fuzzy mathematical morphology on the other.

We have various plans for application of graded fuzzy concept lattices in the research of real-world problems. We think that the use of graded preconcept lattices could be more appropriate for certain practical problems, in particular, those ones that are related to risk analysis due to the possibility to establish a more flexible relationship between the fuzzy set of objects and corresponding fuzzy set of attributes. Specifically, one of areas where the use of fuzzy graded preconcept lattices could be very appropriate is the analysis of pandemic spread of an infection. Graded fuzzy preconcept lattices can serve as a basis for introducing a model for estimated total number of infected inhabitants vs estimated levels of hospitalised inhabitants which have been assessed based on expert opinion. At present we are working on a paper "Application of Graded Fuzzy Preconcept Lattices in Risk Analysis" where such model will be developed. In the process of developing this model, the analysis of application of different t-norms in the calculation of the degree of conceptuality has revealed differences in the impact of the selected t-norms on the assessment of the crisis severity. The obtained results can be further applied in the implementation of public restrictions aimed at containment of Covid-19 and other pandemics. The results of our research can be extended to comparing opinions from different experts. The aggregation of multiple opinions analysed by means of fuzzy preconcept lattices can be also a subject to the further research.

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