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On a Graded Version of Stochastic Dominance

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Abstract

A random variable is said to stochastically dominate another random variable if the cumulative distribution function of the former is smaller than or equal to the cumulative distribution function of the latter. In this paper, we present a graded version of stochastic dominance by measuring the part of the real line in which the inequality holds. Interestingly, when a finite and non-null supermodular fuzzy measure is considered, this graded version of stochastic dominance is proven to be a fuzzy order relation w.r.t. the Lukasiewicz t-norm. We also discuss the use of the Lebesgue measure for random variables with bounded support and present different alternatives for random variables with unbounded support.

Keywords: Stochastic order, Fuzzy order relation, Fuzzy measure, Graded stochastic dominance.

1 Introduction

Stochastic orders [15, 20] have been a popular study subject in Probability Theory and Statistics for comparing whether one random variable is greater than another random variable. Probably the easiest way for comparing two random variables is based on their expected utilities with respect to a given utility function [14]. A more involved option, called stochastic dominance [9], is based on the comparison of their cumulative distribution functions. More specifically, if the cumulative distribution function of a random variable is smaller than the cumulative distribution function of another random variable it means that the first random variable takes larger values than the second random variable with a higher probability. Interestingly, a random variable stochastically dominates another random variable if and only if the expected utility of the first random variable is always greater than or equal to the expected utility of the second random variable, regardless of the considered utility function [10]. This is one of many reasons why stochastic dominance is considered to be the most prominent stochastic order, even though it results in a partial (and not total) order on the set of random variables.

In order to further refine the partial order given by the stochastic dominance, several alternatives have been proposed [2]. Aside of the aforementioned expected utilities and other examples such as probabilistic preference [12], special attention deserves the notion of statistical preference [18, 19] which is based on a reciprocal relation expressing the winning probabilities between the random variables. Interestingly, for independent random variables, stochastic dominance implies statistical preference, however, for non-independent random variables, statistical preference exploits the joint distribution of the random variables. This has served as a source of inspiration for introducing an alternative definition of stochastic dominance that exploits this joint distribution of the random variables [13].

Some authors have referred to an alternative definition of statistical preference as a graded version of stochastic dominance [11] in the sense that it provides a value in the unit interval that equals one in case the first random variable stochastically dominates the second random variable (and both random variables are independent). In this paper, we follow a different direction and partition the real line into two parts: (i) the part in which the cumulative distribution function of the first random variable is smaller than or equal to the cumulative distribution function of the second random variable, and (ii) the part in which the cumulative distribution function of the first random variable is greater than the cumulative distribution function of the second random variable. By measuring the first part by means of a finite and non-null supermodular fuzzy measure, we obtain a fuzzy order relation w.r.t. the Lukasiewicz t-norm.

The remainder of the paper is structured as follows. Section 2 presents some basic notions on triangular norms, fuzzy relations and (fuzzy) measures that are necessary for the correct understanding of the paper. Section 3 presents a way of constructing a fuzzy order relation on a set of measurable functions. This fuzzy order relation is proven to lead to a graded notion of stochastic dominance when applied to the cumulative distribution functions of random variables in Section 4. In particular, we present a natural illustration of the fuzzy order relation when the Lebesgue measure and random variables with bounded support is considered. Some preliminary work on different alternatives to the Lebesgue measure in the context of random variables with unbounded support are presented in Section 5. We end with some conclusions in Section 6.

2 Preliminaries

2.1 Triangular norms

A triangular norm (t-norm, for short) $*$ on $[0,1]$ is a function $*:[0,1]\times[0,1]\to[0,1]$ that is commutative ($a*b=b*a$), associative ($a*(b*c)=(a*b)*c$), order-preserving ($a\leq b$ implies that $a*c\leq b*c$) and has neutral element 1 ($a*1=a$). Here, we pay particular attention to the Lukasiewicz t-norm $*_{\mathrm{L}}$, defined by $x*_{\mathrm{L}}y=\max(0,x+y-1)$. For more details on t-norms, we refer to [7].

2.2 Fuzzy relations

A (binary) fuzzy relation R on X is a mapping $R:X\times X\to[0,1]$. A fuzzy relation is called a crisp relation if $\mathrm{Im}(R)=\{0,1\}$. In such case, we also use the notation $(x,y)\in R$ meaning that $R(x,y)=1$. Given a t-norm $*$, a fuzzy relation is said to be:

- reflexive if $R(x,x)=1$, for any $x\in X$;
- symmetric if $R(x,y)=R(y,x)$, for any $x,y\in X$;
- $*$-transitive if $R(x,y)*R(y,z)\leq R(x,z)$, for any $x,y,z\in X$.

A reflexive, symmetric and $*$-transitive relation is called a $*$-equivalence relation.

Given a t-norm $*$ and a $*$-equivalence relation E, a fuzzy relation is said to be:

- E-reflexive if $E(x,y)\leq R(x,y)$, for any $x,y\in X$;
- $*$-E-antisymmetric if $R(x,y)*R(y,x)\leq E(x,y)$, for any $x,y\in X$.

An E-reflexive, $*$-E-antisymmetric and $*$-transitive relation is called a $*$-E-order relation (in the sense of Bodenhofer [4]). For more details on fuzzy relations, we refer to [5].

2.3 Measures and fuzzy measures

A measure space $(\mathscr{X},\Sigma)$ is formed by a set $\mathscr{X}$ and a σ-algebra Σ on $\mathscr{X}$. A function $\varphi:\mathscr{X}\to\mathbb{R}$ is called Σ-measurable (or, simply, measurable if no confusion can occur) if $\{\varphi\leq\alpha\}=\{x\in\mathscr{X}\mid\varphi(x)\leq\alpha\}\in\Sigma$, for any $\alpha\in\mathbb{R}$.

A function $\mu:\Sigma\to[0,+\infty]$ is called a fuzzy measure if the following two conditions hold:

(i) $\mu(\emptyset)=0$,

(ii) $\mu(A)\leq\mu(B)$, for any $A,B\in\Sigma$ such that $A\subseteq B$.

A fuzzy measure μ is said to be:

- non-null if $\mu(\mathscr{X})>0$;
- finite if $\mu(\mathscr{X})<+\infty$;
- supermodular if, for any $A,B\in\Sigma$, it holds that
$$\mu(A\cup B)+\mu(A\cap B)\geq\mu(A)+\mu(B);$$
- countable additive if, for any countable collection $\{A_n\}_{n\in\mathbb{N}}$ of pairwise disjoint sets in Σ, it holds that:
$$\mu\left(\bigcup_{n\in\mathbb{N}}A_n\right)=\sum_{n\in\mathbb{N}}\mu(A_n).$$

A fuzzy measure that is countable additive is simply called a measure. Note that any measure is a supermodular fuzzy measure. For more details on measures and fuzzy measures, we refer to [3].

3 Fuzzy order relations on sets of measurable functions

Let $(\mathscr{X},\Sigma)$ be a measure space and $\mathscr{F}$ be a set of measurable functions from $\mathscr{X}$ to $\mathbb{R}$. A function $\varphi\in\mathscr{F}$ is said to be smaller than or equal to another function $\varphi'\in\mathscr{F}$ if $\varphi(x)\leq\varphi'(x)$ for any $x\in\mathscr{X}$. For any $\varphi,\varphi'\in\mathscr{F}$, we consider the notations $\mathscr{X}_{\varphi=\varphi'}=\{x\in\mathscr{X}\,|\,\varphi(x)=\varphi'(x)\}$ and $\mathscr{X}_{\varphi\leq\varphi'}=\{x\in\mathscr{X}\,|\,\varphi(x)\leq\varphi'(x)\}$. Note that, in the upcoming sections, $\mathscr{X}$ will actually be the real line $\mathbb{R}$ (with Σ being the Borel algebra on $\mathbb{R}$) and $\mathscr{F}$ will be the set of all possible cumulative distribution functions of a random variable.

Proposition 1. *Consider a finite and non-null supermodular fuzzy measure $\mu : \Sigma \to [0,+\infty[$.*

(i) The fuzzy relation $E_\mu : \mathscr{F} \times \mathscr{F} \to [0,1]$

$$E_\mu(f,g) = \frac{\mu(\mathscr{X}_{f=g})}{\mu(\mathscr{X})}$$

is a $*_L$-equivalence relation.

(ii) The fuzzy relation $R_\mu : \mathscr{F} \times \mathscr{F} \to [0,1]$ defined as

$$R_\mu(f,g) = \frac{\mu(\mathscr{X}_{f\leq g})}{\mu(\mathscr{X})}$$

is a $*_L$-E_μ-order relation.

Proof. (i) Reflexivity. For any $f \in \mathscr{F}$, it holds that $\mathscr{X}_{f=f} = \mathscr{X}$ and, therefore,

$$E_\mu(f,f) = \frac{\mu(\mathscr{X}_{f=f})}{\mu(\mathscr{X})} = \frac{\mu(\mathscr{X})}{\mu(\mathscr{X})} = 1\,.$$

Symmetry. For any $f,g \in \mathscr{F}$, it holds that $\mathscr{X}_{f=g} = \mathscr{X}_{g=f}$ and, therefore,

$$E_\mu(f,g) = \frac{\mu(\mathscr{X}_{f=g})}{\mu(\mathscr{X})} = \frac{\mu(\mathscr{X}_{g=f})}{\mu(\mathscr{X})} = E_\mu(g,f)\,.$$

$*_L$-transitivity. For any $f,g,h \in \mathscr{F}$, it holds that $\mathscr{X}_{f=g} \cap \mathscr{X}_{g=h} \subseteq \mathscr{X}_{f=h}$ and, therefore,

$$\begin{aligned}\mu(\mathscr{X}_{f=h}) &\geq \mu(\mathscr{X}_{f=g} \cap \mathscr{X}_{g=h})\\ &\geq \mu(\mathscr{X}_{f=g}) + \mu(\mathscr{X}_{g=h}) - \mu(\mathscr{X}_{f=g} \cup \mathscr{X}_{g=h})\\ &\geq \mu(\mathscr{X}_{f=g}) + \mu(\mathscr{X}_{g=h}) - \mu(\mathscr{X})\,.\end{aligned}$$

Since $\mu(\mathscr{X}_{f=h}) \geq 0$, we obtain that

$$\begin{aligned}E_\mu(f,h) &\geq \max\big(0, E_\mu(f,g) + E_\mu(g,h) - 1\big)\\ &= E_\mu(f,g) *_L E_\mu(g,h)\,.\end{aligned}$$

(ii) E_μ-reflexivity. For any $f,g \in \mathscr{F}$, it holds that $\mathscr{X}_{f=g} \subseteq \mathscr{X}_{f\leq g}$ and, therefore,

$$E_\mu(f,g) = \frac{\mu(\mathscr{X}_{f=g})}{\mu(\mathscr{X})} \leq \frac{\mu(\mathscr{X}_{f\leq g})}{\mu(\mathscr{X})} = R_\mu(f,g)\,.$$

$*_L$-E_μ-antisymmetry. For any $f,g \in \mathscr{F}$, it holds that

$$\begin{aligned}&\mu(\mathscr{X}_{f\leq g}) + \mu(\mathscr{X}_{g\leq f})\\ &\quad\leq \mu(\mathscr{X}_{f\leq g} \cap \mathscr{X}_{g\leq f}) + \mu(\mathscr{X}_{f\leq g} \cup \mathscr{X}_{g\leq f})\\ &\quad= \mu(\mathscr{X}_{f=g}) + \mu(\mathscr{X})\,.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}E_\mu(f,g) &= \max\left(0, \frac{\mu(\mathscr{X}_{f=g})}{\mu(\mathscr{X})}\right)\\ &\geq \max\left(0, \frac{\mu(\mathscr{X}_{f\leq g})}{\mu(\mathscr{X})} + \frac{\mu(\mathscr{X}_{g\leq f})}{\mu(\mathscr{X})} - 1\right)\\ &= R_\mu(f,g) *_L R_\mu(g,f)\,.\end{aligned}$$

$*_L$-transitivity. For any $f,g,h \in \mathscr{F}$, it holds that $\mathscr{X}_{f\leq g} \cap \mathscr{X}_{g\leq h} \subseteq \mathscr{X}_{f\leq h}$ and, therefore,

$$\begin{aligned}\mu(\mathscr{X}_{f\leq h}) &\geq \mu(\mathscr{X}_{f\leq g} \cap \mathscr{X}_{g\leq h})\\ &\geq \mu(\mathscr{X}_{f\leq g}) + \mu(\mathscr{X}_{g\leq h}) - \mu(\mathscr{X}_{f\leq g} \cup \mathscr{X}_{g\leq h})\\ &\geq \mu(\mathscr{X}_{f\leq g}) + \mu(\mathscr{X}_{g\leq h}) - \mu(\mathscr{X})\,.\end{aligned}$$

Since $\mu(\mathscr{X}_{f\leq h}) \geq 0$, we obtain that

$$\begin{aligned}R_\mu(f,h) &\geq \max\big(0, R_\mu(f,g) + R_\mu(g,h) - 1\big)\\ &= R_\mu(f,g) *_L R_\mu(g,h)\,.\end{aligned}$$

□

Remark 1. *Obviously, the fact that $f = g$ implies that $E_\mu(f,g) = 1$ for any μ. Similarly, the fact that $f \leq g$ implies that $R_\mu(f,g) = 1$ for any μ. Furthermore, if μ is such that $\mu(A) = 0$ for any $A \neq \mathscr{X}$, then both E_μ and R_μ are crisp relations that respectively coincide with the relations $=$ and $\leq$ on $\mathscr{F}$.*

Remark 2. *The result remains valid if μ is a finite and non-null measure since any measure is a supermodular fuzzy measure (see [3]).*

Remark 3. *The result does not necessarily hold for t-norms greater than the Lukasiewicz t-norm. For instance, consider $\mathscr{X} = [0,1]$, the Borel algebra Σ on $\mathscr{X}$ and the Lebesgue measure $\mu : \Sigma \to [0,1]$. For $f(x) = x$, $g(x) = 0.25$ and $h(x) = x^2$, it holds that $R_\mu(f,g) = 0.25$, $R_\mu(g,h) = 0.5$ but $R_\mu(f,h) = 0$. Therefore, R_μ is not $*_P$-transtive, with $*_P$ being the product t-norm (defined as $x *_P y = xy$).*

4 Graded stochastic dominance

4.1 Definition

The notion of stochastic dominance has been widely studied in the context of Statistics for comparing random variables. Formally, given a probablity space $(\Omega, \Sigma_\Omega, P)$, a random variable is a measurable function from Ω to $\mathbb{R}$. The cumulative distribution function $F_X : \mathbb{R} \to [0,1]$ of a random variable X is defined as $F_X(x) = P(X \leq x)$. A random variable X is said to (first-order) stochastically dominate another random variable Y, denoted by $X \succeq_{\mathrm{FSD}} Y$, if $F_X \leq F_Y$. As mentioned in Remark 1, $X \succeq_{\mathrm{FSD}} Y$ can also be defined in terms of the fuzzy order relation defined on the precedent section, as follows: $X \succeq_{\mathrm{FSD}} Y$ if $R_\mu(F_X, F_Y) = 1$ with μ being any finite and non-null supermodular fuzzy measure $\mu : \Sigma \to]0,+\infty[$ such that $\mu(A) = 0$ for any $A \neq \mathscr{X}$. Interestingly, the choice of a more elaborate supermodular fuzzy measure returns a graded version of stochastic dominance in which $R_\mu(F_X, F_Y)$ represents the degree in which X (first-order) stochastically dominates Y.

As a natural result, we study random variables that are identically distributed but shifted, i.e., $Y = X + t$ for some $t \neq 0$. Note that this is equivalent to $F_Y(x) = F_X(x - t)$ for any $x \in \mathbb{R}$. Therefore, since cumulative distribution functions are increasing, it holds that $R_\mu(F_Y, F_X) = 1$ if $t > 0$ and $R_\mu(F_X, F_Y) = 1$ if $t < 0$. Moreover, if F_X and F_Y are strictly increasing, it follows that $E_\mu(F_X, F_Y) = 0$ and $R_\mu(F_X, F_Y) = 1 - R_\mu(F_X, F_Y)$.

4.2 The case of the Lebesgue measure for random variables with bounded support

Probably, one of the most interesting cases is that in which μ is the Lebesgue measure. Obviously, since the Lebesgue measure is not finite in general, we need to restrict to the Lebesgue measure on a bounded set, for instance the common bounded support of a family of random variables.

Here, we consider beta-distributed random variables, whose support is the unit interval (thus, bounded). We recall that a random variable X has a beta distribution with parameters α and β, denoted as $X \rightsquigarrow B(\alpha, \beta)$, if its probability density function is, for any $x \in [0, 1]$,

$$f_X(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{K_{\alpha,\beta}},$$

where $K_{\alpha,\beta} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is the normalizing constant with Γ denoting the Gamma function.

As an illustrative example, we consider $X \rightsquigarrow B(0.5, 0.5)$ and $Y \rightsquigarrow B(2, 2)$. In Figure 1, the cumulative distribution functions of both random variables are plotted. Note that none of the cumulative distribution functions is smaller than or equal to the other one. This implies that X does not stochastically dominate Y and that Y does not stochastically dominate X. However, it can be seen that $R_\mu(F_X, F_Y) = R_\mu(F_Y, F_X) = 0.5$ since both cumulative distribution functions intersect at $x = 0.5$, being F_X smaller than F_Y on $]0.5, 1[$ and F_Y smaller than F_X on $]0, 0.5[$. Actually, since the probability density function of any beta-distributed random variable such that $\alpha = \beta$ is symmetric with respect to 0.5, it holds that $R_\mu(F_X, F_Y) = R_\mu(F_Y, F_X) = 0.5$ for any two random variables $X \rightsquigarrow B(\alpha, \alpha)$ and $Y \rightsquigarrow B(\alpha', \alpha')$.

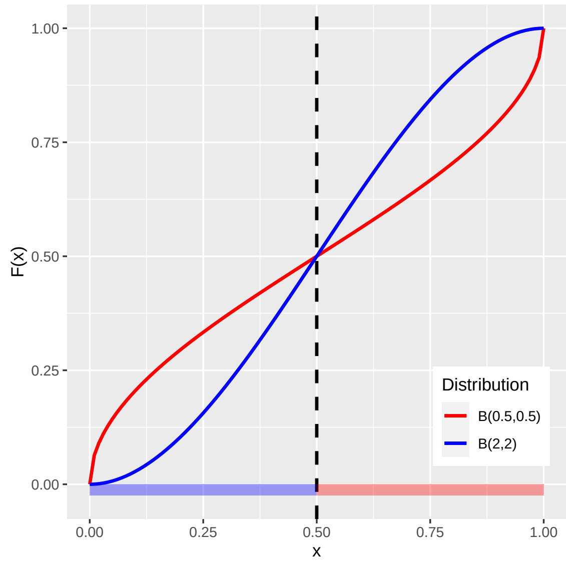

Figure 1: Illustration of graded stochastic dominance between two beta-distributed random variables. The cumulative distribution function F_X of the random variable $X \rightsquigarrow B(0.5, 0.5)$ is plotted in red, whereas the cumulative distribution function F_Y of the random variable $Y \rightsquigarrow B(2, 2)$ is plotted in blue.

5 A more general framework

5.1 Definition

As mentioned in the precedent section, it is necessary that the support of the random variables is bounded for the Lebesgue measure to be finite. Obviously, this is a big drawback since, for instance, it is not possible to compare Gaussian random variables. In this section, we propose an alternative definition of the relations $R_\mu(f, g)$ and $E_\mu(f, g)$ where, instead of a single μ, a family of finite and non-null fuzzy measures $\mu_{\mathscr{F}} = \{\mu_{f,g}\}_{f,g \in \mathscr{F}}$ is considered, being $\mu_{f,g}$ a finite and non-null fuzzy measure dependant on a fixed pair of measurable functions f and g. Formally, the fuzzy relations $E_{\mu_{\mathscr{F}}} : \mathscr{F} \times \mathscr{F} \to [0, 1]$ and $R_{\mu_{\mathscr{F}}} : \mathscr{F} \times \mathscr{F} \to [0, 1]$ are defined as follows

$$E_{\mu_{\mathscr{F}}}(f, g) = \frac{\mu_{f,g}(\mathscr{X}_{f=g})}{\mu_{f,g}(\mathscr{X})},$$

$$R_{\mu_{\mathscr{F}}}(f, g) = \frac{\mu_{f,g}(\mathscr{X}_{f \leq g})}{\mu_{f,g}(\mathscr{X})}.$$

Unfortunately, $E_{\mu_{\mathscr{F}}}$ does not need to be a $*_L$-equivalence relation and $R_{\mu_{\mathscr{F}}}$ does not need to be a $*_L$-$E_{\mu_{\mathscr{F}}}$-order relation in general. As a counterexample, consider $X \rightsquigarrow B(0.5, 0.5)$, $Y \rightsquigarrow B(2, 2)$, $Z = 0.2$ and fix $\mu_{F_X, F_Y} = \mu_{F_Y, F_Z}$ as the Lebesgue measure on $[0, 1]$ and μ_{F_X, F_Z} as the (supermodular) fuzzy measure such that $\mu_{F_X, F_Z}(A) = 0$ if $A \neq [0, 1]$ and $\mu_{F_X, F_Z}([0, 1]) = 1$. We see that $R_{\mu_{\mathscr{F}}}(F_X, F_Z) = 0 < 0.5 + 0.8 - 1 = R_{\mu_{\mathscr{F}}}(F_X, F_Y) *_L R_{\mu_{\mathscr{F}}}(F_Y, F_Z)$. Thus, $R_{\mu_{\mathscr{F}}}$ is not $*_L$-transitive.

5.2 The case of measures induced by the random variables

A natural way for defining $\mu_{f,g}$ exploits the properties of the underlying probability space. In particular, for any two random variables X and Y, we may define the function $\mu_{F_X,F_Y} : \Sigma \to [0,2[$ as follows:

$$\mu_{F_X,F_Y}(A) = P(X \in A) + P(Y \in A).$$

Please note that μ_{F_X,F_Y} is a finite and non-null measure on $\mathbb{R}$. Consider the family $\mu_{\mathscr{F}}$ of all measures μ_{F_X,F_Y}, where X and Y are random variables.

In this subsection, we illustrate the resulting fuzzy relation in the case of Gaussian random variables. We recall that a random variable X has a Gaussian distribution with mean m and standard deviation σ, denoted as $X \rightsquigarrow N(m,\sigma)$, if its probability density function is, for any $x \in \mathbb{R}$,

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}.$$

The cumulative distribution function is given by

$$F_X(x) = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{x-m}{\sigma\sqrt{2}}\right),$$

where $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\,dt$.

It is straightforward to see that for two Gaussian random variables $X \rightsquigarrow N(m_X,\sigma_X)$ and $Y \rightsquigarrow N(m_Y,\sigma_Y)$, it holds that X stochastically dominates Y if and only if $m_X > m_Y$ and $\sigma_X = \sigma_Y$. Here, we expand this result with respect to the relation $R_{\mu_{\mathscr{F}}}$:

- If $m_X = m_Y$ and $\sigma_X = \sigma_Y$, then
$$R_{\mu_{F_X,F_Y}}(F_X,F_Y) = 1;$$
- If $m_X = m_Y$ and $\sigma_X \neq \sigma_Y$, then
$$R_{\mu_{F_X,F_Y}}(F_X,F_Y) = 0.5;$$
- If $m_X < m_Y$ and $\sigma_X = \sigma_Y$, then
$$R_{\mu_{F_X,F_Y}}(F_X,F_Y) = 0;$$
- If $m_X > m_Y$ and $\sigma_X = \sigma_Y$, then
$$R_{\mu_{F_X,F_Y}}(F_X,F_Y) = 1;$$
- If $m_X \neq m_Y$ and $\sigma_X < \sigma_Y$, then
$$R_{\mu_{F_X,F_Y}}(F_X,F_Y) = \frac{1}{2} + \frac{\mathrm{erf}\left(\frac{t_0-m_X}{\sigma_X\sqrt{2}}\right) + \mathrm{erf}\left(\frac{t_0-m_Y}{\sigma_Y\sqrt{2}}\right)}{4},$$
with $t_0 = \frac{m_Y\sigma_X - m_X\sigma_Y}{\sigma_X-\sigma_Y}$;
- If $m_X \neq m_Y$ and $\sigma_X > \sigma_Y$, then
$$R_{\mu_{F_X,F_Y}}(F_X,F_Y) = \frac{1}{2} - \frac{\mathrm{erf}\left(\frac{t_0-m_X}{\sigma_X\sqrt{2}}\right) + \mathrm{erf}\left(\frac{t_0-m_Y}{\sigma_Y\sqrt{2}}\right)}{4},$$
with $t_0 = \frac{m_Y\sigma_X - m_X\sigma_Y}{\sigma_X-\sigma_Y}$.

For the proof of the last item, note that the unique intersection point of F_X and F_Y is $t_0 = \frac{m_Y\sigma_X - m_Y\sigma_X}{\sigma_X-\sigma_Y}$. Therefore, it holds that $F_X(t) < F_Y(t)$ for any $t \in]-\infty,t_0[$, $F_X(t_0) = F_Y(t_0)$ and $F_X(t) > F_Y(t)$ for any $t \in]t_0,+\infty[$. The remainder of the items are straightforward or analogous to the last one.

As an illustrative example, we consider $X \rightsquigarrow N(0,1)$ and $Y \rightsquigarrow N(1,2)$. In Figure 2, the cumulative distribution functions of both random variables are plotted. Note that none of the cumulative distribution functions is smaller than or equal to the other one and, thus, X does not stochastically dominate Y and Y does not stochastically dominate X. Therefore, it holds that $t_0 = -1$ and

$$R_{\mu_{F_X,F_Y}}(F_X,F_Y) \approx 0.15255,$$
$$R_{\mu_{F_Y,F_X}}(F_Y,F_X) \approx 0.84745.$$

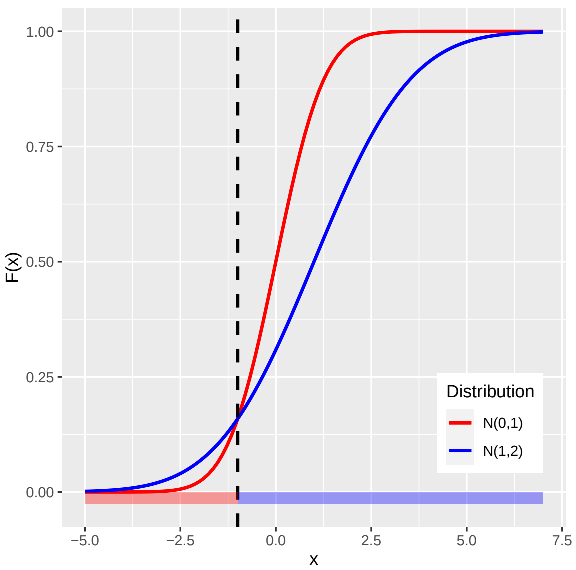

Figure 2: Illustration of graded stochastic dominance between two Gaussian random variables. The cumulative distribution function F_X of the random variable $X \rightsquigarrow N(0,1)$ is plotted in red, whereas the cumulative distribution function F_Y of the random variable $Y \rightsquigarrow N(1,2)$ is plotted in blue.

5.3 Some connections with goodness-of-fit tests

Goodness-of-fit tests have been popularly studied in the field of Statistics for checking whether a sample provides from a theoretical distribution or whether two samples provide from the same distribution. The intuitive idea behind most goodness-of-fit tests is to compute a test statistic representing how much both (empirical) distributions differ. Typically, the values of the test statistic are tabulated, thus allowing to perform the statistical test. Three prominent such goodness-of-fit tests are the Kolmogorov-Smirnov test [8, 21], the Cramer-Von Mises test [6, 22] and the Anderson-Darling test [1]. In this section, we use these three tests as a source of inspiration for defining some alternative fuzzy measures.

Inspired by the Kolmogorov-Smirnov test, we may define finite and non-null fuzzy measures on $\mathbb{R}$ based on the maximal difference between the cumulative distribution functions. Formally, for any two random variables X and Y, we consider the function $\mu^{\mathrm{KS}}_{F_X,F_Y} : \Sigma \to [0,1[$ as follows:

$$\mu^{\mathrm{KS}}_{F_X,F_Y}(A) = \sup_{x\in A} |F_X(x) - F_Y(x)|\,,$$

with the convention that $\mu^{\mathrm{KS}}_{F_X,F_Y}(\emptyset) = 0$ and $0 < \mu^{\mathrm{KS}}_{F_X,F_X}(\mathbb{R}) < +\infty$. Note that $\mu^{\mathrm{KS}}_{F_X,F_Y}$ is a fuzzy measure, but it is not supermodular.

In this case $R_{\mu^{\mathrm{KS}}_{F_X,F_Y}}(F_X,F_Y) = 1$ if the maximum absolute difference between F_X and F_Y is attained on the subset of $\mathbb{R}$ in which $F_X \leq F_Y$. Otherwise, $R_{\mu^{\mathrm{KS}}_{F_X,F_Y}}(F_X,F_Y)$ equals the quotient of the maximum absolute difference between F_X and F_Y on the subset of $\mathbb{R}$ in which $F_X \leq F_Y$ and the maximum absolute difference between F_X and F_Y on $\mathbb{R}$.

Continuing with the illustrative example of $X \rightsquigarrow N(0,1)$ and $Y \rightsquigarrow N(1,2)$ in Figure 2, it holds that the maximum absolute difference between F_X and F_Y on $]-\infty,-1]$ is 0.04492 at $x \approx -1.8475$ and the maximum absolute difference between F_X and F_Y on $[-1,+\infty[$ is 0.34514 at $x \approx 1.1809$. Therefore, it holds that

$$R_{\mu^{\mathrm{KS}}_{F_X,F_Y}}(F_X,F_Y) \approx 0.13015\,,$$
$$R_{\mu^{\mathrm{KS}}_{F_Y,F_X}}(F_Y,F_X) = 1\,.$$

Analogously, inspired by the Cramer-Von Mises and Anderson-Darling tests, we may define finite and non-null fuzzy measures on $\mathbb{R}$ based on the (weighted) average difference between the cumulative distribution functions. Formally, for any two random variables X and Y, we consider the function $\mu^{w}_{F_X,F_Y} : \Sigma \to [0,1[$ as follows:

$$\mu^{w}_{F_X,F_Y}(A) = \int_A \big(F_X(x) - F_Y(x)\big)^2 w(x)\, dH(x)\,,$$

where $w(x)$ is a weighting function, H is a linear combination of F_X and F_Y and, for technical reasons, the convention $0 < \mu^{w}_{F_X,F_X}(\mathbb{R}) < +\infty$ needs to be imposed. The case $w(x) = 1$ relates to the Cramer-Von Mises test, whereas the case $w(x) = \big(H(x)(1-H(x))\big)^{-1}$ relates to the Anderson-Darling test. Note that $\mu^{w}_{F_X,F_Y}$ is a measure in both cases as a result of the Radon-Nikodym theorem [16, 17].

Continuing with the illustrative example of $X \rightsquigarrow N(0,1)$ and $Y \rightsquigarrow N(1,2)$ in Figure 2, it holds that

$$\int_{-\infty}^{-1} \big(F_X(x) - F_Y(x)\big)^2 dH(x) = 0.0001658\,,$$
$$\int_{-1}^{+\infty} \big(F_X(x) - F_Y(x)\big)^2 dH(x) = 0.0473429\,,$$

where $H = (F_X + F_Y)/2$.

Therefore, it holds that

$$R_{\mu^{1}_{F_X,F_Y}}(F_X,F_Y) \approx \frac{0.0001658}{0.0001658 + 0.0473429} \approx 0.0035\,,$$
$$R_{\mu^{1}_{F_Y,F_X}}(F_Y,F_X) \approx \frac{0.0473429}{0.0001658 + 0.0473429} \approx 0.9965\,.$$

6 Conclusions

We have presented a graded version of stochastic dominance for comparing random variables by measuring the subset of the support in which the cumulative distribution function of the first random variable is smaller than or equal to the cumulative distribution function of the second random variable. In the case of random variables with a bounded support, the definition based on the Lebesgue measure seems to be quite natural and to agree with intuition. Unfortunately, some problems arise for random variables with an unbounded support (e.g., for Gaussian random variables). Section 5 aims at solving this problem and at presenting some links with statistical goodness-of-fit tests. In the near future, the study of transitivity-like properties for the fuzzy relations presented in Section 5 will be addressed.

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References

[1] T. Anderson, D. Darling, A test of goodness of fit, Journal of the American Statistical Association 49 (268) (1954) 765–769.

[2] B. D. Baets, Three approaches to the comparison of random variables, in: Data Science for Financial Econometrics, Springer, Cham, 2020, pp. 93–98.

[3] G. Beliakov, S. James, J.-Z. Wu, Discrete Fuzzy Measures: Computational Aspects, Vol. 382 of Studies in Fuzziness and Soft Computing, Springer International Publishing, 2020.

[4] U. Bodenhofer, Representations and constructions of similarity-based fuzzy orderings, Fuzzy Sets and Systems 137 (2003) 113–136.

[5] R. Bělohlávek, Fuzzy Relational Systems: Foundations and Principles, Kluwer Academic Publishers, New York, 2002.

[6] H. Cramér, On the composition of elementary errors. first paper: Mathematical deductions, Scandinavian Actuarial Journal 1928 (1) (1928) 13–74.

[7] E. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrecht, 2000.

[8] A. Kolmogorov, Sulla determinazione empirica di una legge di distribuzione, Giornale dell'Istituto Italiano degli Attuari 4 (1933) 83–91.

[9] E. Lehmann, Ordered families of distributions, The Annals of Mathematical Statistics 26 (1955) 399–419.

[10] H. Levy, Stochastic Dominance: Investment Decision Making under Uncertainty, Kluwer Academic Publishers, Boston, 1998.

[11] K. D. Loof, H. D. Meyer, B. D. Baets, Graded stochastic dominance as a tool for ranking the elements of a poset, in: Soft Methods for Integrated Uncertainty Modelling, Springer, 2006, pp. 53–60.

[12] I. Montes, S. Montes, B. D. Baets, Multivariate winning probabilities, Fuzzy Sets and Systems 362 (2019) 129–143.

[13] I. Montes, J. Salamanca, S. Montes, A modified version of stochastic dominance involving dependence, Statistics & Probability Letters 165 (2020) 108848.

[14] O. Morgenstern, J. Von Neumann, Theory of Games and Economic Behavior, Princeton University Press, Princeton, 1953.

[15] A. Müller, D. Stoyan, Comparison Methods for Stochastic Models and Risks, John Wiley and Sons, Hoboken, 2002.

[16] O. Nikodym, Sur une généralisation des intégrales de MJ Radon, Fundamenta Mathematicae 15 (1) (1930) 131–179.

[17] J. Radon, Theorie und anwendungen der absolut additiven mengenfunktionen, Hölder, Wien, 1913.

[18] B. D. Schuymer, H. D. Meyer, B. D. Baets, A fuzzy approach to stochastic dominance of random variables, in: Lecture Notes on Artificial Intelligence, Vol. 2715, 2003, pp. 253–260.

[19] B. D. Schuymer, H. D. Meyer, B. D. Baets, S. Jenei, On the cycle-transitivity of the dice model, Theory and Decision 54 (3) (2003) 261–285.

[20] M. Shaked, J. Shanthikumar, Stochastic Orders, Springer Science & Business Media, New York, 2007.

[21] N. Smirnov, Table for estimating the goodness of fit of empirical distributions, The Annals of Mathematical Statistics 19 (2) (1948) 279–281.

[22] R. Von Mises, Wahrscheinlichkeit, Statistik und Wahrheit, Vol. 3, Springer, 1928.