

On Some Functional Equations Related to Alpha Migrative t-conorms

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Abstract

In this contribution, we analyse in details the recently introduced definition of migrative t-conorms [see *Fuzzy implications: alpha migrativity and generalised laws of importation*, M. Baczyński, B. Jayaram, R. Mesiar, 2020]. We also focus on some general functional equations, which might be obtained from such a notion. We concentrate on some particular well-known families of fuzzy implications and show solutions of those equations among this kind of fuzzy implication functions.

Keywords: Fuzzy connectives, T-conorm, Fuzzy implication, Migrativity.

1 Introduction

A notion of α -migrativity appeared in 2007, where authors dealt with the problem of a convex combination of t-norms [8]. It is useful from an application point of view since it shows when “proportional” change of one of the coordinates is equivalent of such a change for another argument. Therefore it is important which aggregation functions can ensure this property. Below we present the original definition of α -migrativity, which is a starting point for further investigations.

Definition 1.1 ([6, Definition 1]). Let $\alpha \in (0, 1)$. A binary operator $T: [0, 1]^2 \rightarrow [0, 1]$ is said to be α -migrative if it satisfies

$$T(\alpha x, y) = T(x, \alpha y), \quad x \in [0, 1].$$

A function T is called migrative if it is α -migrative for any $\alpha \in [0, 1]$.

Although a convex combination of two t-norms is not a t-norm, a notion of migrativity has been widely investigated in a context of different fuzzy connectives ([13])

and related concepts like bimigrativity (see [11]), or migrativity over t-norms [12]. One of the earliest papers provides the following fact.

Theorem 1.2 ([5, Corollary 2]). *There are no migrative t-conorms, uninorms or nullnorms (in the sense of Definition 1.1).*

The above sentence is, of course, correct but only under the consideration of Definition 1.1. However, other authors investigated migrativity for different fuzzy connectives – with another definition. Namely, in [3] fuzzy implications were examined. That particular definition of α -migrativity is associated with the graphical interpretation of it. Moreover, the authors explored functional equations connected with such migrativity. The above-mentioned article was an inspiration for us to examine t-conorms, especially when in [3] authors formulated the desired definition.

The paper is organized as follows. Section 2 contains some important definitions and theorems, while the main results are collected in Section 3.

2 Preliminaries

To make this work more self-contained, we placed some important definitions here.

Definition 2.1 ([9]). Let $n \in \mathbb{N}$. An aggregation function in $[0, 1]^n$ is a function $A^{(n)}: [0, 1]^n \rightarrow [0, 1]$ which is nondecreasing (in each variable) and it satisfies $A^{(n)}(0, \dots, 0) = 0$ and $A^{(n)}(1, \dots, 1) = 1$.

Definition 2.2 ([4, Definition 1.9]). An aggregation function $A^{(n)}: [0, 1]^n \rightarrow [0, 1]$ has disjunctive behavior (or is disjunctive) if for every $x = (x_1, \dots, x_n)$ it is bounded by

$$A(x) \geq \max(x) = \max(x_1, \dots, x_n).$$

Definition 2.3 ([7, 10]). A function $T: [0, 1]^2 \rightarrow [0, 1]$ is called a **triangular norm (t-norm in short)**, if it is associative, commutative and non-decreasing operator with a neutral element 1.

Definition 2.4 ([10]). A function $S: [0, 1]^2 \rightarrow [0, 1]$ is called a **triangular conorm** (**t-conorm** in short), if it is associative, commutative and non-decreasing operator with a neutral element 0.

Definition 2.5 ([15]). A function $U: [0, 1]^2 \rightarrow [0, 1]$ is called a **uninorm**, if it is an associative, commutative and non-decreasing operator with a neutral element $e \in [0, 1]$, i.e., such that $U(x, e) = x, x \in [0, 1]$. A uninorm U such that $U(0, 1) = 0$ is called **conjunctive** and if $U(0, 1) = 1$, then it is called **disjunctive**.

Definition 2.6 ([2, 10]). A non-increasing function $N: [0, 1] \rightarrow [0, 1]$ is called a **fuzzy negation**, if $N(0) = 1, N(1) = 0$. Moreover, a fuzzy negation N is called **strict** if it is strictly decreasing and continuous, and **strong** if it is an involution, i.e., $N(N(x)) = x$, for all $x \in [0, 1]$.

Theorem 2.7 ([7, Proposition 1.9]). Let T be a t-norm and N be a strict negation. Then the function $S: [0, 1]^2 \rightarrow [0, 1]$ given by

$$S(x, y) = N^{-1}(T(N(x), N(y))), \quad x, y \in [0, 1], \quad (1)$$

is a t-conorm. We say that S is an N -dual t-conorm to the t-norm T .

Definition 2.8 ([2, 7]). A function $I: [0, 1]^2 \rightarrow [0, 1]$ is called a **fuzzy implication**, if it is non-increasing with respect to the first variable, non-decreasing with respect to the second variable and if $I(0, 0) = I(1, 1) = 1$ and $I(1, 0) = 0$. The family of fuzzy implications will be denoted by $\mathcal{F}\mathcal{I}$.

Definition 2.9 ([2]). We say that a fuzzy implication I satisfies

- (i) the **identity principle**, if

$$I(x, x) = 1, \quad x \in [0, 1], \quad (\text{IP})$$

- (ii) the **left neutrality property**, if

$$I(1, y) = y, \quad y \in [0, 1]. \quad (\text{NP})$$

Definition 2.10 ([2, Definition 2.4.1]). A function $I: [0, 1]^2 \rightarrow [0, 1]$ is called an (S, N) -implication, if there exist a t-conorm S and a fuzzy negation N such that

$$I(x, y) = S(N(x), y), \quad x, y \in [0, 1].$$

If I is generated from a t-conorm S and a fuzzy negation N , then it will be denoted by $I_{S, N}$.

Definition 2.11 ([2, Definition 5.3.1]). A function $I: [0, 1] \rightarrow [0, 1]$ is called a (U, N) -operation, if there exist a uninorm U and a fuzzy negation N such that

$$I(x, y) = U(N(x), y), \quad x, y \in [0, 1].$$

A (U, N) -operator $I_{U, N} \in \mathcal{F}\mathcal{I}$ if and only if U is a disjunctive uninorm (see [2, Theorem 5.3.3]).

3 Main results

Here, we recall some concepts presented in [3]. The definition of α -migrative t-conorm is based on some particular graphical interpretation of this property (see also Fig. 1).

Definition 3.1 ([3, Definition 4.1]). Let $\alpha \in (0, 1)$ be fixed. A t-conorm S is said to be α -migrative, if it satisfies

$$S(1 - \alpha + \alpha x, y) = S(x, 1 - \alpha + \alpha y), \quad (2)$$

for all $x, y \in [0, 1]$. If S is α -migrative for every $\alpha \in (0, 1)$, then S is said to be migrative.

Of course, Eq. (2) is always satisfied when $\alpha = 0$ or $\alpha = 1$. Indeed, for $\alpha = 0$ we have

$$S(1, y) = 1 = S(x, 1)$$

and for $\alpha = 1$ we have

$$S(x, y) = S(x, y).$$

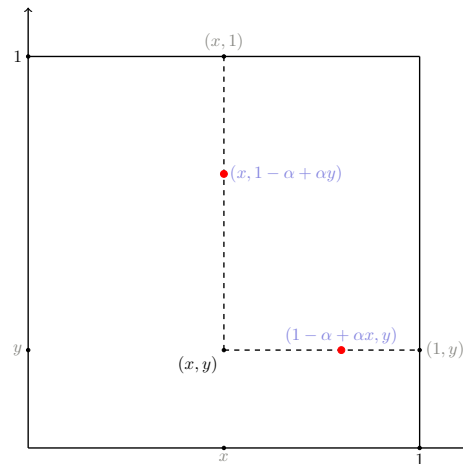


Figure 1: The graphical interpretation of α -migrative t-conorm (see Definition 3.1)

First, let us formulate simple facts.

Proposition 3.2. Let T be an α -migrative t-norm for some $\alpha \in (0, 1)$ (in the sense of Definition 1.1). If S is the N_C -dual t-conorm to T , where $N_C(x) = 1 - x$, for all $x \in [0, 1]$, then S is α -migrative (in the sense of Definition 3.1).

Proof. Let T be an α -migrative t-norm for some $\alpha \in (0, 1)$ and S be the N_C -dual t-conorm to T . Then we

have

$$\begin{aligned} S(1 - \alpha + \alpha x, y) &= 1 - T(1 - (1 - \alpha + \alpha x), 1 - y) \\ &= 1 - T(\alpha(1 - x), 1 - y) \\ &= 1 - T(1 - x, \alpha(1 - y)) \\ &= 1 - T(1 - x, 1 - (1 - \alpha + \alpha y)) \\ &= S(x, 1 - \alpha + \alpha y), \end{aligned}$$

for any $x, y \in [0, 1]$. □

We can also show the reverse of Proposition 3.2; the N_C -dual t-norm to α -migrative t-conorm is also α -migrative. Based on [5, Theorem 7] we know that the only migrative t-norm is the product t-norm. Hence the following fact is easy to obtain.

Corollary 3.3. S_P is the only migrative t-conorm.

Furthermore, inspired by investigations done in [3], we can consider the following functional equation, which is nothing else but the other form of the formula from Definition 3.1:

$$S(I_{RC}(\alpha, x), y) = S(x, I_{RC}(\alpha, y)), \quad \alpha, x, y \in [0, 1], \tag{3}$$

where I_{RC} is the Reichenbach implication given by $I_{RC}(x, y) = 1 - x + xy$. Hence, as the Pexider type of the above equation, we can analyse the following functional equation

$$S_1(I_1(\alpha, x), y) = S_2(x, I_2(\alpha, y)), \tag{4}$$

satisfied for all $\alpha, x, y \in [0, 1]$, where S_1, S_2 are t-conorms (or any generalisations of the classical disjunction) and I_1, I_2 are fuzzy implications (or any generalisation of the classical implication).

Let us start with some necessary conditions for functions S_1, S_2, I_1, I_2 to satisfy (4).

Proposition 3.4. Let S_1, S_2 be aggregation functions, $I_1, I_2 \in \mathcal{F}\mathcal{I}$. If (S_1, S_2, I_1, I_2) satisfies (4), then

- (i) 1 is an annihilator of S_1, S_2 ,
- (ii) if I_1, I_2 satisfies (NP), then $S_1 = S_2$.

Proof. (i) : Let us take $x = 1$ and $\alpha = 0$ in (4). Then we have

$$\begin{aligned} S_1(I_1(0, 1), y) = S_2(1, I_2(0, y)) &\iff S_1(1, y) = S_2(1, 1) \\ &\iff S_1(1, y) = 1, \end{aligned}$$

for all $y \in [0, 1]$, so 1 is an annihilator of S_1 .

Now, let us put $y = 1$ and $\alpha = 0$ in (4). Then we have

$$\begin{aligned} S_1(I_1(0, x), 1) = S_2(x, I_2(0, 1)) &\iff S_1(1, 1) = S_2(x, 1) \\ &\iff 1 = S_2(x, 1), \end{aligned}$$

for all $x \in [0, 1]$, so 1 is an annihilator of S_2 .

(ii) : Let us take $\alpha = 1$ in (4). Then we obtain

$$S_1(I_1(1, x), y) = S_2(x, I_2(1, y)) \iff S_1(x, y) = S_2(x, y)$$

for all $x, y \in [0, 1]$. □

Next results can be obtained in a natural way.

Proposition 3.5. Let N be a fuzzy negation and U be a uninorm. Then the quadruple $(U, U, I_{U,N}, I_{U,N})$ satisfies (4).

Proposition 3.6. Let $I_1, I_2 \in \mathcal{F}\mathcal{I}$ and S_1 (S_2) be binary functions. Next, let us assume that the quadruple (S_1, S_2, I_1, I_2) satisfies (4). If S_1 (S_2 , respectively) is a t-conorm, then I_1 (I_2 , respectively) is an (S_2, N_{I_2}) -implication ((S_1, N_{I_1}) -implication, respectively).

After these results, it is clear that we should focus on (U, N) -implications. Note that we can expand Definition 2.11 and talk about (A, N) -operators, where A is just an aggregation function. Also, the following result is obvious.

Proposition 3.7 ([14, Theorem 33]). If A is a disjunctive aggregation function and N is a fuzzy negation, then the operator $I: [0, 1]^2 \rightarrow [0, 1]$ given by the formula $I(x, y) = A(N(x), y)$ is a fuzzy implication.

Moreover, from the above investigations, it seems that usually in Eq. (4) we assume that $I_1 = I_2$. This does not always have to be true. Let us analyse the following functions.

Example 3.8. Let us take the classical fuzzy negation N_C and the following aggregation operator

$$A(x, y) = \begin{cases} 0, & x = y = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Then

$$I_{A, N_C}(x, y) = \begin{cases} 0, & x = 1, y = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Also, let us consider the drastic (S, N) -implication given by the formula

$$I_D(x, y) = \begin{cases} 1, & x = 0, \\ y, & \text{otherwise.} \end{cases}$$

Then we have $A(I_{(A,N)}(\alpha, x), y) = A(x, I_D(x, y))$, so the quadruple (A, A, I_{A, N_C}, I_D) satisfies (4).

Also, it is not true that always (according to the above notation) $I_1 = I_{S_1, N}$ for some N .

Example 3.9. Again, let us take I_D and any t-conorm S . Then it can be easily shown that the quadruple (S, S, I_D, I_D) satisfies (4).

Let us go back to (A, N) -operators. If we consider that $I = I_{A, N}$ in (4), then we obtain the following functional equation:

$$S_1(A_1(N(\alpha), x), y) = S_2(x, A_2(N(\alpha), y)),$$

satisfied for $\alpha, x, y \in [0, 1]$. With the assumptions of the commutativity of A_1, A_2 and the surjectivity of N , it leads us to the following functional equation of generalized associativity:

$$S_1(A_1(x, z), y) = S_2(x, A_2(z, y)), \quad x, y, z \in [0, 1]. \quad (GA)$$

This equation has been already investigated by Aczél et al. [1, Theorem 1]. However, the authors required there functions S_1, S_2, A_1, A_2 with properties, which were ensured by the existence of a quasigroup. Therefore, we should add properties that can substitute those assumptions for fuzzy connectives and still give a valid result. To obtain it, we restrict our considerations to some particular aggregation functions.

Definition 3.10. We say that an aggregation function $A: [0, 1]^2 \rightarrow [0, 1]$ is “quasi-multiplicative” generated if there exist continuous bijections $f, g, h: [0, 1] \rightarrow [0, 1]$ such that

$$A(x, y) = h(f(x) \cdot g(y)), \quad x, y \in [0, 1], \quad (QM)$$

where f, g, h are all increasing (or all decreasing) with

$$h^{-1}(0) = f(0)g(0) \text{ and } h^{-1}(1) = f(1)g(1).$$

Note that strict t-norms and t-conorms are subclasses of “quasi-multiplicative” generated functions. Thus, we have the following interesting result for the introduced family of aggregation functions.

Theorem 3.11. Let S_1, S_2, A_1, A_2 be “quasi-multiplicative” generated aggregation functions. Then the following statements are equivalent:

(i) The quadruple (S_1, S_2, A_1, A_2) satisfies (GA).

(ii) $A_1(x, y) = \varphi^{-1}(G(\xi(x), \lambda(y))),$
 $S_1(x, y) = G(\varphi(x), \psi(y)),$
 $A_2(x, y) = \rho^{-1}(G(\lambda(x), \psi(y))),$
 $S_2(x, y) = G(\xi(x), \rho(y)),$

for $x, y \in [0, 1]$, where $\varphi, \psi, \xi, \rho, \lambda: [0, 1] \rightarrow [0, 1]$ are continuous bijections and $G: [0, 1]^2 \rightarrow [0, 1]$ is an associative function.

Let us finish our manuscript with the following interesting examples.

Example 3.12. (i) Let

$$\begin{aligned} \varphi(x) &= x^2, \\ \psi(x) &= \xi(x) = \lambda(x) = x, \\ \rho(x) &= x^3, \end{aligned}$$

for $x \in [0, 1]$ and

$$G(x, y) = x + y - xy, \quad x, y \in [0, 1].$$

Then

$$\begin{aligned} S_1(x, y) &= x^2 + y - x^2y, \\ S_2(x, y) &= x + y^3 - xy^3, \\ A_1(x, y) &= \sqrt{x + y - xy}, \\ A_2(x, y) &= \sqrt[3]{x + y - xy}, \end{aligned}$$

for $x, y \in [0, 1]$. From our main new result (Theorem 3.11) we have that the quadruple (S_1, A_1, S_2, A_2) satisfies (GA) (in fact, this observation can be easily checked). Moreover, the quadruple $(S_1, S_2, I_{A_1, N_C}, I_{A_2, N_C})$ satisfies (4).

(ii) Note, that if G is, for instance, the product t-norm, then with the above $\varphi, \psi, \xi, \rho, \lambda$, the Eq. (GA) is satisfied by the quadruple (S_1, A_1, S_2, A_2) , where

$$\begin{aligned} S_1(x, y) &= x^2y, \\ S_2(x, y) &= xy^3, \\ A_1(x, y) &= \sqrt{xy}, \\ A_2(x, y) &= \sqrt[3]{xy^3}, \end{aligned}$$

for $x, y \in [0, 1]$.

4 Conclusions

In this paper, we have discussed the property of migrativity of t-conorms. Also, we have shown how it is connected with the notion of generalized associativity. Moreover, we have given some solutions to such a functional equation.

References

- [1] J. Aczél, V.D. Belousov, M. Hosszú, Generalized associativity and bisymmetry on quasigroups, Acta Mathematica Academiae Scientiarum Hungarica 11 (1963) 127–136.
- [2] M. Baczyński, B. Jayaram, Fuzzy Implications, Vol. 231 of Studies in Fuzziness and Soft Computing, Springer, Berlin Heidelberg, 2008.
- [3] M. Baczyński, B. Jayaram, R. Mesiar, Fuzzy implications: alpha migrativity and generalised laws of importation, Information Sciences 531 (2020) 87–96.
- [4] G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions: A Guide for Practitioners, Springer-Verlag, Berlin Heidelberg, 2007.

- [5] H. Bustince, J. Montero, R. Mesiar, Migrativity of aggregation functions, *Fuzzy Sets and Systems* 160 (2009) 766–777.
- [6] F. Durante, P. Sarkoci, A note on the convex combinations of triangular norms, *Fuzzy Sets and Systems* 159 (2008) 77–80.
- [7] J. Fodor, M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Support*, Kluwer Academic Publishers, Dordrecht, 1994.
- [8] J. Fodor, I.J. Rudas, On continuous triangular norms that are migrative, *Fuzzy Sets and Systems* 158 (2007) 1692–1697.
- [9] M. Grabisch, J. Marichal, R. Mesiar, E. Pap, Aggregation Functions, Vol. 127 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 2009.
- [10] E.P. Klement, R. Mesiar, E. Pap, *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, 2000.
- [11] C. Lopez-Molina, B. De Baets, H. Bustince, E. Induráin, A. Stupňanová, R. Mesiar, Bimigrativity of binary aggregation functions, *Information Sciences* 274 (2014) 225–235.
- [12] M. Mas, M. Monserrat, D. Ruiz-Aguilera, J. Torrens, Migrative uninorms and nullnorms over t-norms and t-conorms, *Fuzzy Sets and Systems* 261 (2015) 20–32.
- [13] R. Mesiar, H. Bustince, J. Fernandez, On the α -migrativity of semicopulas, quasi-copulas, and copulas, *Information Sciences* 180 (2010) 1967–1976.
- [14] A. Pradera, G. Beliakov, H. Bustince, B. De Baets, A review of the relationships between implication, negation and aggregation functions from the point of view of material implication, *Information Sciences* 329 (2016) 357–380.
- [15] R.R. Yager, A. Rybalov, Uninorm aggregation operators, *Fuzzy Sets and Systems* 80 (1996) 111–120.