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# Similarity-based Reasoning from the Perspective of Extensionality

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## Abstract

Similarity-based reasoning systems consider the notion of similarity as crucial for their motivation and the whole design. However, similarity can be seen not only as a general notion but also as a particular family of binary fuzzy relations modeling a fuzzy equality. Moreover, these relations are tightly connected to another crucial notion – to the extensionality of fuzzy sets. This contribution focuses on this natural bridge and studies similarity-based reasoning fuzzy inference systems from the point of view of extensionality. Robustness of these systems is the first property that is considered in the newly set-up framework. However, the potential impact and topics of interest are much wider.

**Keywords:** Similarity-based reasoning, Similarity, Extensionality, Extensional fuzzy numbers, Approximate reasoning

## 1 Introduction

Fuzzy inference systems are undoubtedly the crucial fuzzy models that initiated huge development of theoretical results as well as practical applications. The idea of inferring conclusions based on a knowledge base formulated by fuzzy (IF-THEN) rules and observations found its application domains in expert systems, automatic control, decision-making systems, and other areas. Since the introduction of first models initiated by the seminal work of Mamdani and Assilian [17], the development went through fuzzy relational models [6], some closely related models [19, 20], interpolative reasoning [3, 16], and it did naturally include the idea of incorporating similarity as the key concept [7, 23].

Broadly speaking, out of the above-mentioned ones, there are two major types of fuzzy inference systems, in particular, Fuzzy Relational Systems and Similarity-Based Reasoning (SBR). While the former has seen sustained interest from researchers that led to a deep knowledge about the preservation of distinct properties of these systems, the latter one, although equally interesting, still leaves room for the investigation of natural questions and perspectives. This work is an attempt at offering one such perspective, wherein we view the output of an SBR system in terms of the extensionality of the fuzzy sets involved.

Note that the term 'similarity' in the nomenclature refers to how the inference intrinsically depends on the similarity between the given input and the antecedents. However, extensionality deals with a different kind of similarity – that defined on the domain of the antecedents.

It is well known that a collection of fuzzy sets $\{A_i\} \subseteq \mathscr{F}(X)$ does give rise to a similarity relation on $X$ [13]. In this work, we show that the entire process of the SBR can be seen as reasoning with extensional fuzzy sets when some appropriate functions are employed.

## 2 Preliminaries

For the whole paper, we fix a residuated lattice $\langle [0,1], \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ [9] as the underlying structure of membership values and operations on them. Note that $\otimes$ is a left-continuous t-norm [15] and $\rightarrow$ its adjoint residuated fuzzy implication [1].

Furthermore, we fix some non-empty input and output universes $X$ and $Y$, respectively. For the sake of simplicity and comprehension, we may restrict our choice of $X$ and $Y$ to closed real intervals. Such a restriction is practical and not restrictive, having in mind the fact that we deal with fuzzy inference systems that map some "values" from the input space to "values" from the output space. Indeed, this setting formalizes the SISO case, and the generalization to the MISO case is straightforward. Let $\mathbb{N}_n$ denote the set $\{1,\dots,n\}$, $n \in \mathbb{N}$.

### 2.1 Similarity-based reasoning

Similarity-based reasoning systems [5, 7, 8, 18, 22, 23] belong to very interesting logically motivated inference systems with a strong background in fuzzy approximate reasoning. Their structure is slightly more flexible than, say, the more standard fuzzy relational ones, where a single fuzzy relation gathers all knowledge base information. However, they all possess a clear idea behind the inferring principles. The idea is that similar inputs should lead to similar outputs, or more particularly, inputs similar to antecedents should lead to outputs similar to corresponding consequents. This principle is not surprising as it actually turns out to be implicitly involved in other systems as well. For example, this "continuous" behavior of a system is equivalent to its preservation of modus ponens, formally expressed in its interpolativity (solvability of the related system of fuzzy relation equations), whenever we deal with fuzzy relational inference systems, see [21, 24]. The relationship to fuzzy relational inference systems can be shown easily, as, for some specific choices of operations, the similarity-based reasoning inference systems become equivalent to the usual settings of fuzzy relational inference systems [4]. The main difference lies in the fact that the main principle of inference described above is apriori implemented in the construction of the similarity-based reasoning, not being a consequence of the preservation of another crucial property.

Let us briefly recall the main principles and components of the similarity-based reasoning and let us fix the notation for the rest of the article. Let us be given a fuzzy rule base consisting of pairs of antecedent and consequent fuzzy sets $(A_i, B_i)$, where $A_i \in \mathscr{F}(X)$, $B_i \in \mathscr{F}(Y)$, and $i \in \mathbb{N}_n$. Let us consider an input $A' \in \mathscr{F}(X)$. The similarity-based reasoning can be split into three phases.

**Step 1: Matching Antecedent(s) to the Input:** The first phase consists in the determination of the similarity degree between the input $A'$ and each particular antecedent fuzzy set $A_i, i \in \mathbb{N}_n$. This determination is done by a *matching function* $M\colon \mathscr{F}(X) \times \mathscr{F}(X) \to [0,1]$. We may recall some typical examples, e.g., the *Morsi & Fahmy matching function* [18]:

$$M_{\mathbf{MF}}(A', A_i) = \bigwedge_{x \in X} (A'(x) \to A_i(x)) \,, \tag{1}$$

or the most usual *Zadeh matching function* [26]:

$$M_{\mathbf{Z}}(A', A_i) = \bigvee_{x \in X} (A'(x) \otimes A_i(x)) \,. \tag{2}$$

Note that $M_Z$ was originally designed with the minimum t-norm $\wedge$; however, we can freely consider a strong conjunction $\otimes$ from any residuated lattice. The original version will then become a special case when we consider the Gödel algebra.

**Step 2: Modifying the Consequent(s):** Let us denote the similarity value obtained by a matching function as $s_i = M(A', A_i)$, where $i \in \mathbb{N}_n$. Now, the corresponding consequent $B_i$ is modified using $s_i$ with the help of a modification function $J\colon [0,1]^2 \to [0,1]$, and is given, for any $y \in Y$ and $i \in \mathbb{N}_n$, by

$$B_i'(y) = J(s_i, B_i(y)) = J(M(A', A_i), B(y)) \,.$$

In [23], the following modification functions have been used:

(i) $J_{\mathbf{ML}}(s, B(y)) = \min\{1, B(y)/s\}$ for $s > 0$ and $J_{\mathbf{ML}}(s, B(y)) = 1$ otherwise, $y \in Y$;

(ii) $J_{\mathbf{MVR}}(s, B(y)) = s \cdot B(y)$ , $y \in Y$.

In [5] and [18], $J$ is taken to be a fuzzy implication operator. In fact, $J_{\mathbf{ML}}$ is the Goguen implication [1].

**Step 3: Aggregating the Modified Consequents:**

We infer the final output by aggregating the modified consequents $B_i'$ using an associative operator $G\colon [0,1]^2 \to [0,1]$:

$$B'(y) = G_{i=1}^n \left( J\left(M(A', A_i), B_i(y)\right)\right), \quad y \in Y.$$

Usually, $G$ is a $t$-norm, $t$-conorm, or a uninorm [15].

An SBR fuzzy inference system can be represented by the quadruple $\mathbb{F} = \{\mathscr{R}(A_i, B_i), M, J, G\}$, where

- $\mathscr{R}$ is a fuzzy rule base formed from collections of fuzzy sets $\{A_i\}$ and $\{B_i\}$ on $X$ and $Y$, respectively,
- $M$ is a matching function,
- $J$ is a modification function,
- $G$ is an aggregation function.

Thus, an SBR system forms a mapping $\psi : \mathscr{F}(X) \to \mathscr{F}(Y)$ determined by $\mathbb{F}$.

## 3 Extensional hulls and bases

### 3.1 Similarity relations and extensionality

**Definition 3.1** Fuzzy relation $S : X^2 \to [0,1]$ is a $\otimes$-*similarity* on $X$ if the following properties are fulfilled for any $a, b, c \in X$:

a) $S(a,a) = 1$ , (reflexivity)

b) $S(a,b) = S(b,a)$ , (symmetry)

c) $S(a,b) \otimes S(b,c) \leq S(a,c)$ . ($\otimes$-transitivity)

Let us consider the fixed underlying residuated algebraic structure. Then by a similarity space $(X,S)$ we refer to a set $X$ with a similarity relation $S$ w.r.t. $\otimes$ defined on it.

Let $\otimes$ be the Łukasiewicz t-norm. Then, $S_p(x,y) = (1 - p|x-y|) \vee 0$ for $x,y \in \mathbb{R}$, $p > 0$, is a $\otimes$-similarity on $\mathbb{R}$ called Łukasiewicz similarity.

The most important property related to similarity relations is the extensionality.

**Definition 3.2** [2] Let $(X,S)$ be a similarity space. If a fuzzy set $A \in \mathscr{F}(X)$ meets the following condition for arbitrary $a,b \in X$:

$$A(a) \otimes S(a,b) \leq A(b) , \tag{3}$$

then it is called to be *extensional* with respect to $S$.

### 3.2 Extensional hulls

Extensional fuzzy sets can be intuitively seen as "nice" fuzzy sets that gather similar elements of the given universe. Hence, the chosen algebra and the chosen similarity $S$ play crucial roles for the final shape of the extensional fuzzy sets. The least extensional fuzzy superset of a given fuzzy set is called *extensional hull*.

**Definition 3.3** Let $S$ be a $\otimes$-similarity on $X$ and let $E_S(X) = \{B \in \mathscr{F}(X) \mid B \text{ is extensional w.r.t. } S\}$. Consider $A \in \mathscr{F}(X)$. Then the *extensional hull of A w.r.t. S* is the fuzzy set $\mathrm{HULL}_S(A) \in \mathscr{F}(X)$:

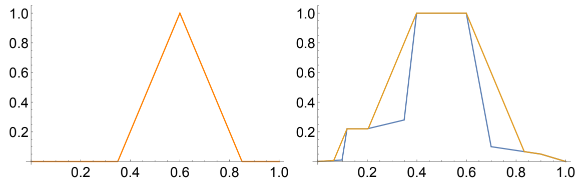
$$\mathrm{HULL}_S(A) = \bigwedge \{B \mid A \subseteq B \,\&\, B \in E_S(X)\} .$$

The extensional hull may serve as a sort of closest extensional upper approximation of a given fuzzy set that is not "nice", i.e., that does not meet our extensionality requirement. The proposition below shows how it can be determined from the original fuzzy set and the given similarity.

**Proposition 3.1** *[13] Let $(X,S)$ be a similarity space and $A \in \mathscr{F}(X)$. Then*

$$\mathrm{HULL}_S(A)(a) = \bigvee_{b \in X} (A(b) \otimes S(a,b)) , \quad a \in X .$$

If we consider a singleton fuzzy set $A \in \mathscr{F}(X)$ given by $A(a) = 1$ and $A(x) = 0$ for $x \neq a$ as a representation of a crisp number $a \in X$, we may construct its extensional hull and neglect the difference between $A$ and $a$. We get $\mathrm{HULL}_S(a)(x) = S(a,x)$, and obtain an object called fuzzy point [13, 14] or *extensional fuzzy number* [11, 12]. We denote this fuzzy set by $a_S$, i.e., $a_S = \mathrm{HULL}_S(a)$.




(a) (b)

Figure 1: Extensional hulls (orange) of (a) a singleton at 0.6 and (b) a general fuzzy set (blue) w.r.t. the Łukasiewicz similarity with $p = 4$.

### 3.3 Extensional bases

If we are given an $A \in E_S(X)$, then $\mathrm{HULL}_S(A) = A$. Moreover, let $S_1, S_2$ be two $\otimes$-similarities on $X$ such that $S_1(a,b) \leq S_2(a,b)$ for arbitrary $a,b \in X$. For example, this holds if $S_1, S_2$ are Łukasiewicz similarities with $p_1$, $p_2$ such that $p_1 > p_2$. Then for arbitrary $A \in E_{S_2}(X)$ it holds that also that $A \in E_{S_1}(X)$, see (3). Therefore, for $A \notin E_{S_1}(X)$, we obtain $\mathrm{HULL}_{S_1}(A) \supset A$; however, for $B \in E_{S_2}(X)$, we obtain $\mathrm{HULL}_{S_1}(B) = \mathrm{HULL}_{S_2}(B) = B$. In other words, the extensional hull "extends" the given fuzzy set to a wider one but not to a too wide one, where the size is determined by the used similarity. However, if we consider too wide fuzzy sets, they remain too wide. Constructing their hulls is not resizing them to a pre-given width, since it acts this way only on smaller fuzzy sets.

We may ask for the dual approach that would make smaller fuzzy sets from too wide ones. The duality principle can be used towards such a construction of the new concept of an *extensional base* of a fuzzy set.

**Definition 3.4** Let $(X,S)$ be a similarity space and $A \in \mathscr{F}(X)$. Then the *extensional base of A w.r.t. S* is the fuzzy set $\mathrm{BASE}_S(A) \in \mathscr{F}(X)$:

$$\mathrm{BASE}_S(A) = \bigvee \{B \mid A \supseteq B \,\&\, B \in E_S(X)\} .$$

Natural question arises, in particular, whether the extensional base behaves analogously to the extensional hull. The following characterization presents a base in a light analogous to that in which hulls were presented in [13].

**Proposition 3.2** *Let $(X,S)$ be a similarity space and*

$A \in \mathscr{F}(X)$. *Then*

$$(i)\, \mathrm{BASE}_S(A)(a) = \bigwedge_{b\in X} (S(a,b) \to A(b)) \ , \ a \in X \ ,$$
$$(ii)\, \mathrm{BASE}_S(A) \in E_S(X) \ ,$$
$$(iii)\, \mathrm{BASE}_S(\mathrm{BASE}_S(A)) = \mathrm{BASE}_S(A) \ .$$

*Sketch of the proof:* Let us denote by $\bar{A}$ the fuzzy set

$$\bar{A}(a) = \bigwedge_{b\in X} (S(a,b) \to A(b)) \ .$$

Then we may write

$$\bar{A}(a) \le S(a,a) \to A(a) = A(a) \ ,$$

hence, $\bar{A} \subseteq A$. Furthermore,

$$\begin{aligned} S(a,b) \to \bar{A}(b) &= S(a,b) \to \bigwedge_c (S(b,c) \to A(c)) \\ &= \bigwedge_c (S(a,b) \to (S(b,c) \to A(c))) \\ &= \bigwedge_c ((S(a,b) \otimes S(b,c)) \to A(c)) \\ &\ge \bigwedge_c (S(a,c) \to A(c)) = \bar{A}(a) \ . \end{aligned}$$

Hence, $\bar{A} \in E_S(X)$, which together with $\bar{A} \subseteq A$ leads to $\bar{A} \subseteq \mathrm{BASE}_S(A)$.

Consider $B \in E_S(X)$ s.t. $B \subseteq A$. Then, for any $b \in X$,

$$B(a) \le S(a,b) \to B(b) \le S(a,b) \to A(b) \ ,$$

hence,

$$B(a) \le \bigwedge_b (S(a,b) \to A(b)) = \bar{A}(a) \ .$$

Consequently, we get $B \subseteq \bar{A}$; thus, $\mathrm{BASE}_S(A) \subseteq \bar{A}$, which together with the above leads to $\mathrm{BASE}_S(A) = \bar{A}$. □

**Corollary 3.3** *Let* $A \in E_S(X)$. *Then*

$$\mathrm{BASE}_S(A) = A = \mathrm{HULL}_S(A) \ .$$

### 3.4 Relationship to the matching functions

Let us consider the antecedent $A$ to be an extensional hull of $a \in \mathbb{R}$, i.e., let $A = \mathrm{HULL}_S(a) = a_S = S(a,\cdot)$. If the matching function is $M_{\mathbf{Z}}$, then, for any input $A' \in \mathscr{F}(X)$, we get

$$\begin{aligned} M_{\mathbf{Z}}(A',A) = M_{\mathbf{Z}}(A',a_S) &= \bigvee_{x\in X} \big(A'(x) \otimes \mathrm{HULL}_S(a)(x)\big) \\ &= \bigvee_{x\in X} \big(A'(x) \otimes S(a,x)\big) = \mathrm{HULL}_S(A')(a) \ . \end{aligned}$$

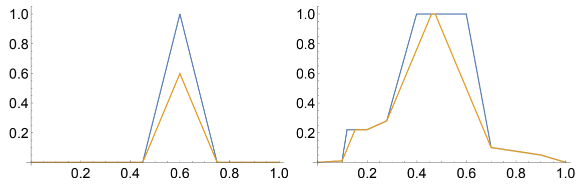



Figure 2: Extensional bases (orange) of (a) a triangle (blue) and (b) a general fuzzy set (blue) w.r.t. the Łukasiewicz similarity with $p = 4$.

It means that the matching function $M_{\mathbf{Z}}$ applied to an input and an antecedent that is the extensional hull of $a \in \mathbb{R}$ results in nothing else than the membership degree of $a$ in the extensional hull of the input $A'$. The following lemma and corollary show that this can be generalized for an arbitrary extensional fuzzy set in a similarity space $(X,S)$.

**Lemma 3.4** *Let* $(X,S)$ *be a similarity space. Then, for arbitrary* $A, A' \in \mathscr{F}(X)$:

$$M_{\mathbf{Z}}(A',\mathrm{HULL}_S(A)) = M_{\mathbf{Z}}(\mathrm{HULL}_S(A'),A) \ . \qquad (4)$$

*Sketch of the proof:* Since $\otimes$ is $\vee$-preserving, we have

$$\begin{aligned} M_{\mathbf{Z}}(A',\mathrm{HULL}_S(A)) &= \bigvee_{x\in X} \big(A'(x) \otimes \mathrm{HULL}_S(A)(x)\big) \\ &= \bigvee_{x\in X} \Big(A'(x) \otimes \bigvee_{t\in X} (A(t) \otimes S(t,x))\Big) \\ &= \bigvee_{t\in X} \Big(A(t) \otimes \bigvee_{x\in X} (A'(x) \otimes S(x,t))\Big) \\ &= \bigvee_{t\in X} (A(t) \otimes \mathrm{HULL}_S(A')(t)) \\ &= M_{\mathbf{Z}}(\mathrm{HULL}_S(A'),A) \ . \end{aligned}$$

□

**Corollary 3.5** *Let* $(X,S)$ *be a similarity space. If* $A \in E_S(X)$, *then, for any* $A' \in \mathscr{F}(X)$, *we have* $M_{\mathbf{Z}}(\mathrm{HULL}_S(A'),A) = M_{\mathbf{Z}}(A',A)$.

An analogous property can be expected for the extensional base if we deal with an appropriate matching function. Consider the following matching function:

$$M_{\mathbf{DJS}}(A',A_i) = \bigwedge_{x\in X} (A_i(x) \to A'(x)). \qquad (5)$$

It is, at first sight, similar to $M_{\mathbf{MF}}$, but with a reversed order of $A'$ and $A$ inside. This might seem a bit unintuitive. Indeed, a narrow (e.g., a singleton) input in the middle of a wide antecedent will lead to the $M_{\mathbf{DJS}}$

determining the similarity to be zero. However, as we will show, this concept will turn to be useful and fully fitting in a specific, yet not unusual setting.

Let us again consider a similarity space $(X,S)$ and let the antecedent $A$ be the extensional hull of $a \in \mathbb{R}$, i.e., $A = \mathrm{HULL}_S(a) = a_S = S(a,\cdot)$. If the matching function is $M_{\mathbf{DJS}}$, then, for any input $A' \in \mathscr{F}(X)$, we get

$$M_{\mathbf{DJS}}(A',A) = \bigwedge_{x\in X} (\mathrm{HULL}_S(a)(x) \to A'(x)) = \bigwedge_{x\in X} (S(a,x) \to A'(x)) = \mathrm{BASE}_S(A')(a) .$$

It means that $M_{\mathbf{DJS}}$ of an input and an antecedent that is the hull of $a \in \mathbb{R}$ is nothing else but the membership degree of $a$ in the extensional base of the input $A'$.

Analogously, we can mimic the results formulated in Lemma 3.4 for extensional bases and the $M_{\mathbf{DJS}}$ matching function.

**Lemma 3.6** *Let $(X,S)$ be a similarity space. Then, for arbitrary $A,A' \in \mathscr{F}(X)$:*

$$M_{\mathbf{DJS}}(A', \mathrm{HULL}_S(A)) = M_{\mathbf{DJS}}(\mathrm{BASE}_S(A'),A) . \quad (6)$$

*Sketch of the proof:*

$$\begin{aligned} M_{\mathbf{DJS}}(A', \mathrm{HULL}_S(A)) &= \bigwedge_{x\in X} \big(\mathrm{HULL}_S(A)(x) \to A'(x)\big) \\ &= \bigwedge_{x\in X} \left(\bigvee_{t\in X} (A(t)\otimes S(t,x)) \to A'(x)\right) \\ &= \bigwedge_{x\in X}\bigwedge_{t\in X} \big((A(t)\otimes S(x,t)) \to A'(x)\big) \\ &= \bigwedge_{x\in X}\bigwedge_{t\in X} \big(A(t) \to \big(S(x,t) \to A'(x)\big)\big) \\ &= \bigwedge_{t\in X} \left(A(t) \to \bigwedge_{x\in X} \big(S(x,t) \to A'(x)\big)\right) \\ &= M_{\mathbf{DJS}}(\mathrm{BASE}_S(A'),A) . \end{aligned}$$

□

Lemma 3.6 directly leads to the following corollary.

**Corollary 3.7** *Let $(X,S)$ be a similarity space. If $A \in E_S(X)$ then, for any $A' \in \mathscr{F}(X)$, we have $M_{\mathbf{DJS}}(\mathrm{BASE}_S(A'),A) = M_{\mathbf{DJS}}(A',A)$.*

We leave a deeper discussion on the impact of the provided results, namely, of Corollaries 3.5 and 3.7, for the next section.

If one considers the Morsi-Fahmy matching function $M_{\mathbf{MF}}$ given by (1), one sees that the extensional base plays an analogous role as the extensional hull for the Zadeh's matching function, which is captured by the following result.

**Lemma 3.8** *Let $(X,S)$ be a similarity space. Then, for arbitrary $A,A' \in \mathscr{F}(X)$:*

$$M_{\mathbf{MF}}(A', \mathrm{BASE}_S(A)) = M_{\mathbf{MF}}(\mathrm{HULL}_S(A'),A) .$$

*Sketch of the proof:* Using an analogous technique to the proof of Lemma 3.6. □

The impact of Lemma 3.8 is analogous to the previous ones, i.e., under the assumption that the antecedent is extensional, we obtain a sort of robustness (non-sensitivity) of the matching function with respect to the input and its extensional hull.

**Corollary 3.9** *Let $(X,S)$ be a similarity space. If $A \in E_S(X)$, then, for any $A' \in \mathscr{F}(X)$, we have $M_{\mathbf{MF}}(\mathrm{HULL}_S(A'),A) = M_{\mathbf{MF}}(A',A)$.*

## 4 SBR from the perspective of extensional fuzzy numbers

### 4.1 Robustness of SBR systems

As has already been remarked, the SBR inference derives its name due to its dependence on the "similarity" between a given input and the antecedents of the IF-THEN rules. However, largely, this inference scheme has remained imperceptive of the similarity in the sense of fuzzy equivalence (relation) on the underlying domain $X$.

The results presented in the preceding section focusing on the relationship between extensionality and matching functions allow us to formulate easily further results with the direct impact on the *robustness* (in the sense of Klawonn & Castro). First, we need to prepare the environment by the following lemma.

**Lemma 4.1** *Let $(X,S)$ be a similarity space. Let us consider an SBR fuzzy inference system $\mathbb{F} = \{\mathscr{R}(A_i,B_i),M,J,G\}$ such that $M = M_{\mathbf{Z}}$ and all $A_i \in \mathscr{F}(X)$ would be given as $A_i = \mathrm{HULL}_S(a_i)$ for some crisp values $a_i \in X$. Then, for arbitrary $A' \in \mathscr{F}(X)$,*

$$M_{\mathbf{Z}}(\mathrm{HULL}_S(A'),A_i) = M_{\mathbf{Z}}(A',A_i) . \quad (7)$$

*Sketch of the proof:* See Lemma 3.4. □

The most usual fuzzy relation inference systems apply so-called *First-Aggregate-Then-Infer* (FATI) strategy that gathers all rules into a single fuzzy relation, while the SBR systems in principle apply the so-called *First-Infer-Then-Aggregate* (FITA) strategy [10]. It means that each rule is used to infer the partial result independently of the others, and only after this step is applied to all rules, the partial results are aggregated to the final result. Consequently, SBR can be seen as a concept built from separate blocks. If the first block – the matching function – generates the same matching/similarity values for two inputs, then we necessarily obtain the same outputs for both.

For a particular setting above, we can formulate the following corollary.

**Corollary 4.2** *Let $(X,S)$ be a similarity space. Let us consider an SBR fuzzy inference system $\mathbb{F} = \{\mathscr{R}(A_i,B_i),M,J,G\}$ such that $M = M_{\mathbf{Z}}$ and all $A_i \in \mathscr{F}(X)$ would be given as $A_i = \mathrm{HULL}_S(a_i)$ for some crisp values $a_i \in X$. Then, for arbitrary $A' \in \mathscr{F}(X)$,*

$$\psi(\mathrm{HULL}_S(A')) = \psi(A') \ .$$

Analogous results may be mimicked for the setting using the $M_{\mathbf{DJS}}$ matching function in connection with extensional bases.

**Lemma 4.3** *Let $(X,S)$ be a similarity space. Let us consider an SBR fuzzy inference system $\mathbb{F} = \{\mathscr{R}(A_i,B_i),M,J,G\}$ such that $M = M_{\mathbf{DJS}}$ and all $A_i \in \mathscr{F}(X)$ would be given as $A_i = \mathrm{HULL}_S(a_i)$ for some crisp values $a_i \in X$. Then, for arbitrary $A' \in \mathscr{F}(X)$,*

$$M_{\mathbf{DJS}}(\mathrm{BASE}_S(A'),A_i) = M_{\mathbf{DJS}}(A',A_i) \ . \quad (8)$$

*Sketch of the proof:* See Lemma 3.6. □

And again, the direct corollary can be formulated as follows.

**Corollary 4.4** *Let $(X,S)$ be a similarity space. Let us consider an SBR fuzzy inference system $\mathbb{F} = \{\mathscr{R}(A_i,B_i),M,J,G\}$ such that $M = M_{\mathbf{DJS}}$ and all $A_i \in \mathscr{F}(X)$ would be given as $A_i = \mathrm{HULL}_S(a_i)$ for some crisp values $a_i \in X$. Then, for arbitrary $A' \in \mathscr{F}(X)$,*

$$\psi(\mathrm{BASE}_S(A')) = \psi(A') \ .$$

The results formulated in Corollaries 4.2 and 4.4 provide a clear interpretation related to the above-recalled robustness [13] of fuzzy inference systems. Let all the antecedent fuzzy sets be extensional fuzzy numbers w.r.t a fixed similarity relation. Then the result of the SBR system with the Zadeh's matching function is independent on whether it processes the original input or its extensional hull w.r.t. the same similarity relation. Klawonn & Castro claim that it means that the imprecision stored in the antecedents is unavoidable; thus, it makes no sense to "measure" more precisely. Even if we process precise crisp numbers (singletons), we obtain the same results as if we process extensional fuzzy numbers of the same width as the antecedents. Analogously, if we use the DJS matching function, the inferred result is equivalent for the processed input as well as for its extensional base. This is indeed an analogous, yet dual, robustness – it is not suitable in the context of narrow fuzzy sets, where it could lead to the subnormality, while it seems to be very natural and helpful for the case of very wide input fuzzy sets, see Fig. 2.

Although the setting that requires antecedents to be extensional fuzzy numbers w.r.t. a fixed similarity relation is not restrictive from the engineering point of view at all, a natural question arises, whether the results may be even generalized. Below, we present such a generalization.

**Theorem 4.5** *Let us consider an SBR system $\mathbb{F} = \{\mathscr{R}(A_i,B_i),M,J,G\}$. Consider the collection of antecedent fuzzy sets $\mathscr{A} = \{A_i\}_{i=1}^n \subseteq \mathscr{F}(X)$. Let $S_{\mathscr{A}} \in \mathscr{F}(X \times X)$ be given by*

$$S_{\mathscr{A}}(x,y) = \bigwedge_{A_i \in \mathscr{A}} (A_i(x) \leftrightarrow A_i(y)) \ .$$

*Then*

$$\begin{aligned} M_{\mathbf{Z}}(A',A_i) &= M_{\mathbf{Z}}(\mathrm{HULL}_{S_{\mathscr{A}}}(A'),A_i) \ , \\ M_{\mathbf{DJS}}(A',A_i) &= M_{\mathbf{DJS}}(\mathrm{BASE}_{S_{\mathscr{A}}}(A'),A_i) \ , \\ M_{\mathbf{MF}}(A',A_i) &= M_{\mathbf{MF}}(\mathrm{HULL}_{S_{\mathscr{A}}}(A'),A_i) \ . \end{aligned}$$

*Sketch of the proof:* Based on Theorem 3.1 from [13], we know that $S_{\mathscr{A}}$ is the greatest $\otimes$-similarity relation on $X$ s.t. every $A_i$ is extensional w.r.t. this $S_{\mathscr{A}}$. Clearly, for any $A_i \in \mathscr{A}$ the following holds: $A_i = \mathrm{HULL}_{S_{\mathscr{A}}}(A_i) = \mathrm{BASE}_{S_{\mathscr{A}}}(A_i)$. Hence, using the results above, we obtain the claim. □

**Theorem 4.6** *Let us consider an SBR fuzzy inference system $\mathbb{F} = \{\mathscr{R}(A_i,B_i),M,J,G\}$ such that $M = M_{\mathbf{Z}}$. Then*

$$\psi(\mathrm{HULL}_{S_{\mathscr{A}}}(A')) = \psi(A') \ .$$

**Theorem 4.7** *Let us consider an SBR fuzzy inference system $\mathbb{F} = \{\mathscr{R}(A_i,B_i),M,J,G\}$ such that $M = M_{\mathbf{DJS}}$. Then*

$$\psi(\mathrm{BASE}_{S_{\mathscr{A}}}(A')) = \psi(A') \ .$$

**Theorem 4.8** *Let us consider an SBR fuzzy inference system $\mathbb{F} = \{\mathscr{R}(A_i,B_i),M,J,G\}$ such that $M = M_{\mathbf{MF}}$. Then*

$$\psi(\mathrm{HULL}_{S_{\mathscr{A}}}(A')) = \psi(A') \ .$$

### 4.2 Interpretation of the results

The above results show that, implicitly, for some matching functions, an SBR inference system respects the similarity relation imposed by the antecedents; thus, it is robust in its behavior.

For the comprehension of the different (dual) function of the hulls and bases, we recall explanatory Figs. 1 and 2. The hull extends the given fuzzy set to an extensional one, i.e., to a fuzzy set “covering” similar values from $X$, see Fig. 1. Dually, the base reduces the given fuzzy set, see Fig. 2. If a given fuzzy set is extensional w.r.t. the given similarity, neither the hull nor the base make any change.

Indeed, the base may reduce the fuzzy set too much, e.g., to a subnormal fuzzy set, see Fig. 2(a). This is not surprising; however, the proper and wanted effect of the base may be seen in Fig. 3. Similarly to the unwanted effect of the base, even the hull may lead to an inappropriate widening effect that occurs in the case of too wide similarity. Both concepts can be equally useful, each of them needs to be well-chosen for particular fuzzy sets and related similarity.

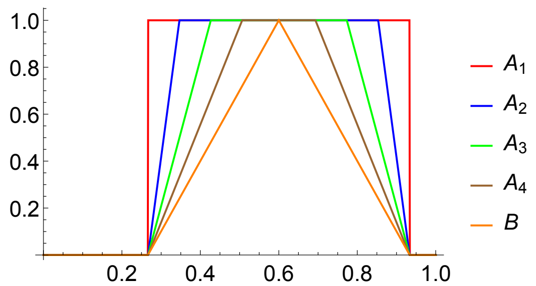



Figure 3: All fuzzy sets $A_i$, $i = 1,\dots,4$, produce the same extensional base $B$, i.e., $\mathrm{BASE}_{S_p}(A_i) = B$, for the Łukasiewicz similarity $S_p$ with $p = 3$.

Let us also briefly comment on the results formulated in Theorems 4.6-4.8, basically stemming from Lemma 3.4, Lemma 3.6, and Lemma 3.8, respectively.

The results related to the hulls applied to an input, i.e., formulated in Theorem 4.6 and Theorem 4.8, may be interpreted analogously to the results for the fuzzy relational systems published in [13], and later on in [24, 25]. In particular, it does not make sense to measure inputs more precisely, as the imprecision stored in the antecedent fuzzy sets is unavoidable. Due to the FITA inference strategy, this holds also for any SBR system with $M_{\mathbf{Z}}$ or $M_{\mathbf{MF}}$ matching function. The result related to the base applied to an input, i.e., formulated in Theorem 4.7, has a dual interpretation. It can be understood in a way that it makes no sense to make the input (or its core) wider than given by the similarity used in the antecedents, as the results will be equivalent. This dual yet philosophically equivalent nonsensitivity to changes in inputs is a desirable property that can be viewed as a system robustness.

However, we may shed also another light on the results from a bit different perspective. Instead of understanding it as telling us what does not make sense to insert into an SBR system as its input, i.e., either too narrow or too wide fuzzy sets, we may view it as what we can dare to insert into the system without any impact on its output. In particular, considering $M_{\mathbf{Z}}$ and $M_{\mathbf{MF}}$, we do not have to think of applying some optional fuzzification block at the beginning of the inference system, because as long as the fuzzified input would be within the inclusion of the extensional hull w.r.t. the similarity used for antecedents fuzzy sets, the output would not change. As a consequence, we may freely process crisp inputs represented as singletons. And, dually, for $M_{\mathbf{DJS}}$, there is no reason to apply some optional block reducing the vagueness of the inputs. We may freely process, e.g., intervals, the result will not be different from processing the extensional fuzzy numbers located at the centres of these intervals.

## 5 Conclusions and future work

In this paper, we studied similarity-based reasoning framework from the point of view of the important property of extensionality of fuzzy sets. Similarity-based reasoning systems can be characterized by three functions (matching, modification, and aggregation ones); they apply the FITA strategy. Consequently, the robustness of these systems can be characterized in terms of the matching function that measures the similarity of inputs to antecedent fuzzy sets.

Following the ideas and results of [13], which provided the notion of the extensional hull and showed its importance for fuzzy relational systems and their robustness, we introduced in this paper the dual notion of the extensional base of a fuzzy set. We showed that, in combination with the appropriate matching function, it allows us to obtain robustness for “wide” input fuzzy sets in an analogous way as for extensional hulls and “narrow” inputs.

In our future work, we will pursue further this approach and study the interplay between the matching functions proposed in this paper and the remaining components of SBR systems, namely, modification and aggregation functions. The goal is to propose an end-to-end SBR system respecting the notion of extensionality in all its components, and study its further properties, including interpolativity, continuity, and monotonicity.

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