ATLANTIS PRESS
Atlantis Studies in Uncertainty Modelling, volume 3
Joint Proceedings of the 19th World Congress of the International Fuzzy Systems Association (IFSA), the 12th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT), and the 11th International Summer School on Aggregation Operators (AGOP)

Generalized Peterson's Syllogisms with the Quantifier "At Least Several (A Few; A Little)"

Petra Murinová and Vilém Novák
University of Ostrava,
Institute for Research and Applications of Fuzzy Modeling NSC IT4Innovations,
30. dubna 22 701 03 Ostrava 1, Czech Republic,
petra.murinova@osu.cz,vilem.novak@osu.cz

Abstract

In this article, we follow up on previous publications in which we studied generalized Peterson's syllogisms. The main objective of this paper is to define a new intermediate quantifier "At least <Several, A few, A little>" and analyze new forms of valid syllogisms.

Keywords: Fuzzy natural logic, Generalized intermediate quantifiers, Generalized Peterson's syllogisms

1 Introduction

In our previous papers (see, e.g., [5, 6, 8, 11, 12] and others), we developed the theory of intermediate quantifiers and focused on several selected ones, namely *all, almost all, most, many* and *some*. The quantifiers *all* and *some* are classical. Nontrivial intermediate ones are *almost all, most* and *many*. All of them characterize a big part of the universe of discourse. There are, however, also intermediate quantifiers whose meaning corresponds to the opposite, i.e., a small part of the universe. Typical examples of them are *several, a few*, and *a little*. The first two quantifiers can be joined with a countable noun while the latter with an uncountable one. There is a difference between *a few* and *few* (*a little* and *little*) in a wider context when the former is a simple quantifier characterizing a certain number or amount corresponding to some, while the latter also carries a negative evaluation expressing non-satisfaction with a given amount. Because we do not have (so far) formal means to express such evaluations, we will consider only the first cases. It is interesting that *some* is more specific: it always means *at least one*, but possibly also *all* which cannot hold for "few, several, or many".

In connection with this, also extended syllogistic reasoning by adding new quantifiers was proposed, see for example, [3, 2, 15, 16, 18, 19, 20]). Dubois et al. work in these publications with quantifiers represented as crisp closed intervals (*more than a half* $= [0.5, 1]$, *around three*$= [2, 4]$). Pereira-Fariña et al. continue in [16] by analyzing logical syllogisms with more premises. This group of authors proposes a *general inference schema for syllogistic reasoning* which is based on transformation of the syllogistic reasoning problem into an equivalent optimization problem.

Generalization of the syllogistic reasoning in the four classical figures in which the classical quantifiers are replace by the generalized (fuzzy) ones was introduced by Peterson in [17]. It was formally analyzed by Novák and Murinovä in [4, 7].

The main goal of this paper is to continue the study of new forms of generalized syllogisms with new quantifiers "At least, Several, A few, A little".

2 Preliminaries

2.1 Fuzzy type theory

The theory of intermediate quantifiers has been developed in higher-order fuzzy logic (fuzzy type theory; Ł-FTT) with Łukasiewicz MV-algebra of truth values. The basic syntactical objects of Ł-FTT are classical, namely the concepts of *type* and *formula* (see [1]). The atomic types are ϵ (elements) and o (truth values). General types are defined as follows: if α, β are types then $(\beta\alpha)$ is a type. We denote types by Greek letters and the set of all types by *Types*. The set of all formula of a type α is denoted by $Types_\alpha$.

The *language J* of Ł-FTT consists of variables $x_\alpha, \ldots$, special constants $c_\alpha, \ldots$ ($\alpha \in Types$), the symbol λ, and brackets. We will consider the following concrete special constants: $\mathbf{E}_{(o\alpha)\alpha}$ (fuzzy equality) for every $\alpha \in Types$, $\mathbf{C}_{(oo)o}$ (conjunction), $\mathbf{D}_{(oo)}$ (delta operation on truth values) and the description operator $\iota_{\varepsilon(o\varepsilon)}$.

Formulas are formed of variables, constants (each of specific type), and the symbol λ. Thus, each formula A is assigned a type and we write it as A_α. A set of formulas of type α is denoted by $Form_\alpha$. The set of all formulas is $Form = \bigcup_{\alpha\in Types} Form_\alpha$[1].

If $B \in Form_{\beta\alpha}$ and $A \in Form_\alpha$ then $(BA) \in Form_\beta$. Similarly, if $A \in Form_\beta$ and $x_\alpha \in J$, $\alpha \in Types$, is a variable then $(\lambda x_\alpha A) \in Form_{\beta\alpha}$.

Interpretation of formulas is the following. If $\mathscr{M}$ is a model then $\mathscr{M}(A_o) \in M_o$ is a truth value, $\mathscr{M}(A_\varepsilon) \in M_\varepsilon$ is some element and $\mathscr{M}(A_{\beta\alpha}) : M_\alpha \to M_\beta$ is a function. For example, $\mathscr{M}(A_{o\alpha}) : M_\alpha \to M_o$ is a fuzzy set and $\mathscr{M}(A_{(o\alpha)\alpha}) : M_\alpha \times M_\alpha \to M_o$ a fuzzy relation.

The following theorem will be used.

Theorem 2.1 ([9, 10]) *Let T be a consistent theory, $A \in Form_o$ and $\mathbf{u}_{1,\alpha}, \dots, \mathbf{u}_{n,\alpha}$, $\alpha \in Types$ be new special constants that do not belong to the language $J(T)$. If $T \vdash (\exists x_{1,\alpha}) \cdots (\exists x_{n,\alpha}) \mathbf{\Delta} A$ then $T \cup \{A_{x_{1,\alpha},\dots,x_{n,\alpha}}[\mathbf{u}_{1,\alpha}, \dots, \mathbf{u}_{n,\alpha}]\}$ is a conservative extension of T.*

The special (derived) formulas Υ_{oo} and $\hat{\Upsilon}_{oo}$ will be used below: $\Upsilon_{oo} A_o$ says that A_o in every model has a non-zero truth value and $\hat{\Upsilon}_{oo} A_o$ says that A_o has a general truth value (i.e., neither false 0, nor true 1)[2].

We will also need the following crisp predicate of sharp ordering of formulas from $Form_o$:

$$<_{(oo)o} := \lambda x_o \lambda y_o \cdot \mathbf{\Delta}(x_o \Rightarrow y_o) \,\&\, \neg\mathbf{\Delta}(x_o \equiv y_o).$$

Clearly, if $\mathscr{M}$ is a model then $\mathscr{M}_p(x_o < y_o) = 1$ iff $\mathscr{M}_p(x_o) < \mathscr{M}_p(y_o)$ for arbitrary assignment $p \in \mathrm{Asg}(\mathscr{M})$.

2.2 Evaluative linguistic expressions

The theory of intermediate quantifiers is based on the theory of evaluative linguistic expressions that are expressions of natural language such as *small, medium, big, very short, more or less deep, quite roughly strong, extremely high*, etc. There is a formal theory of them presented in detail in [10] and less formally including formulas for the direct computation in [14].

The semantics of evaluative linguistic expressions is formulated in a special formal theory T^{Ev} of Ł-FTT. Its language J^{Ev} has the following special symbols:

(i) The constants $\top, \bot \in Form_o$ for truth and falsity and $\dagger \in Form_o$ for the middle truth value.

(ii) A special constant $\sim \in Form_{(oo)o}$ for an additional fuzzy equality on the set of truth values L.

(iii) A set of special constants $\boldsymbol{\nu}, \ldots \in Form_{oo}$ for linguistic hedges. The J^{Ev} is supposed to contain the following special constants: $\{Ex, Si, Ve, ML, Ro, QR, VR\}$ that represent the linguistic hedges (*extremely, significantly, very, roughly, more or less, rather, quite roughly, very roughly*, respectively).

(iv) A set of triples of additional constants $(\mathbf{a}_{\boldsymbol{\nu}}, \mathbf{b}_{\boldsymbol{\nu}}, \mathbf{c}_{\boldsymbol{\nu}}), \ldots \in Form_o$ where each hedge $\boldsymbol{\nu}$ is uniquely associated with one triple of these constants.

The evaluative expressions are construed by special formulas $Sm \in Form_{oo(oo)}$ (*small*), $Me \in Form_{oo(oo)}$ (*medium*), $Bi \in Form_{oo(oo)}$ (*big*), and $Ze \in Form_{oo(oo)}$ (*zero*) that can be extended by several selected linguistic hedges. Recall that a *hedge*, which is often an adverb such as "very, significantly, about, roughly", etc. is in general construed by a formula $\boldsymbol{\nu} \in Form_{oo}$ with specific properties. To classify that a given formula is a hedge, we introduced a formula $Hedge \in Form_{o(oo)}$. Then $T^{\mathrm{Ev}} \vdash Hedge\, \boldsymbol{\nu}$ means that $\boldsymbol{\nu}$ is a hedge. We refer the reader to [10] for the technical details. We assume that the following is provable: $T^{\mathrm{Ev}} \vdash Hedge\, \boldsymbol{\nu}$ for all $\boldsymbol{\nu} \in \{Ex, Si, Ve, ML, Ro, QR, VR\}$.

The evaluative linguistic expression is represented in the theory T^{Ev} by one of the following formulas:

$$Sm\,\boldsymbol{\nu}, Me\,\boldsymbol{\nu}, Bi\,\boldsymbol{\nu}, Ze\,\boldsymbol{\nu} \in Form_{oo} \qquad (1)$$

where $\boldsymbol{\nu}$ is a hedge.

For example, $SmVe$ is a formula construing the evaluative expression "very small". We will also consider an *empty hedge* $\bar{\boldsymbol{\nu}}$ that is always present in front of *small, medium* and *big* if no other hedge is given. We also apply the connective $\mathbf{\Delta}_{oo}$ that represents the expression "utmost". Crucial in the model of the semantics of evaluative expressions is an *additional fuzzy equality* $\sim$ using which semantics of the formulas Sm, Me, Bi, Ze is defined. We also assume that $\sim$ is separated.

If, for explanation, we do not need to know the concrete evaluative expression then we will use the metavariable *Ev*.

Evaluative expressions characterize certain imprecisely determined positions on a bounded linearly ordered scale. As there is an infinite number of scales, we must first specify the *context*, in which we characterize them (cf. [10]). This is formally represented by

[1] To improve readability of formulas, we quite often write the type only once in the beginning of the formula and then omit it. Alternatively, we write $A \in Form_\alpha$ to emphasize that A is a formula of type α and do not repeat its type again.

[2] Their formal definitions are $\Upsilon_{oo} \equiv \lambda z_o \cdot \neg\mathbf{\Delta}(\neg z_o)$ and $\hat{\Upsilon}_{oo} \equiv \lambda z_o \cdot \neg\mathbf{\Delta}(z_o \vee \neg z_o)$.

a function $w : E \to M$. Less formally, we define the context as a triple of numbers $v_L, v_S, v_R \in M$ such that $v_L < v_S < v_R$ (the ordering on M is induced by w). Then $x \in w$ iff $x \in [v_L, v_S] \cup [v_S, v_R]$.[3] The context is *standard* if $v_L = 0, v_S = 0.4, v_R = 1$.

The concept of context naturally leads to the concepts of intension and extension. For simplification, we usually omit in the theory of intermediate quantifiers the context and, hence, intension of the expressions (1) is identical to their extension that reduces to a simple fuzzy set in the universe of truth values. More details about the theory of evaluative linguistic expressions can be found in the above cited literature.

Let $\boldsymbol{\nu}_{1,oo}, \boldsymbol{\nu}_{2,oo}$ be two hedges, i.e., $T^{\mathrm{Ev}} \vdash Hedge\, \boldsymbol{\nu}_{1,oo}$ and $T^{\mathrm{Ev}} \vdash Hedge\, \boldsymbol{\nu}_{2,oo}$. We define a relation of partial ordering of hedges by

$$\ll_{o(oo)(oo)} \equiv \lambda p_{oo} \lambda q_{oo} \cdot (\forall z_o) \boldsymbol{\Delta}(pz \Rightarrow qz). \qquad (2)$$

Theorem 2.2 ([13]) *(a)*

$$T^{Ev} \vdash \boldsymbol{\Delta} \ll Ex \ll Si \ll Ve \ll \bar{\boldsymbol{\nu}} \ll ML \ll Ro \ll QR \ll VR. \qquad (3)$$

(b) Let $\boldsymbol{\nu}_{1,oo}, \boldsymbol{\nu}_{2,oo}$ be hedges such that $T^{Ev} \vdash \boldsymbol{\nu}_{1,oo} \ll \boldsymbol{\nu}_{2,oo}$. Then for any context w

$$T^{Ev} \vdash (Sm\,\boldsymbol{\nu}_1)w \subseteq (Sm\,\boldsymbol{\nu}_2)w,$$
$$T^{Ev} \vdash (Me\,\boldsymbol{\nu}_1)w \subseteq (Me\,\boldsymbol{\nu}_2)w,$$
$$T^{Ev} \vdash (Bi\,\boldsymbol{\nu}_1)w \subseteq (Bi\,\boldsymbol{\nu}_2)w.$$

A special evaluative expression is "zero". A mathematical model of its meaning in the standard context is

$$\widetilde{Ze}_{oo} \equiv \lambda z_o\, \widetilde{Sm}\,\boldsymbol{\nu}_{Ze}\, z, \qquad (4)$$

and $Ze_{(o\alpha)(\alpha o)} \equiv \lambda w_{\alpha o} \lambda x_\alpha (Sm\,\boldsymbol{\nu}_{Ze})wx$ if a general context w is considered. The special hedge $\boldsymbol{\nu}_{Ze}$ has in the ordering $\ll$ the position

$$T^{\mathrm{Ev}} \vdash \boldsymbol{\Delta} \ll \boldsymbol{\nu}_{Ze} \ll Ex.$$

Hence,

$$T^{\mathrm{Ev}} \vdash Ze\, w_{\omega o} \subseteq (Sm\, Ex)w_{\omega o}$$

due to Theorem 2.2(b). This expression is specific in the sense, that "zero" is considered in a fuzzy sense. Namely, that also values very close to zero are taken as being zero. Hence, to define its mathematical model, we put $c_{\boldsymbol{\nu}_{Ze}} = \bot$. Then, using the general definition, we obtain

$$c_{Ze} \equiv \iota\, \lambda x_o \cdot \boldsymbol{\Delta}(LH\, x \equiv \mathbf{c}_{\boldsymbol{\nu}_{Ze}}) \qquad (5)$$

which gives $c_{Ze} = \bot$[4].

To characterize properties of *Ze*, we must introduce the following axiom for arbitrary hedge $\boldsymbol{\nu}$:

(NZc) $z_o < \mathbf{c}_{\boldsymbol{\nu}} \equiv \neg\boldsymbol{\Delta}(\boldsymbol{\nu}\, z_o)$.

This axiom says that a truth value z_o is strictly smaller than $\mathbf{c}_{\boldsymbol{\nu}}$ iff application of the hedge $\boldsymbol{\nu}$ to it is strictly smaller than $\top$. This is clear from interpretation of $\mathbf{c}_{\boldsymbol{\nu}}$ because this constant represents a threshold over which the function interpreting $\boldsymbol{\nu}$ is equal to 1. The following can be proved:

$$T^{\mathrm{Ev}} \vdash (\widetilde{Ze}\,\boldsymbol{\nu}_{Ze})\bot \equiv \top,$$

which says that the fundamental property of *Ze* is provable (i.e., the membership degree of zero in *Ze* is 1).

Below, we introduce the evaluative expression "*positively* ⟨*hedge*⟩*small*" that is formally defined by

$$^{+}Sm\,\boldsymbol{\nu} \equiv \lambda t_o \cdot (Sm\,\boldsymbol{\nu})t_o\, \&\, \neg(Ze\,\boldsymbol{\nu}_{Ze})t_o. \qquad (6)$$

Clearly, $^{+}Sm\,\boldsymbol{\nu} \in Form_{oo}$.

3 The theory of intermediate quantifiers

Recall that the meaning of intermediate quantifiers lays between the meaning of the classical quantifiers $\forall$ and $\exists$. They are modeled by selected formulas of a special formal theory T^{Ev} of Ł-FTT. These formulas express quantification over the universe represented by a fuzzy set whose size is characterized by a measure due to the following definition. Formal definitions of operations on fuzzy sets can be found in [11].

Definition 3.1 *Let $R \in Form_{o(o\alpha)(o\alpha)}$ be a formula where $\alpha \in Types$ is an arbitrary type.*

(i) A formula $\mu \in Form_{o(o\alpha)(o\alpha)}$ defined by

$$\mu_{o(o\alpha)(o\alpha)} \equiv \lambda z_{o\alpha} \lambda x_{o\alpha} (Rz_{o\alpha})x_{o\alpha} \qquad (7)$$

represents a measure on fuzzy sets *in the universe of type $\alpha \in Types$ if it has the following properties:*

(M1) $\boldsymbol{\Delta}(x_{o\alpha} \subseteq z_{o\alpha})\, \&\, \boldsymbol{\Delta}(y_{o\alpha} \subseteq z_{o\alpha})\, \&\, \boldsymbol{\Delta}(x_{o\alpha} \subseteq y_{o\alpha}) \Rightarrow ((\mu z_{o\alpha})x_{o\alpha} \Rightarrow (\mu z_{o\alpha})y_{o\alpha})$,

(M2) $\boldsymbol{\Delta}(x_{o\alpha} \subseteq z_{o\alpha}) \Rightarrow ((\mu z_{o\alpha})(z_{o\alpha} \setminus x_{o\alpha}) \equiv \neg(\mu z_{o\alpha})x_{o\alpha})$,

[3]We write the interval $[v_L, v_R]$ as the union of intervals $[v_L, v_S] \cup [v_S, v_R]$ to emphasize the role of the typically medium value v_S.

[4]Do not confuse $\mathbf{c}_{\boldsymbol{\nu}_{Ze}}$ and c_{Ze}. While the former is a parameter of the linguistic hedge $\boldsymbol{\nu}_{Ze}$ for "zero", the latter is the corresponding threshold on the universe, which, in this case, is the set of truth values. If we consider a context $w = \langle 0, v_S, v_R \rangle$ then c_{Ze} would equal 0 on the scale $[0, v_S] \cup [v_S, v_R]$.

(M3) $\mathbf{\Delta}(x_{o\alpha} \subseteq y_{o\alpha}) \,\&\, \mathbf{\Delta}(x_{o\alpha} \subseteq z_{o\alpha}) \,\&\, \mathbf{\Delta}(y_{o\alpha} \subseteq z_{o\alpha}) \Rightarrow ((\mu z_{o\alpha})x_{o\alpha} \Rightarrow (\mu y_{o\alpha})x_{o\alpha})$

where $x_{o\alpha}, y_{o\alpha}, z_{o\alpha}$ are variables representing fuzzy sets.

(ii) The following formula characterizes measurable fuzzy sets *of a given type α:*

$$\mathbf{M}_{o(o\alpha)} \equiv \lambda z_{o\alpha} \cdot \neg\mathbf{\Delta}(z_{o\alpha} \equiv \emptyset_{o\alpha}) \,\&\, \mathbf{\Delta}(\mu z_{o\alpha})z_{o\alpha} \,\&\, (\forall x_{o\alpha})(\forall y_{o\alpha})\mathbf{\Delta}((M1) \,\&\, (M3)) \,\&\, (\forall x_{o\alpha})\mathbf{\Delta}(M2) \quad (8)$$

where (M1)–(M3) are the axioms from (i).

On the basis of this definition we can prove that the empty (fuzzy) set has zero measure.

Lemma 3.2 *Let $\vdash \mathbf{M}(z_{o\alpha})$. Then for any $u \in Form_{o\alpha}$*

$$\vdash \mathbf{\Delta}(u_{o\alpha} \equiv \emptyset_{o\alpha}) \Rightarrow ((\mu z_{o\alpha})u_{o\alpha} \equiv \bot).$$

Proof 3.3 *$\vdash (\mu z_{o\alpha})\emptyset_{o\alpha} \equiv \bot$ by [9, Lemma 6]. The lemma then follows from the deduction theorem by adding $x_{o\alpha} \equiv \emptyset_{o\alpha}$ as the assumption.*

Lemma 3.4 *Let $\vdash \mathbf{M}(z_{o\alpha})$. Then for any $u \in Form_{o\alpha}$ and $x \in Form_\alpha$*

$$T^{Ev} \vdash \Upsilon\neg(Ze\boldsymbol{\nu})((\mu z_{o\alpha})u_{o\alpha}) \Rightarrow \Upsilon(\exists x_\alpha)(u_{o\alpha}x).$$

Proof 3.5 *(L.1)* $\vdash \neg((\mu z_{o\alpha})u_{o\alpha} \equiv \bot) \Rightarrow \neg\mathbf{\Delta}(u_{o\alpha} \equiv \emptyset_{o\alpha})$ *(Lemma 3.2, contraposition)*

(L.2) $T^{Ev} \vdash \Upsilon(\neg Ze\boldsymbol{\nu})(\mu z_{o\alpha})u_{o\alpha} \Rightarrow (c_{Ze} < (\mu z_{o\alpha})u_{o\alpha})$ *(proof of [9, Lemma 5(g)])*

(L.3) $T^{Ev} \vdash c_{Ze} \equiv \bot$ *(by (5))*

(L.4) $T^{Ev} \vdash \Upsilon(\neg Ze\boldsymbol{\nu})(\mu z_{o\alpha})u_{o\alpha} \Rightarrow \neg\mathbf{\Delta}((\mu z_{o\alpha})u_{o\alpha} \equiv \bot)$ *(L.2, L.3, definition of "$<$", properties of Ł-FTT)*

(L.5) $T^{Ev} \vdash \Upsilon(\neg Ze\boldsymbol{\nu})(\mu z_{o\alpha})u_{o\alpha} \Rightarrow \neg\mathbf{\Delta}(u_{o\alpha} \equiv \emptyset_{o\alpha})$ *(L.1, L.4, properties of Ł-FTT)*

(L.6) $T^{Ev} \vdash \Upsilon(\neg Ze\boldsymbol{\nu})(\mu z_{o\alpha})u_{o\alpha} \Rightarrow \neg\mathbf{\Delta}(\forall x)\neg(u_{o\alpha}x)$ *(L.5, def. of $\emptyset$, properties of Ł-FTT)*

(L.7) $T^{Ev} \vdash \Upsilon(\neg Ze\boldsymbol{\nu})(\mu z_{o\alpha})u_{o\alpha} \Rightarrow \neg\mathbf{\Delta}\neg(\exists x)u_{o\alpha}x$ *(L.6, properties of Ł-FTT)*

(L.8) $T^{Ev} \vdash \Upsilon(\neg Ze\boldsymbol{\nu})(\mu z_{o\alpha})u_{o\alpha} \Rightarrow \Upsilon(\exists x)u_{o\alpha}x$ *(L.7, def. of Υ, properties of Ł-FTT)*

By this lemma, if the measure of a fuzzy set $u_{o\alpha}$ is non-zero then there exists an element x_α in $u_{o\alpha}$ with non-zero membership degree.

For the definition of the intermediate quantifier, we need a special operation "cut of a fuzzy set $y \in Form_{o\alpha}$", given a fuzzy set $z \in Form_{o\alpha}$:

$$y|z \equiv \lambda x_\alpha \cdot zx \,\&\, \mathbf{\Delta}(\Upsilon(zx) \Rightarrow (yx \equiv zx)). \quad (9)$$

Interpretation of this formula in a model is the following: if fuzzy sets $B, Z \underset{\sim}{\subset} M$ are given, then the operation $B|Z$ "cuts" B by taking only those $m \in M$ whose membership $B(m)$ is equal to $Z(m)$, otherwise $(B|Z)(m) = 0$. If there is no such element then $B|Z = \emptyset$. We can thus take various fuzzy sets Z to "pick up proper elements" from B.

Definition 3.6 *Let $\mathscr{S} \subseteq Types$ be a selected set of types and $P = \{R \in Form_{o(o\alpha)(o\alpha)} \mid \alpha \in \mathscr{S}\}$ be a set of new constants. Let T be a consistent extension of the theory T^{Ev} in the language $J(T) \supseteq J^{Ev} \cup P$. We say that the theory T contains* intermediate quantifiers *w.r.t. the set of types $\mathscr{S}$ if for all $\alpha \in \mathscr{S}$ the following is provable:*

(i)

$$T \vdash (\exists z_{o\alpha})\mathbf{M}_{o(o\alpha)}z_{o\alpha}. \quad (10)$$

(ii)

$$T \vdash (\forall z_{o\alpha})(\exists x_{o\alpha})(\mathbf{M}_{o(o\alpha)}z_{o\alpha} \Rightarrow (\mathbf{\Delta}(x_{o\alpha} \subseteq z_{o\alpha}) \,\&\, \hat{\Upsilon}((\mu z_{o\alpha})x_{o\alpha})). \quad (11)$$

In the sequel, we will denote the theory due to Definition 3.6 by T^{IQ} and fix a selected set of types $\mathscr{S}$.

Definition 3.7 *Let $Ev \in Form_{oo}$ be a formula representing some evaluative linguistic expression, $z \in Form_{o\alpha}$, $x \in Form_\alpha$ be variables and $A, B \in Form_{o\alpha}$ be formulas such that $T^{IQ} \vdash \mathbf{M}_{o(o\alpha)}B$, $\alpha \in \mathscr{S}$. An intermediate quantifier of type $\langle 1,1 \rangle$ is one of the following formulas:*

$$(Q^{\forall}_{Ev}x)(B,A) \equiv (\exists z)[(\forall x)((B|z)x \Rightarrow Ax) \wedge Ev((\mu B)(B|z))], \quad (12)$$

$$(Q^{\exists}_{Ev}x)(B,A) \equiv (\exists z)[(\exists x)((B|z)x \wedge Ax) \wedge Ev((\mu B)(B|z))]. \quad (13)$$

Either of the quantifiers (12) or (13) construes the sentence

$$\langle \textit{Quantifier} \rangle \textit{ B's are A.}$$

Formula $B_{o\alpha}$ in (i)–(iii) represents a universe of quantification.

If we replace the metavariable *Ev* in (12) or (13) by a formula representing a specific evaluative linguistic expression we obtain definition of the concrete intermediate quantifier.

Definition of serveral intermediate quantifiers used below is the following.

- **(A)** "All *B*'s are *A*": $(Q^{\forall}_{Bi\mathbf{\Delta}}x)(B,A)$
- **(P)** "Almost all *B*'s are *A*": $(Q^{\forall}_{Bi\,Ex}x)(B,A)$
- **(T)** "Most *B*'s are *A*": $(Q^{\forall}_{Bi\,Ve}x)(B,A)$
- **(K)** "Many *B*'s are *A*": $(Q^{\forall}_{\neg Sm}x)(B,A)$
- **(I)** "Some *B*'s are *A*": $(Q^{\exists}_{Bi\mathbf{\Delta}}x)(B,A)$

By the results of [9], we can prove the following lemma.

Lemma 3.8 *Let $\mathscr{M} \models T^{IQ}$ be a model, p an assignment and $B,z \in Form_{o\alpha}$ be formulas. Let $Ev \in \{Bi\mathbf{\Delta}, BiSi, BiVe, \neg(Sm\,\bar{\mathbf{v}})\}$. If $\mathscr{M}_p(Ev((\mu B)(B|z))) > 0$ then there exists an element $m \in M_\alpha$, such that*

$$\mathscr{M}_p((\exists x)(B|z)x) \geq \mathscr{M}_p(B|z)(m) > 0 \qquad (14)$$

holds in any model $\mathscr{M} \models T^{IQ}$ and for every assignment $p \in$ Asg.

Proof 3.9 *Let $\mathscr{M} \models T^{IQ}$ be a model. If $\mathscr{M}_p(Ev((\mu B)(B|z))) > 0$ then*

$$\mathscr{M}_p((\mu B)(B|z)) > 0$$

because none of the considered evaluative expressions allows zero value of this measure. Using Lemma 3.2 we conclude that the fuzzy set $\mathscr{M}_p(B|z)$ is non empty, i.e., there is $m \in M_\alpha$ such that $\mathscr{M}_p(B|z)(m) > 0$. This implies (14).

□

Let us remark the question whether presupposition in both quantifiers should be considered. Indeed, the empirical observation reveals necessity to exclude empty fuzzy set $\mathscr{M}(B_{o\alpha}|z_{o\alpha})$. For example:

> "There are several cows grazing in the meadow"

cannot mean that there was no cow in the meadow. Therefore, we will define the quantifiers "A few, Several, A little" as follows.

Definition 3.10 *Let T^{IQ} be a theory containing intermediate quantifiers w.r.t. a set of types $\mathscr{S}$, $z \in Form_{o\alpha}$, $x \in Form_\alpha$ and $A,B \in Form_{o\alpha}$ be the same as in Definition 3.7. The following are new quantifiers:*

- **(F)** *"A few (A little) B's are A": $(Q^{\forall}_{+Sm\,Si}x)(B,A)$.*
- **(V)** *"A few (A little) B's are not A": $(Q^{\forall}_{+Sm\,Si}x)(B,\neg A)$.*
- **(S)** *"Several B's are A": $(Q^{\forall}_{+Sm\,Ve}x)(B,A)$.*
- **(Z)** *"Several B's are not A": $(Q^{\forall}_{+Sm\,Ve}x)(B,\neg A)$.*

Note that there is no formal difference between "A few" and "A little" in this definition. The difference consists in the used measure and manifests itself only when a specific model $\mathscr{M}$ is considered. Namely, if the support of the fuzzy set $\mathscr{M}(B_{o\alpha})$ is countable with the corresponding measure then **(F)** or **(V)** construes "A few". If it is uncountable then it construes "A little".

Note further that there is a difference between the quantifier "Few" and the quantifier "A few". From the Peterson's square of opposition it follows that if we say "Few students were successful in the test" then the statement cannot be true if "All students were successful in the test", because quantifiers "All" and "'Few " are contraries. On the other hand if we say that "All students were successful in the test" then also "A few students were successful in the test", because the quantifier "All" is superaltern of the quantifier "A few". For the precise definitions see [9].

Definition 3.11 *Let $(Q^{\forall}_{Ev}x)(B,A) \in \{A\,few,\ Several,\ A\,little\}$. Let $z,z' \in Form_{o\alpha}$, $x \in Form_\alpha$ be variables and $A,B \in Form_{o\alpha}$ be formulas such that $T^{IQ} \vdash \mathbf{M}_{o(o\alpha)}B$, $\alpha \in \mathscr{S}$. An intermediate quantifier construing the sentence*

At least ⟨Quantifier⟩ B's are A

is formula

$$(\mathbf{AtL}\,Q^{\forall}_{Ev}x)(B,A) \equiv (\exists z)(\exists z')[((\forall x)((B|z)x \Rightarrow Ax) \wedge Ev((\mu B)(B|z))) \vee (\mathbf{\Delta}(z \subseteq z') \wedge (\forall x)((B|z')x \Rightarrow Ax))]. \qquad (15)$$

By this definition, if a fuzzy set $z_{o\alpha}$ assures $(Q^{\forall}_{Ev}x)(B,A)$ in some degree, we assume that there can be a fuzzy set $z'_{o\alpha}$ larger than $z_{o\alpha}$ whose cut of B also has the property A. By the monotonicity of μ, $\vdash (\mu B)(B|z) \Rightarrow (\mu B)(B|z')$. For the simplicity we denote by $((\mathbf{AtL^a})$⟨Quantifier⟩ affirmative and by $((\mathbf{AtL^n})$⟨Quantifier⟩ negative quantifier, respectively.

4 New forms of generalized Peterson's syllogisms

4.1 Formalization of syllogisms

In general, we can define syllogism as follows. A *syllogism* is a triple of formulas $\langle P_{o,1}, P_{o,2}, C_o \rangle$ where P_1 is a *major premise*, P_2 a *minor premise* and C is a *conclusion*. We say that the syllogism is *valid* if $T^{\mathrm{IQ}} \vdash P_1 \,\&\, P_2 \Rightarrow C$.

A typical syllogisms of Figure III is

P_1 : Almost all jokes are funny.
P_2 : Most jokes are old.
C : Some old jokes are funny.

4.2 New forms of generalized syllogisms

In our previous papers [4, 6], we syntactically and semantically analyzed several forms of generalized Peterson's syllogisms. This subsection is devoted to the study of new forms of generalized Peterson's syllogisms with the quantifier "At least <several, a few> B's are A".

Below we summarize valid forms of syllogisms of Figure-I as follows:

Theorem 4.1 *[6] The following syllogisms of Figure-I are valid in T^{IQ}:*

AAA						
AAP	**APP**					
AAT	**APT**	**ATT**				
AAK	**APK**	**ATK**	**AKK**			
AAF	**APF**	**ATF**	**AKF**	**AFF**		
AAS	**APS**	**ATS**	**AKS**	**AFS**	**ASS**	
A*AI	**A*PI**	**A*TI**	**A*KI**	**A*FI**	**A*SI**	**AII**

We begin with the theorem stating that the quantifiers "At least <several, a few> B's are A" have the monotonicity property.

Theorem 4.2 *Let $B_{o\alpha}$ and $A_{o\alpha}$ be formulas. Then the following is provable in T^{IQ}:*

(a) $T^{IQ} \vdash \mathbf{S} \Rightarrow (\mathbf{AtL^a})\mathbf{S}$

(b) $T^{IQ} \vdash ((\mathbf{AtL^a})\mathbf{S})^* \Rightarrow \mathbf{I}$

Proof 4.3 *(a) Using $\vdash A \Rightarrow A \vee B$ we have*

$$T^{IQ} \vdash [((\forall x)(B|z) \Rightarrow Ax) \wedge {}^{+}Sm\,Ve((\mu B)(B|z))] \Rightarrow [((\forall x)(B|z) \Rightarrow Ax) \wedge {}^{+}Sm\,Ve((\mu B)(B|z))] \vee (\boldsymbol{\Delta}(z \subseteq z') \wedge (\forall x)((B|z')x \Rightarrow Ax)) \quad (16)$$

By the quantifier properties we obtain

$$T^{IQ} \vdash \mathbf{S} \Rightarrow (\exists z)(\exists z')[((\forall x)(B|z) \Rightarrow Ax) \wedge {}^{+}Sm\,Ve((\mu B)(B|z)) \vee (\boldsymbol{\Delta}(z \subseteq z') \wedge (\forall x)((B|z')x \Rightarrow Ax))] \quad (17)$$

(b)

$$T^{IQ} \vdash (\forall x)((B|z) \Rightarrow Ax) \wedge {}^{+}Sm\,Ve((\mu B)(B|z)) \Rightarrow (\forall x)((B|z) \Rightarrow Ax) \quad (18)$$

Let us denote by $Ev = {}^{+}Sm\,Ve((\mu B)(B|z))$. By properties of Ł-FTTwe have

$$T^{IQ} \vdash ((\forall x)((B|z) \Rightarrow Ax) \wedge Ev) \,\&\, (\exists x)(B|z)x \Rightarrow (\exists x)((B|z)x \wedge Ax) \quad (19)$$

which gives us using $\vdash ((B|z)x \wedge Ax) \Rightarrow (Bx \wedge Ax)$ and by quantifiers properties

$$T^{IQ} \vdash (\exists z)(((\forall x)((B|z) \Rightarrow Ax) \wedge Ev) \,\&\, (\exists x)(B|z)x) \Rightarrow (\exists x)(Bx \wedge Ax). \quad (20)$$

Furthermore, similarly as above, we know that

$$T^{IQ} \vdash (\forall x)((B|z')x \Rightarrow Ax)) \Rightarrow ((\exists x)(B|z')x)) \Rightarrow ((\exists x)((B|z')x \wedge Ax)). \quad (21)$$

By weakening of $\wedge$ and by quantifiers properties we obtain

$$T^{IQ} \vdash (\exists z)(\exists z')[(\boldsymbol{\Delta}(z \subseteq z') \wedge (\forall x)((B|z')x \Rightarrow Ax)) \,\&\, ((\exists x)(B|z')x))] \Rightarrow (\exists x)(Bx \wedge Ax) \quad (22)$$

From (20) and (22) and by quantifiers properties we obtain

$$T^{IQ} \vdash ((\mathbf{AtL^a})\mathbf{S})^* \Rightarrow (\exists x)(Bx \wedge Ax) \vee (\exists x)(Bx \wedge Ax). \quad (23)$$

Finally, using $\vdash (A \vee A) \equiv A$ we can conclude that

$$T^{IQ} \vdash ((\mathbf{AtL^a})\mathbf{S})^* \Rightarrow (\exists x)(Bx \wedge Ax). \quad (24)$$

Theorem 4.4 *Let $B_{o\alpha}$ and $A_{o\alpha}$ be formulas. Then the following is provable in T^{IQ}:*

(a) $T^{IQ} \vdash \mathbf{Z} \Rightarrow (\mathbf{AtL^n})\mathbf{Z}$

(b) $T^{IQ} \vdash ((\mathbf{AtL^n}\mathbf{Z})^* \Rightarrow \mathbf{O}$

Proof 4.5 *Similarly as in the previous theorem.*

The syllogisms with the quantifiers "a few, several, a little" are limited but still validity of few can be proved similarly as other syllogisms of Figure-I in [4]. Below we introduce the syllogism of Figure-I with the quantifier "At least" in the conclusion.

P_1: All M are Y
P_2: Several X are M
C: At least several X are Y

Validity of this syllogism can be verified as follows:

- To use the monotonicity property from Theorem 4.2.
- To find syntactical proof in Ł-FTT.

Below, we introduce the both variants.

Theorem 4.6 *Let $M, X, Y, z, z' \in Form_{o\alpha}$ be formulas representing fuzzy sets and $x \in Form_\alpha$ be a variable. Then the following syllogism is valid for evaluative expressions $Ev \in \{^+SmSi, ^+SmVe\}$:*

$$\begin{array}{l} P_1(\mathbf{A}) : (\forall x)(Yx \Rightarrow Xx). \\ P_2 : Q^{\forall}_{Ev}(M,Y) \equiv (\exists z)[(\forall x)((M|z)x \Rightarrow Yx) \wedge \\ \qquad\qquad Ev((\mu M)(M|z))]. \\ \hline C : (\mathbf{AtL}Q^{\forall}_{Ev}x)(M,X). \end{array}$$

Proof 4.7 *The following formula is provable:*

$$\vdash ((M|z)x \Rightarrow Yx) \,\&\, (Yx \Rightarrow Xx) \Rightarrow ((M|z)x \Rightarrow Xx).$$

From it follows that

$$\vdash (\forall x)((M|z)x \Rightarrow Yx) \,\&\, (\forall x)(Yx \Rightarrow Xx) \Rightarrow (\forall x)((M|z)x \Rightarrow Xx), \quad (25)$$

and further,

$$T^{IQ} \vdash [(\forall x)((M|z)x \Rightarrow Yx) \wedge Ev((\mu M)(M|z))] \,\&\, (\forall x)(Yx \Rightarrow Xx) \Rightarrow ((\forall x)((M|z)x \Rightarrow Xx) \wedge Ev((\mu M)(M|z)))$$

(using $\vdash (A \Rightarrow B) \Rightarrow ((A \Rightarrow C) \Rightarrow (A \Rightarrow B \wedge C))$).

Using the provable formula $\vdash A \Rightarrow A \vee B$, the transitivity of implication, and the quantifier properties, we finally obtain

$$T^{IQ} \vdash (Q^{\forall}_{Ev}x)(M,Y) \,\&\, (\forall x)(Yx \Rightarrow Xx) \Rightarrow (\exists z)(\exists z')[((\forall x)((M|z)x \Rightarrow Xx) \wedge Ev((\mu M)(M|z))) \vee (\mathbf{\Delta}(z \subseteq z') \wedge (\forall x)((M|z')x \Rightarrow Xx))].$$

Theorem 4.8 *Let* $\mathbf{AAA}, \mathbf{APP}, \mathbf{ATT}, \mathbf{AKK}, \mathbf{AFF}, \mathbf{ASS}$ *be a basic valid forms of syllogisms. Let $Q^{\forall}_{Ev}x)(B,A) \in \{Afew, Several, Alittle\}$. Then the following syllogisms* $\mathbf{AA}(\mathbf{AtL^a}), \mathbf{AP}(\mathbf{AtL^a}), \mathbf{AT}(\mathbf{AtL^a}), \mathbf{AK}(\mathbf{AtL^a})$ $\mathbf{AF}(\mathbf{AtL^a}), \mathbf{AS}(\mathbf{AtL^a})$ *are valid in T^{IQ}.*

Proof 4.9 *It follows from the assumption of valid forms of syllogisms and by the monotonicity Theorem 4.2.*

Theorem 4.10 *Let* $\mathbf{EAE}, \mathbf{EPB}, \mathbf{ETD}, \mathbf{EKG}, \mathbf{EFV}, \mathbf{ESZ}$ *be a basic valid forms of syllogisms. Let $Q^{\forall}_{Ev}x)(B,A) \in \{Afew, Several, Alittle\}$. Then the following syllogisms* $\mathbf{EA}(\mathbf{AtL^n}), \mathbf{EP}(\mathbf{AtL^n}), \mathbf{ET}(\mathbf{AtL^n}), \mathbf{EK}(\mathbf{AtL^n})$ $\mathbf{EF}(\mathbf{AtL^n}), \mathbf{ES}(\mathbf{AtL^n})$ *are valid in T^{IQ}.*

5 Conclusion

In this paper, we continued to study generalized Peterson's syllogisms with the new quantifier "At least <several, a few, a little> B's are A". We syntactically proved new forms of valid syllogisms.

Further, we will extend this work in several directions. We will build on these results and continue our studies of new forms of syllogisms of other figures. In the next scientific part we will analyze graded structures (square, cube, hexagon, etc.) of opposition in fuzzy natural logic with the new quantifier.

Acknowledgement

The work was supported from ERDF/ESF by the project "Centre for the development of Artificial Inteligence Methods for the Automotive Industry of the region" No. CZ.02.1.01/0.0/0.0/17-049/0008414.

References

[1] P. Andrews, An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof, Kluwer, Dordrecht, 2002.

[2] D. Dubois, L. Godo, H. López de Mántras, R. Prade, Qualitative reasoning with imprecise probabilities, Journal of Intelligent Information Systems 2 (1993) 319–363.

[3] D. Dubois, H. Prade, On fuzzy syllogisms, Comput.Intell. 4 (1988) 171–179.

[4] P. Murinová, V. Novák, A formal theory of generalized intermediate syllogisms, Fuzzy Sets and Systems 186 (2013) 47–80.

[5] P. Murinová, V. Novák, Analysis of generalized square of opposition with intermediate quantifiers, Fuzzy Sets and Systems 242 (2014) 89–113.

[6] P. Murinová, V. Novák, The structure of generalized intermediate syllogisms, Fuzzy Sets and Systems 247 (2014) 18–37.

[7] P. Murinová, V. Novák, Generalized conjunctive syllogisms with more premisses in fuzzy naturral

logic, Uncertainty Modelling in Knowledge Engineering and Decision Making 10 (2016) 282–288.

[8] P. Murinová, V. Novák, Syllogisms and 5-square of opposition with intermediate quantifiers in fuzzy natural logic, Logica universalis 10 (2) (2016) 339–357.

[9] P. Murinová, V. Novák, The theory of intermediate quantifiers in fuzzy natural logic revisited and the model of "many", Fuzzy Sets and Systems 388 (2020) 56–89.

[10] V. Novák, A comprehensive theory of trichotomous evaluative linguistic expressions, Fuzzy Sets and Systems 159 (22) (2008) 2939–2969.

[11] V. Novák, A formal theory of intermediate quantifiers, Fuzzy Sets and Systems 159 (10) (2008) 1229–1246.

[12] V. Novák, P. Murinová, On the properties of measure in the theory of intermediate quantifiers and the quantifier "many", in: Proc. SSCI-FOCI 2017, Honolulu, USA, 2017.

[13] V. Novák, P. Murinová, A formal model of the intermediate quantifiers "a few", "several" and "a little", in: Proc. IFSA-NAFIPS 2019, Lafayette, USA, 2019.

[14] V. Novák, I. Perfilieva, A. Dvořák, Insight into Fuzzy Modeling, Wiley & Sons, Hoboken, New Jersey, 2016.

[15] M. Pereira-Fariña, F. Díaz-Hermida, A. Bugarín, On the analysis of set-based fuzzy quantified reasoning using classical syllogistics, Fuzzy Sets and Systems 214 (2013) 83–94.

[16] M. Pereira-Fariña, J. C. Vidal, F. Díaz-Hermida, A. Bugarín, A fuzzy syllogistic reasoning schema for generalized quantifiers, Fuzzy Sets and Systems 234 (2014) 79–96.

[17] P. Peterson, Intermediate Quantifiers. Logic, linguistics, and Aristotelian semantics, Ashgate, Aldershot, 2000.

[18] D. G. Schwartz, Dynamic reasoning with qualified syllogisms, Artif.Intell. 93 (1997) 103–167.

[19] L. A. Zadeh, A computational approach to fuzzy quantifiers in natural languages, Computers and Mathematics 9 (1983) 149–184.

[20] L. A. Zadeh, Syllogistic reasoning in fuzzy logic and its applications to usuality and reasoning with dispositions, IEEE Trans. Syst. Man Cybern. 15 (1985) 754–765.