Atlantis Studies in Uncertainty Modelling, volume 3
Joint Proceedings of the 19th World Congress of the International Fuzzy Systems Association (IFSA), the 12th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT), and the 11th International Summer School on Aggregation Operators (AGOP)

On Some Generalizations of Homogeneity of Aggregation Functions

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Abstract

Positive homogeneity of aggregation functions is one of important properties reflecting the ratio scales in decision making. We recall, introduce and discuss some generalizations of this property, including k-homogeneity, quasi-homogeneity and end-point linearity.

Keywords: Aggregation function, Homogeneity, Quasi-homogeneity, End-point linearity, homogeneity of order k.

1 Introduction

The standard homogeneity of n-dimensional real functions means that $f(\lambda \mathbf{x}) = \lambda f(\mathbf{x})$ for any real λ such that $\mathbf{x}$ and $\lambda \mathbf{x}$ are from the domain of f (a convex subset of $\mathbb{R}^n$ containing $\mathbf{0}$). Considering aggregation functions, i.e., monotone real functions $A : [0,1]^n \to [0,1]$ satisfying the boundary conditions $A(\mathbf{0}) = 0$ and $A(\mathbf{1}) = 1$ it is obvious that $\lambda < 0$ has no sense to be considered, i.e., we can deal with positive homogeneity only. Even this can be reduced, considering $\lambda \in [0,1]$ only. Hence, an n-ary aggregation function A (for more details concerning aggregation functions see [3, 8]) is homogeneous if and only if $A(\lambda \mathbf{x}) = \lambda A(\mathbf{x})$ for all $\lambda \in [0,1]$ and $\mathbf{x} \in [0,1]^n$. More details concerning homogeneous aggregation functions can be found in [13] or [8, Chapter 7], where the homogeneity of aggregation functions is also called as ratio scale invariance. The basic characterization of homogeneous aggregation functions is the next one (see, e.g., Theorem 1 in [13]).

Theorem 1.1 *Let $C : [0,1]^n \to [0,1]$ be an aggregation function. Define $H^C : [0,1]^n \to [0,1]$ by*

$$H^C(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{0}, \\ \max(\mathbf{x}) \cdot C\left(\frac{\mathbf{x}}{\max(\mathbf{x})}\right) & \text{otherwise.} \end{cases}$$

Then H^C is an aggregation function if and only if

$$\frac{C(\mathbf{x})}{C(\mathbf{y})} \geq \min\left(\frac{x_1}{y_1}, \ldots, \frac{x_n}{y_n}\right),$$

for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$ such that $\mathbf{x} \leq \mathbf{y}$ and there is $i \in \{1, \ldots, n\}$ such that $x_i = y_i = 1$ (the convention $\frac{0}{0} = 1$ is considered whenever necessary).

It is evident that, under the constraints of Theorem 1.1, H^C is a homogeneous aggregation function, and $H^C = C$ whenever C is a homogeneous aggregation function. In our contribution, we will deal with binary aggregation function only, i.e., $n = 2$ will be fixed. Due to Theorem 1.1, we have the next result.

Corollary 1.1 *Let $A : [0,1]^2 \to [0,1]$ be an aggregation function. Then A is homogeneous if and only if*

$$A(x,y) = \begin{cases} 0 & \text{if } x = y = 0, \\ x \cdot h\left(\frac{y}{x}\right) & \text{if } x \geq y \text{ and } x \neq 0, \\ y \cdot g\left(\frac{x}{y}\right) & \text{if } y \geq x \text{ and } y \neq 0, \end{cases}$$

where $h(y) = A(1,y)$ and $g(x) = A(x,1)$ satisfy $h(1) = g(1) = 1$, $\frac{h(y)}{y}$ and $\frac{g(x)}{x}$ are decreasing on $[0,1]$ (with convention $\frac{0}{0} = 1$).

The aim of our contribution is a deeper study of some generalizations of homogeneity of aggregation functions. Some of them were already studied for particular cases of aggregation functions and we recall some of known results. In some cases, our results are original and promising for the further study.
The contribution is organized as follows. In the next section, we recall the homogeneity of order k and several related aggregation functions. Section 3 is devoted to quasi-homogeneous aggregation functions. In Section 4, we introduce and discuss end-point linear aggregation functions. Finally, some concluding remarks are added.

2 Aggregation functions which are homogeneous of order k

An aggregation function $A : [0,1]^2 \to [0,1]$ is homogeneous of order k, or, in short k-homogeneous, where $k > 0$, if

$$A(\lambda x, \lambda y) = \lambda^k A(x,y)$$

for all $\lambda, x, y \in [0,1]$.

Theorem 2.1 ([17]) *Let $A : [0,1]^2 \to [0,1]$ be an aggregation function. Then it is k-homogeneous for some $k > 0$ if and only if $A = B^k$, where $B : [0,1]^2 \to [0,1]$ is a homogeneous aggregation function.*

Observe that the 1-homogeneity is just standard homogeneity, and that for any $k, r > 0$, if A is a k-homogeneous aggregation function then A^r is kr-homogeneous aggregation function. We recall some well known results for special types of aggregation functions (for their description and characterization see, e.g., [3, 8]).

Theorem 2.2 ([1])

i) Let $T : [0,1]^2 \to [0,1]$ be a t-norm. Then it is k-homogeneous for some $k > 0$ if and only if either $k = 1$ and T is the minimum t-norm, $T = T_M$, or $k = 2$ and T is the product t-norm, $T = T_P$.

ii) Let $S : [0,1]^2 \to [0,1]$ be a k-homogenous t-conorm for some $k > 0$. Then $k = 1$ and S is maximum t-conorm, $S = S_M$.

Theorem 2.3 ([12]) *Let $C : [0,1]^2 \to [0,1]$ be a copula. Then it is k-homogeneous for some $k > 0$ if and only if $1 \le k \le 2$ and $C = C_\theta$ is a member of the Cuandras-Augé family with $\theta = 2 - k$,*

$$C(x,y) = x^{k-1} y^{k-1} \min(x^{2-k}, y^{2-k}).$$

3 Quasi-homogeneous aggregation functions

Ebanks in [6] has introduced a weaker form of homogeneity, namely the quasi-homogeneity.

Definition 3.1 *An aggregation function $A : [0,1]^2 \to [0,1]$ is quasi-homogenous (or (φ, f)-quasi homogenous) if and only if there is a strictly monotone continuous function $\varphi : [0,1] \to \mathbb{R}$ and a function $f : [0,1] \to [0,1]$ such that for all $\lambda, x, y \in [0,1]$ it holds*

$$A(\lambda \mathbf{x}, \lambda y) = \varphi^{-1}\left(f(\lambda)\varphi(A(x,y))\right).$$

Obviously, if φ is the identity function and $f(\lambda) = \lambda^k$, then the quasi-homogeneity is just the k-homogeneity (and if $k = 1$, then we get the standard homogeneity). We recall few interesting results concerning the quasi-homogeneity of aggregation functions.

Theorem 3.1 ([6]) *A t-norm T is quasi-homogeneous if $T = T_M$, or $T = T_P$, or T is a strict Schweizer-Sklar t-norm given by*

$$T(x,y) = (x^\beta + y^\beta - 1)^{\frac{1}{\beta}} \text{ for some } -\infty < \beta < 0$$

and then T is (φ, f)-quasi-homogeneous, where $f(x) = x^c$ and $\varphi(x) = \left(\frac{x^\beta + 1}{2}\right)^{\frac{c}{\beta}}$ for an arbitrary chosen $c > 0$.

Theorem 3.2 ([11]) *Let $C : [0,1]^2 \to [0,1]$ be a binary function with continuous diagonal $\delta(x) = C(x,x)$. Then C is a quasi-homogeneous copula if and only if δ is a strictly increasing convex function and C is given by*

$$C(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ \delta\left(\max(x,y)\delta^{-1}\left(\frac{\min(x,y)}{\max(x,y)}\right)\right) & \text{else.} \end{cases}$$

Note that in this case $f(x) = x^c$ and $\varphi(x) = \left(\delta^{-1}(x)\right)^c$ for some arbitrary chosen $c > 0$.

4 End-point linear aggregation functions

Homogeneity of aggregation functions has an interesting geometric interpretation. Indeed, for any $(x,y) \neq (0,0)$, consider the maximal segment s in $[0,1]^2$ containing points $\mathbf{0} = (0,0)$ and (x,y). Obviously, each point $(u,v) \in s$ has a form $(u,v) = (\lambda x, \lambda y)$ for $\lambda \in \left[0, \frac{1}{\max(x,y)}\right]$, and then $A(u,v) = \lambda A(x,y)$ once A is homogeneous.

Due to the boundary condition $A(\mathbf{0}) = 0$, we see that then $A|s$ is a linear function on s, and that $\mathbf{0}$ is one end-point of s. This observation has led us to the following generalization of the homogeneity of aggregation functions.

Definition 4.1 *Let $A : [0,1]^2 \to [0,1]$ be an aggregation function and $\mathbf{u} \in [0,1]^2$. Then A is called $\mathbf{u}$-end-point linear if for any segment s in $[0,1]^2$ such that $\mathbf{u}$ is an end-point of s, the restricted function $A|s$ is linear.*

Obviously, standard homogeneity is just the $\mathbf{0}$-end point linearity. The next result is easy to be verified.

Lemma 4.1 *Let $A : [0,1]^2 \to [0,1]$ be an $\mathbf{u}$-end-point linear aggregation function for some $\mathbf{u} = (u_1, u_2) \in$*

$[0,1]^2$. *Then the dual aggregation function* $A^d : [0,1]^2 \to [0,1]$ *given by* $A^d(x,y) = 1 - A(1-x, 1-y)$ *is* $(1-u_1, 1-u_2)$*-end-point linear.*

As a corollary of Lemma 4.1 we see that for a homogeneous aggregation function A, its dual A^d is **1**-end-point linear.
The total end-point linearity, i.e., **u**-end-point linearity for any $\mathbf{u} \in [0,1]^2$ is equivalent to the linearity of the considered aggregation function.

Theorem 4.1 *For an aggregation function* $A : [0,1]^2 \to [0,1]$, *the following are equivalent:*

(i) A is **u***-end-point linear for any point* $\mathbf{u} \in [0,1]^2$;

(ii) A is the weighted arithmetic mean, i.e.,

$$A(x,y) = wx + (1-w)y \qquad \text{for some } w \in [0,1].$$

Now, we discuss the most important conjunctive aggregation function applied in many domains. They all are binary aggregation functions on $[0,1]$ such that $e = 1$ is their neutral element, i.e., $A(x,1) = A(1,x) = x$ for any $x \in [0,1]$ (i.e., they extend the classical boolean conjunction). These special conjunctors are called *semicopulas* [5]. Recall that a commutative and associative semicopula is called *a triangular norm* [9, 14]. A 1-Lipschitz semicopula A, i.e.,

$$|A(x,y) - A(x',y')| \le |x - x'| + |y - y'|$$

for all $x, y, x', y' \in [0,1]$, is called *a quasi-copula* [2]. Finally, a super-modular semicopula, i.e.,

$$A(\mathbf{x} \vee \mathbf{y}) + A(\mathbf{x} \wedge \mathbf{y}) \ge A(\mathbf{x}) + A(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in [0,1]^2$, is called *a copula* [12, 16].

Theorem 4.2 *Let* $A : [0,1]^2 \to [0,1]$ *be a* $\mathbf{u} = (u_1, u_2)$*-end-point linear function, with* $\mathbf{u} \in]0,1[^2$ *such that* $e = 1$ *is its neutral element and* $a = 0$ *is its annihilator (i.e.,* $A(0,x) = A(x,0) = 0$ *for all* $x \in [0,1]$. *Let* $A(u_1,u_2) = \alpha$. *Then:*

(i) A is a semicopula if and only if $\alpha \le \min(u_1,u_2)$;

(ii) the following are equivalent

- *A is a quasi-copula;*
- *A is a copula;*
- $\max(0, u_1 + u_2 - 1) \le \alpha \le \min(u_1, u_2)$.

Semicopulas (quasi-copulas, copulas) characterized by **u**-end-point linearity, where $A(u_1,u_2) = \alpha$, are depicted in Figure 1.

For triangular norms, we have the next result.

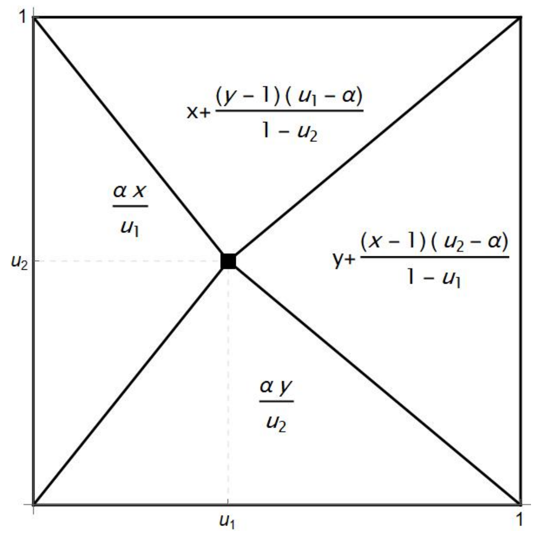

Figure 1: **u**-end-point linear semicopula.

Theorem 4.3 *Under the constraints of Theorem 4.2, A is a triangular norm if and only if*

- *either* $\mathbf{u} = (u,u)$ *for some* $u \in]0,1[$ *and* $\alpha = u$ *(then A is the minimum t-norm,* $A = T_M$*),*

- *or* $\mathbf{u} = (u, 1-u)$ *for some* $u \in]0,1[$ *and* $\alpha = 0$ *(then A is the Lukasiewicz t-norm* $A = T_L(x,y) = \max(x+y-1, 0)$*).*

Based on Lemma 4.1, the related result for particular disjunctive aggregation functions can be obtained by duality. So, e.g., the only **u**-end-point linear triangular conorms [9] are either the smallest t-conorm $S_M = \max$ (and then $\alpha = u, \mathbf{u} = (u,u)$ for some $u \in]0,1[$), or the bounded sum S_L, $S_L(x,y) = \min(x+y, 1)$ (and then $\alpha = 1, \mathbf{u} = (u, 1-u)$ for some $u \in]0,1[$).

5 Concluding remarks

We have discussed and studied some generalizations of the homogeneity of binary aggregation functions. We have presented not only some already known concepts (such as k-homogeneity or quasi-homogeneity) and related examples, but also an original concept of end-point linearity. Up to some general results for end-point linear aggregation functions, we have also added a complete description of end-point linear semicopulas, copulas and triangular norms, concerning $\mathbf{u} \in]0,1[^2$. Note that if $\mathbf{u} = \mathbf{0}$ then **u**-end point linearity is the standard homogeneity, and then the related t-norms and copulas are recalled in Section 2 (for $k = 1$).
Note that the introduced end-point linearity is rather

different concept generalizing the homogeneity than, e.g., migrativity [4]. When thinking on possible applications, we expect our approach can be considered in differential equations domain [7] as well as in any other domains where **0** is the inner point of the considered homogeneous function, recall the symmetric Choquet integral (called also Šipoš integral) [8, 15], as an example. In our next study, we aim to consider some more general classes of functions, such as the pre-aggregation functions [10], or aggregation functions of higher dimensions.

Acknowledgement

The support of the grants APVV-17-0066 and VEGA 1/0468/20 is kindly announced.

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