Bases in Min-Plus Algebra

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ABSTRACT

In classical linear algebra, a basis is a vector set that generates all elements in the vector space and that vector set is a linear independence set. However, the definitions of the linear dependence and independence in min-plus algebra are little more complex given that the min-plus algebra is the linear algebra over the commutative idempotent semiring. The definition of the linear dependence (independence) is used in this paper is Gondran-Minoux linear dependence (independence). A finite set is Gondran-Minoux linearly dependent if the set can be divided into two sets that form a linear space with an intersection which is not a zero vector. We will define the concept of the bases in min-plus algebra. In this paper also defined the concept of a weak bases and will be shown that the linear dependence (independence) is not needed to form a weak basis. In the last part of the research result’s are proven that every basis in a semimodules in min-plus algebra is a weak basis.

Keywords: Basis, Linearly independent, Weak basis.

1. INTRODUCTION

A semiring \((S, +, \times)\) is a set \(S\) with addition (+) and multiplication (\(\times\)) operations, such that \((S, +)\) is a commutative semigroup, \((S, \times)\) is semigroup, and \(\times\) distributes over +. The linear algebra over the semiring \(\mathbb{R}_\text{max} = \mathbb{R} \cup (-\infty)\) with two binary operations of maximization (\(\oplus\)) and addition (\(\otimes\)) is called max-plus algebra [1]. In \(\mathbb{R}_\text{max}\) the neutral (identity) element for \(\oplus\) and \(\otimes\) are \(e = -\infty\) and \(e = 0\). Research on max-plus algebra applications has been carried out by several researchers, see [2][3]. Some researchers also expand the concept in max-plus algebra into interval max-plus algebra, see [4][5]. \(\mathbb{R}_\text{max}\) is idempotent commutative semiring [6] and can be referred to as idempotent semifield [7]. If \(\mathbb{R}_\text{max}^n\) is defined as \(\mathbb{R}_\text{max}^n = \{(x_1, x_2, \ldots, x_n)\mid x_i \in \mathbb{R}_\text{max}\}\), \(\mathbb{R}_\text{max}^n\) is a semimodules over the semiring \(\mathbb{R}_\text{max}\) [6]. The elements \(x\) in \(\mathbb{R}_\text{max}^n\) are called max-plus vectors [8].

A subset \(F\) of a semimodule \(V\) over \(\mathbb{R}_\text{max}\) spans \(V\) if every \(x \in V\) is a finite linear combination of all element in \(F\). A vector set that generates all elements in the vector space and linearly independent then that vector set is called a basis. The are several concept of linear dependence in max-plus algebra as described in [9][10][11]. In [12], they give examples of vector sets in \(\mathbb{R}_\text{max}^n\) that are weakly linearly independent but Gondran-Minoux linearly dependent and Gondran-Minoux linearly independent but tropically linearly dependent. In this article, we will use the concept of linear dependence in Gondran-Minoux sense such that a finite set is said to be linearly dependent if the set can be divided into two sets that form a linear space with an intersection which is not a zero vector [9].

In the field of mathematics studies there are another semiring beside max-plus algebra is min-plus algebra. Min-plus algebra is the linear algebra over the semiring \(\mathbb{R}_\text{min} = \mathbb{R} \cup (+\infty)\) that equipped with two binary operations minimization (\(\ominus\)) and addition (\(\otimes\)) with the neutral (identity) element for \(\ominus\) and \(\otimes\) are \(e = +\infty\) and \(e = 0\) [1]. Min-plus algebra is isomorphic to min-plus algebra [13]. Because there are similarities of structure between max-plus and min-plus algebra, we can transform the concepts in max-plus algebra to min-plus algebra. In this article we will define the concept of bases in min-plus algebra.

First, we will discuss the concept of semimodules \(\mathbb{R}_\text{min}^n\) over semiring \(\mathbb{R}_\text{min}\). Next, we will define the concept of linear dependence in Gondran-Minoux sense in min-plus algebra. Using the concept of linear dependence in Gondran-Minoux sense we will define the concept of bases in min-plus algebra.
2. BASIC NOTATIONS AND DEFINITION

We define the min-plus algebra $\mathbb{R}_{\text{min}}$ by $\mathbb{R}_{\text{min}} = \mathbb{R} \cup \{+\infty\}$ with the binary operations $\oplus$ and $\otimes$. 

There are elements $e'$ and $e$ in $\mathbb{R}_{\text{min}}$ following

i. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ 
   $(a \otimes b) \otimes c = a \otimes (b \otimes c)$

ii. $a \ominus b = b \ominus a$

iii. $(a \oplus b) \oplus c = (a \oplus c) \oplus (b \oplus c)$
    $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$

iv. $a \oplus e' = a = e' \oplus a$
    $a \otimes e = a = e \otimes a$

v. $a \ominus (a) = e = -a \ominus a$

vi. $a \oplus e' = e' = e' \oplus a$ 
    $a \otimes a = a$

To define the linear dependence and basis in min-plus algebra we need the definition of semimodule.

**Definition 2.1.** A semimodule $M$ over the semiring $(S,+)$ is a commutative monoid $(M,+)$ that equipped with scalar multiplication operation

$$ \cdot : S \times M \rightarrow M $$

and for each $\lambda, \mu \in S, \lambda \mu \in M$ following

1. $\lambda \cdot (x + y) = (\lambda \cdot x) + (\lambda \cdot y)$
2. $(\lambda + \mu) \cdot x = (\lambda \cdot x) + (\mu \cdot x)$
3. $(\lambda \cdot x) \cdot x = \lambda \cdot (\mu \cdot x)$
4. $1 \cdot x = x$
5. $0 \cdot x = 0$

Let $\mathbb{R}_{\text{min}}^n = \{(x_1, x_2, ..., x_n)^T | x_i \in \mathbb{R}_{\text{min}}, i = 1,2, ..., n \}$. For each $x, y \in \mathbb{R}_{\text{min}}^n$ and $\lambda \in \mathbb{R}_{\text{min}}$ and define with operation $\oplus'$ and scalar multiplication $\lambda$ such that

$$ x \oplus' y = (x_1 \oplus' y_1, x_2 \oplus' y_2, ..., x_n \oplus' y_n)^T $$

$$ \lambda \cdot x = (\lambda \otimes x_1, \lambda \otimes x_2, ..., \lambda \otimes x_n)^T $$

In [14], we notice that $(\mathbb{R}_{\text{min}}^n, \oplus')$ is a commutative monoid with the neutral element $(e', e', ..., e')^T$ and $\mathbb{R}_{\text{min}}$ satisfies the axiom in Definition 2.1 by operations $\oplus'$ and $\otimes$, therefore $\mathbb{R}_{\text{min}}^n$ is semimodule over the semiring $\mathbb{R}_{\text{min}}$.

**Definition 2.2.** A subset $V$ in $\mathbb{R}_{\text{min}}^n$ is a commutative idempotent semimodule over $\mathbb{R}_{\text{min}}$ if it is closed under $\oplus'$ and scalar multiplication such that $u \oplus' v \in V$ and $a \otimes v \in V$, $\forall u, v \in \mathbb{R}_{\text{min}}^n$ and $a \in \mathbb{R}_{\text{min}}$.

**Definition 2.3.** An element $x$ is a finite linear combination of elements in $F \subseteq V$ if $x = \bigoplus' \lambda_f \otimes f$, such that $\lambda_f \in F$.

3. RESULT AND DISCUSSION

**Definition 3.1.** A subset $F$ of a semimodule $V$ over $\mathbb{R}_{\text{min}}$ generates $V$ if every element $x \in V$ is a finite linear combination of all elements in $F$.

**Definition 3.2.** A generating set is called minimal if it can be divided into two disjoint subsets such that for some $\alpha_i \in \mathbb{R}_{\text{min}}, i \neq k$

$$ \bigoplus' a_i \otimes v_i \neq \bigoplus' a_k \otimes v_k $$

**Definition 3.3.** A family of vectors $\{v_i\}_{i=1}^p$ is a weak basis of a semimodule $V$ if it is a minimal generating set.

We will define the concept of linear dependence (independence) in the Gondran-Minoux sense in min-plus algebra based on the analogy in [9].

**Definition 3.4.** Vectors $v_1, v_2, ..., v_p \in \mathbb{R}_{\text{min}}^n$ are called Gondran-Minoux linearly dependent if there exists a finite linear combination $\lambda_i \in I \cup K = \{1,2, ..., p\}$ such that for $a_j \neq e'$ ($j \in I \cup K$)

$$ \bigoplus'_{i \in I} a_i \otimes v_i = \bigoplus'_{k \in K} a_k \otimes v_k $$

If no such $I, K$, and $a_j$ exist, $\{v_1, v_2, ..., v_p\}$ is a linearly independent set.

**Definition 3.5.** Vectors $v_1, v_2, ..., v_p \in \mathbb{R}_{\text{min}}^n$ are called Gondran-Minoux linearly independent if for all disjoint subsets $I$ and $K$, $I \cup K = \{1,2, ..., p\}$ and all $a_j \in \mathbb{R}_{\text{min}}$ 

$$ \bigoplus'_{i \in I} a_i \otimes v_i \neq \bigoplus'_{k \in K} a_k \otimes v_k $$

unless $a_j = e'$, $\forall j \in I \cup K$.

Using the linear dependence (independence) definition in min-plus algebra, then the following definition is obtained.

Let $W \subseteq \mathbb{R}_{\text{min}}^n$ and a nonempty finite subset $U = \{u_1, u_2, ..., u_n\}$ of $W$. For each $u \in W$ can be written as a finite linear combination of all elements of $U$ (denoted by $u \sim U$) as in Definition 2.3 for $u_i \neq w, i = 1,2, ..., n$. The following theorem will explain the reason for there is such an exception $(u_i \neq w)$.

**Theorem 3.1.** For any $x, y \in \mathbb{R}_{\text{min}}$, there is a $\lambda \in \mathbb{R}_{\text{min}}$ such that $x \oplus' \lambda \otimes y = x$.

**Proof.** If $x = (x_1, x_2, ..., x_n)^T$ and $y = (y_1, y_2, ..., y_n)^T$ then for the $\lambda$ we may take any value greater than or equal to the maximum of $(x_1 \otimes y_1, x_2 \otimes y_2, ..., x_n \otimes y_n)$.

**Definition 3.6.** Let $V$ be a semimodule in $\mathbb{R}_{\text{min}}^n$. A finite subset $U \subseteq V$ is called a basis of $V$ if and only if $U$ is a generating set of $V$ and $U$ is linearly independent in other words for each $v \in V$ either $v \notin U$ or $v \sim U$ but not both.

**Definition 3.7.** Suppose that $U \subseteq V$ is a basis of a semimodule $V$. The number of vectors in $U$ is called the dimension of $V$ and denoted by $\text{dim}(V)$.  

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In [15] Wagner stated that every finitely generated semimodule has a weak bases. For any two weak bases have the same number of generators.

**Definition 3.8.** The weak basis cardinality is called the weak rank of the semimodule \( V \) and denoted by \( r_w(V) \).

Consider the definitions, the following examples are given.

**Example 3.1.** Given the set \( P \) in \( \mathbb{R}_m^3 \). \( P \) is defined as \( P = \{(−1,e,e),(e,−1,e),(e,e,−1)\} \). \( P \) is a linear independent set in Gondran-Minoux sense because not exist \( a_j e \neq e \) that satisfies \( \oplus_{i \in I} a_i \otimes v_i = \bigoplus_{k \in K} a_k \otimes v_k \). However, it is seen that \( P \) does not generate \( \mathbb{R}_m^3 \). Therefore \( P \) is not a basis of \( \mathbb{R}_m^3 \).

**Example 3.2.** Let \( X = \{(e,e'),(e',e),(e,e)\} \) be the set in \( \mathbb{R}_m^3 \). Each element of \( \mathbb{R}_m^3 \) is a finite linear combination of \( X \) so that it can be written

\[
\mathbb{R}_m^3 = \{a \otimes (e') \oplus b \otimes (e') \oplus e' \otimes (e') \mid a, b \in \mathbb{R}_m\}.
\]

Therefore it is clear that \( X \) generates \( \mathbb{R}_m^3 \). There is a vector in \( X \) that can be written \( e \otimes (e,e) = e \otimes (e,e') \oplus (e',e) \) which satisfies Definition 3.4. Since \( X \) generates \( \mathbb{R}_m^3 \) but is not linearly independent then \( X \) is not a basis of \( \mathbb{R}_m^3 \). On other side we can show that \( X \) is a weak basis of \( \mathbb{R}_m^3 \) because for any \( a, b \in \mathbb{R}_m \)

\[
eq a \otimes (e,e') \oplus b \otimes (e,e') \oplus e' \otimes (e,e')
\]

with \( r_w(\mathbb{R}_m^3) = 3 \).

**Example 3.3.** Let us consider the following four matrices

\[
R = \begin{pmatrix}
-1 & e' \\
e' & e
\end{pmatrix},
S = \begin{pmatrix}
e' & 1 \\
e & e'
\end{pmatrix},
T = \begin{pmatrix}
e' & e' \\
e' & e'
\end{pmatrix},
U = \begin{pmatrix}
e' & e' \\
e & e'
\end{pmatrix}.
\]

Suppose that \( M = \{R,S,T,U\} \). It will be shown that \( M \) is a basis of \( \mathbb{R}^{2x2}_m \).

i. \( M \) generates \( \mathbb{R}^{2x2}_m \),

\[
\mathbb{R}^{2x2}_m = \left\{(a \otimes b) \mid a, b, c, d \in \mathbb{R}_m\right\} = \left\{(a + 1) \otimes R \oplus (b - 1) \otimes S \oplus (c + 1) \otimes T \oplus d \otimes U \mid a, b, c, d \in \mathbb{R}_m\right\}.
\]

ii. \( M \) is a linearly independent set because not exist \( a_j e \neq e' \) that satisfies \( \bigoplus_{i \in I} a_i \otimes M_i = \bigoplus_{k \in K} a_k \otimes M_k \) for \( j \in I \cup K \). Therefore \( M \) is a weak basis of \( \mathbb{R}^{2x2}_m \) with \( \dim(\mathbb{R}^{2x2}_m) = 4 \).

**Theorem 3.2.** Let \( V \) is a finite semimodule in \( \mathbb{R}^{2x2}_m \) and \( U \subseteq V \) is a basis of \( V \) then \( U \) is a weak basis of \( V \).

Proof. Since \( U \) is a basis of \( V \) then \( U \) is generating set of \( V \) and \( U \) is linearly independent. \( U \) is linearly independent such that it satisfies a minimal generating set. Therefore \( U \) is a weak basis of \( V \). ■

Consider the set in Example 3.3, \( M \) is a basis of \( \mathbb{R}^{2x2}_m \). Using Theorem 3.2, \( M \) can be called a weak basis of \( \mathbb{R}^{2x2}_m \). Also consider the set in Example 3.2, \( X \) is a linearly dependent (not linearly independent) and is a weak basis of \( \mathbb{R}_m^3 \). Because there is a weak basis that is linearly dependent or linearly dependent, then the linear independence is not needed to form a weak basis of a semimodule over semiring \( \mathbb{R}_m \).

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