The Dynamics of Modified Leslie-Gower the Pest-Predator System with Additional Food and Fear Effect

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ABSTRACT

We have studied a dynamic analysis of the prey-predator model that describes the interaction between pests and predators as natural enemies. The model is considering a Modified Leslie-Gower prey-predator model with the nature of fear of the growth of pests and additional food to predator. The predation uses the Holling type II response function by assuming that the predator also needs additional food to survive. We analyze the dynamics of the system includes determining the equilibrium point as well as local stability analysis. It shows that there are four equilibriums in which each equilibrium point exists and three equilibrium points are local asymptotically stable with some sufficient conditions. Numerical simulations were carried out to support the analytical finding. The numerical simulations also indicate that bifurcation occurs and the possible phase portraits diagram has been depicted. The simulation results and discussion of the diagram bifurcation are lucidly presented in the text of the paper.

Keywords: *Dynamics, Pest-predator, Stability, Bifurcation.*

1. INTRODUCTION

Rice is the main food crop grown by farmers in Indonesia. The growth of rice plants in Indonesia is at risk in the form of pest and disease attacks. The main pests of rice plants in Indonesia are field mice. The increasing rat population is a problem for farmers. Rat pest control is an effort to reduce the rat population level as low as possible so that economically the presence of rat pests is not detrimental [1]. Pests have natural enemies in the form of predators. Natural enemies are used to controlling the number of pest populations naturally so that the ecosystem remains stable [2]. The population of owls as a natural enemy is causing fear in rat pests [3]. A high level of fear in rat pests caused by predators results in reduced reproduction of rats so that it can reduce the number of rat pests. The reduction in the number of rat pests makes predators look for alternative food other than rats. Additional food is needed to keep predatory species from becoming extinct so that the balance of the two populations is controlled.

Based on the problem of pest growth, it can be applied to a mathematical model of the interaction of two species. The model describes the dynamics of the spread of pests and natural enemies, namely predators. The pest-predator model considers predation and the nature of the pest against predators. Some scientists are interested in developing a similar model with various assumptions. Srinivasu [4] has studied the Lotka-Volterra predator-prey model with Holling type II functional response. This model assumes that the number of assemblies per predator is proportional to the obtainable additional food. Sen, et. al [5] discussed a two-species model in the presence of forage and harvesting in predators using the Holling type II functional response. Then, Prasad [6] reviewed a model that took into account the parameters of handling time and nutritional value of food additives. Models combining disease in pests with pathogens (bacteria, fungi, and viruses) to infect pests and harvesting have also been introduced [7-8]. The pest-predator model [9] concatenating the effects of Alle on pests and food additives on predators. Sahoo [10] discusses food additives playing an important role in survival in a biological conservation model. Many researchers have studied the modified Leslie-Gower model with different response functions [11-15]. The proposed pest-predator model [16] incorporate an epidemic model and the Leslie-Gower likewise of pest harvesting. Mondal [17] developed a two-species model with the effect of fear and competition among prey. The model involves seasonal parameters in determining the quality and quantity of food additives. Inspired by the model [18] which discussed the Leslie-Gower model and modified by assumpting the additional food to predator in terms of the handling time. In this article, we have studied a prey-predator system with additional food for predator and considered the effect of fear only on the birth rate of the prey population. We also involved additional food to support predator growth in terms of the handling time of Holling type II.

The rest of the paper is sequenced as follows. We construct the pest-predator model to consider the effect of fear for pest and addition food for predator in Section 2. In Section 3, the existence and local stability of the equilibrium point are discussed. Numerical simulations are performed in Section 4 by phase portrait and bifurcation diagram. The last section, we end with concluding in Section 6.

2. MATHEMATICS MODELLING

Based on the model [17] and [18], a prey growth model was constructed by involving the effect of fear without competition between preys and the additional food to predators. The main objective of this paper is to investigate parameters of the maximum value of growth rate of predators with additional food in controlling the balance between populations. The dynamic analysis carried out includes determining the equilibrium point and the type of equilibrium for each solution of the system. In order to construct a mathematical model, the following assumptions have been made in the present study. Specifically, the model to be constructed is described in the following subsection.

2.1 Assumptions of the Model

The pest population grows logistically with an intrinsic growth rate of r and it is assumed that the pest has a fear of predators so that the growth becomes $\frac{rx}{1+fy}$. The growth rate of pests is reduced by the presence of predation or predation of predators on pests. The predation process is known as the functional responses. The parameter α is the maximum value of the growth rate of the predator. The growth rate of pests can be written as follows

$$\frac{dx}{dt} = \frac{rx}{1+fy} - \frac{\alpha xy}{m+x+nA} \quad (1)$$

The growth rate of predators does not only depend on the prey population but also depends on the environmental carrying capacity to maintain the survival of the predators. The predator growth rate model is known as the Leslie-Gower modification which is expressed as $\left(1-\frac{y}{x+k}\right)$. This model considers the presence of additional food for predators to survive if there is no main food. The parameters β represents the efficiency with which the food consumed by the predator gets converted into predator then the maximum growth rate of the predator.

$$\frac{dy}{dt} = \beta y\left(1-\frac{y}{x+k}\right) + \frac{\rho nAy}{m+x+nA} \quad (2)$$

2.2 Mathematical Formulation of the Model

Based on the assumptions described above, the interaction model between pest and predator is obtained as follows

$$\frac{dx}{dt} = \frac{rx}{1+fy} - \frac{\alpha xy}{m+x+nA}$$
$$\frac{dy}{dt} = \beta y\left(1-\frac{y}{x+k}\right) + \frac{\rho nAy}{m+x+nA} \quad (3)$$

with $x(t)$ represents the population density of pests and $y(t)$ represents the population density of predator at time t. The parameters f, k, and β represent the natural of fear of pest, carrying capacity of y, and maximum growth rate of y. The parameter n and A are the parameters which characterize the additional food. All the parameters relevant with the system (3) are positive.

2.3 Equilibrium Pointsl

The dynamic analysis includes the determination of the equilibrium point, the existence of the equilibrium points and local stability analysis. Local stability analysis to determine the type of stability of each equilibrium point. Then the possible positive equilibrium point of the system (3) is obtained. The equilibrium point represents the population density at equilibrium. Therefore, an equilibrium point is said to exist if every element is non-negative.

2.3.1 $E_1 = (0,0)$, where all populations are extinct.
2.3.2 $E_2 = \left(0, \frac{k(nA\beta+nA\rho+\beta m)}{\beta(nA+m)}\right)$, where only predator survives. Represents the extinction of pest populations
2.3.3 $E_3 = (x^*, y^*)$, where pest and predator coexist. The population pest and predator are able to survive.

And x^* is a root of a fourth-degree equation with complicated coefficients. We provide sufficient conditions for the existence of a nonnegative root. for x^* we get equation.

$$H_1(y^*)^4 + H_2(y^*)^3 + H_3(y^*)^2 + H_4(y^*) + H_5 = 0, (4)$$

with $H_1 = a\alpha f$, and $a = \alpha\beta f$

$H_2 = a(2\alpha - r)$
$H_3 = akr - amr - nAr + nA\alpha fr\rho + \alpha^2\beta - \alpha\beta r$
$H_4 = nA\alpha r\rho + \alpha\beta kr - nA\alpha\beta r - \alpha\beta mr$
$H_5 = n^2A^2r^2\rho + nAk\rho r^2 - nAm\rho r^2$
A fourth-degree equation has four roots in the complex domain (4). Omitting to write explicitly its coefficients, in view of their complexity (4). Now we find sufficient conditions for it to have at least one positive root [19]. Depending on the parameters of the system (3), there is two equilibrium point $E_3 = (x_3{}^*, y_3{}^*)$ and $E_4 = (x_4{}^*, y_4{}^*)$.

3. LOCAL STABILITY ANALYSIS

The stability properties of the equilibrium point of system (3) is determined by the roots of the characteristic equation. Here, the Jacobian matrix at an equilibrium point is

$$J = \left[\frac{r}{1+fy} - \frac{\alpha y}{nA+m+x} + \frac{\alpha xy}{(nA+m+x)^2}\ \frac{rfx}{1+fy} - \frac{\alpha x}{nA+m+x}\ \frac{\beta y^2}{(k+x)^2} - \frac{\rho nAy}{(nA+m+x)^2}\ \beta\left(1-\frac{y}{k+x}\right) - \frac{\beta y}{k+x} + \frac{\rho nA}{nA+m+x}\right]$$

The direct evaluation of the Jacobian matrix at E_1 gives

$$J(E_1) = \left[r\ 0\ 0\ \beta + \frac{\rho nA}{nA+m}\right],$$

Theorema 1. The equilibrium point $E_1 = (0,0)$ is always unstable.

Proof.

The characteristic equation of the matrix $J(E_1)$ are $(\lambda - r)(nA\lambda + m\lambda - An\beta + nA\rho + m\beta) = 0$. It is Obvious that one of eigenvalue $\lambda_1 = r$ is positive, the other eigenvalues is always positive $\lambda_2 = \frac{An\beta + nA\rho + m\beta}{nA+m} > 0$. Hence, E_1 is an unstable.

The evaluation of the Jacobian matrix at E_2 gives

$J(E_2) = [a_1\ 0\ a_2\ a_3\ ]$, with

$$a_1 = \frac{r}{\frac{fk(\beta nA+\rho nA+\beta m)}{\beta(nA+m)}+1} - \frac{\alpha k(\beta nA+\rho nA+\beta m)}{\beta(nA+m)^2}$$

$$a_2 = \frac{(\beta nA+\rho nA+\beta m)}{\beta(nA+m)^2} - \frac{nA\rho k(\beta nA+\rho nA+\beta m)}{\beta(nA+m)^3}$$

$$a_3 = \beta\left(1 - \frac{\beta nA+\rho nA+\beta m}{\beta(nA+m)}\right) - \frac{\beta nA+\rho nA+\beta m}{nA+m} + \frac{\rho nA}{nA+m}$$

Theorema 2. The equilibrium point $E_2 = \left(0, \frac{k(nA\beta+nA\rho+\beta m)}{\beta(nA+m)}\right)$ is locally asymptotically stable

Proof.

The equilibrium point E_2 is locally asymptotically stable if $T < 0$ and $D > 0$.

The trace (T) and the determinant (D) of the Jacobian matrix at E_2 are determined by $\lambda^2 - T\lambda + D = 0$, where $T = a_1 + a_3$ and $D = a_1a_3$. The system (3) satisfies the Kolmogorov conditions [20] at $\left(0, \frac{k(nA\beta+nA\rho+\beta m)}{\beta(nA+m)}\right)$ making $D = a_1a_3$ positive. Therefore E_2 is locally asymptotically stable if $T < 0$ and unstable if $T > 0$. We have

$$T = \frac{r}{\frac{fk(\beta nA+\rho nA+\beta m)}{\beta(nA+m)}+1} - \frac{\alpha k(\beta nA+\rho nA+\beta m)}{\beta(nA+m)^2} + \beta\left(1 - \frac{\beta nA+\rho nA+\beta m}{\beta(nA+m)}\right) - \frac{\beta nA+\rho nA+\beta m}{nA+m} + \frac{\rho nA}{nA+m}$$

Here it is important that the term $T < 0$ and D is always positive and this completes the proof.

Similarly, the Jacobian matrix at E_3 is

$$J(E_3) = [a_4\ a_5\ a_6\ a_7\ ]$$

Theorem 3. The equilibrium point $E_3 = (x^*, y^*)$ is locally asymptotically stable

Proof.

Clearly that two eigenvalues are determined by $\lambda^2 - T\lambda + D = 0$, where $T = a_4 + a_6$ and $D = a_4a_7 - a_5a_6$. Therefore E_3 is stable if $T < 0$ and $D > 0$. The proof is complete.

4. NUMERICAL SIMULATIONS

We would like to convey a numerical simulation to illustrate bifurcation phenomena. The study numeric are very important to solving of solution the system (3) which is undertaken by Python 3.7. Numerical simulations are carried out to support the analysis and display the change in the equilibrium point solution through a bifurcation diagram.

Table 1. The set of parameter value used for simulation

Parameter	Description	Value
r	Maximum growth rate of pest	3.22
f	The fear effect of pest to predator	0.31
α	Maximum rate of predator	0.7
m	Half saturation value of the predator	0.0577
n	The relative ability of the predator to detect additional food	0.83
A	Amount of additional food to the predators	1.39
k	Carrying capacity of predator	2.1
β	Maximum growth rate of the predator	0.68
ρ	Proportionality constant	0.322

4.1 Phase Portrait

For simulation, we take the same parameter values in Table 1. Using these parameter values, it can be shown that all equilibrium points E_1, E_2, E_3, and E_4 of the system (3) exist. However, we perform the numerical simulation showing that system (3) the limit cycle undergoes periodic perturbation.

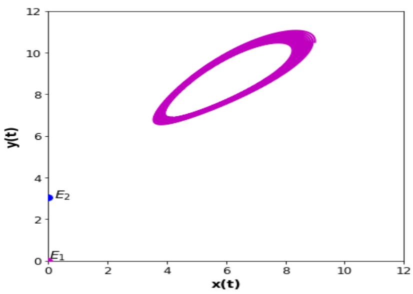

Figure 1a The phase portraits of system (3) which shows Hopf Bifurcation at $\alpha = 0.72993019$.

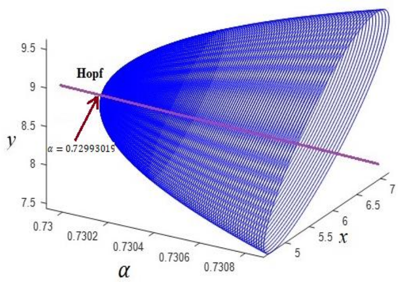

Figure 1b Hopf Bifurcation Diagram with specify of the limit cycle at $\alpha = 0.72993019$.

In Figure 1a, we illustrate that the system (3) gets a limit cycle when parameter value $\alpha = 0.72993019$ with initial condition $N = [x(0) = 5, y(0) = 7.2]$. This occurs quickly since the initial value is taken chose to N. The limit cycle encourages an almost periodic solution when α increases. The equilibrium E_1 (0,0) unstable remains itself. The interior equilibrium $E_3 = (5.763726, 8.479635)$ evolves into the positive periodic solution. We find the Hopf bifurcation when $\alpha^{*1} = 0.72993019$ and a stable limit cycle appears around the interior equilibrium point for $E_3 = (5.763726, 8.479635)$ in Figure 1b. Now, we discuss four different parameters value (α) to illustrate the solution tends to the equilibrium point. The dynamics of the two populations of the present system (3) are obvious at the eight frames shown in the Figure 2a to 5b.

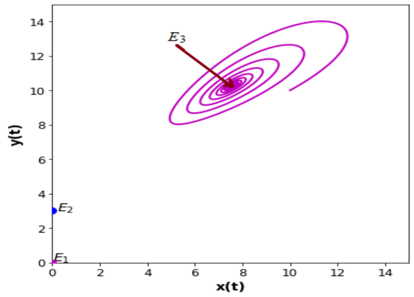

Figure 2a The phase portraits of system (3) which shows E_3 stable.

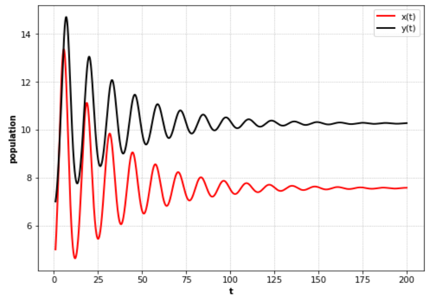

Figure 2b The solution tends to E_3 stable at $\alpha = 0.658321$.

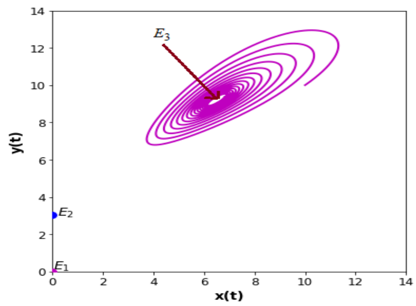

Figure 3a The phase portraits of system (3) which shows E_3 stable.

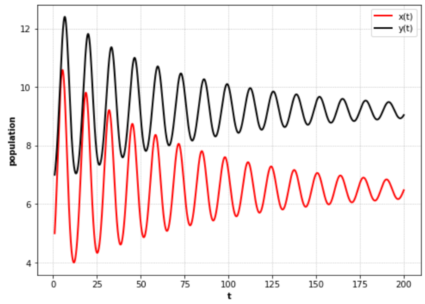

Figure 3b The dynamics tends to E_3 stable at $\alpha = 0.7$.

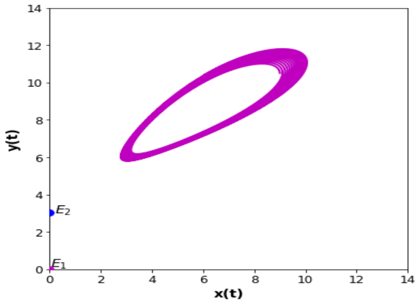

Figure 4a The phase portraits of system (3) which shows E_3 stable.

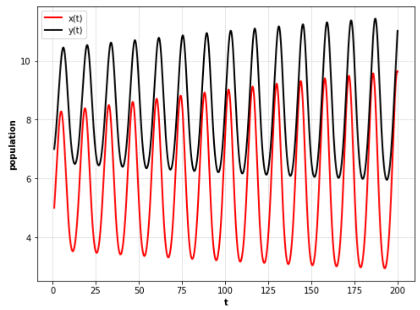

Figure 4b The dynamics of the system (3) at $\alpha = 0.73994233$.

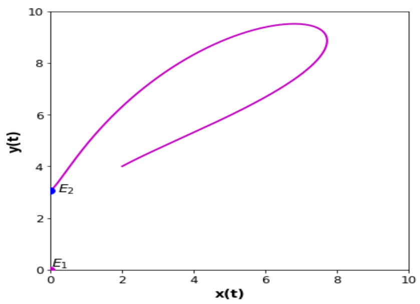

Figure 5a The phase plane diagram of the system (3) stable.

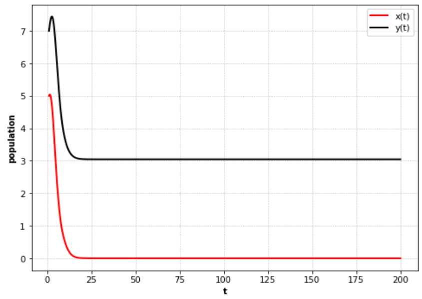

Figure 5b The solution tends to $E_2 = (0, 3.047)$at $\alpha = 0.866893$.

The phase portraits of the system (3) which the solution tends to $E_3 = (5.5284, 8.24679)$ stable when parameter $\alpha = 0.658321$ shown in Figure 2a and 2b. Figure 3a and 3b shows that all the equilibrium points E_1, E_2, E_3, and E_4 exists. This simulation uses the parameters as shown in Table 1, that the stability conditions at the equilibrium point $E_3 = (6.4863, 9.1956)$ fulfill the results of the stability conditions. The red line corresponds to the pest (x) population and the black line correspond to the predator (y) population in Figure (2b – 5b). Numerically, the system (3) indicates a kind dynamic fullness.

4.2 Bifurcations around the interior equilibrium point

Numerical continuity is performed on an equilibrium solution with a variation of one parameter to indicate an indication of a change in the type of stability uses MatCont. Changes in stability indicate the emergence of bifurcations, namely Hopf bifurcation, Saddle Node bifurcation, and Transcritical bifurcation.

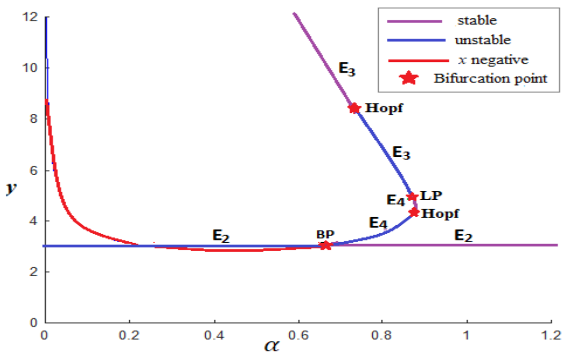

Figure 6 Bifurcation Diagram of the predator population (y) with continuation parameter α.

In numerical continuity, the parameter (α) is to indicate changes in the stability of several the equilibrium point. Figure 6 presents the bifurcation diagram of the solution of the system (3) which is varying the parameter of the maximum rate of the predator predation to the pest population (α).

a. *Saddle node* Bifurcation

In detail the changes that occur at the interior equilibrium point are shown in Figure 6 diagram of bifurcation for the predator population ($\boldsymbol{y}$) with respect to parameter $\boldsymbol{\alpha}$. The results of the numerical continuation of the parameter α show the changes that occur in the stability at the interior equilibrium point ($\boldsymbol{E_3}$). The Bifurcation diagram in Figure 6 shows for $\mathbf{0.72993019} < \boldsymbol{\alpha} < \mathbf{0.866894}$. There are two interior equilibrium point where of them is ($\boldsymbol{E_3}$) unstable while ($\boldsymbol{E_4}$) is stable. Both conditions of the interior equilibrium point crash at $\boldsymbol{\alpha} = \mathbf{0.866894}$, namely *Limit Point* (LP). The phenomenon of the emergence of LP shows that there is a *Saddle-node* (*Fold*) bifurcation which is driven by the parameter of the maximum value of the predator predation rate on the pest. if $\boldsymbol{\alpha} > \mathbf{0.8668935}$, ($\boldsymbol{E_2}$) stable.

b. Double stability phenomenon

Another interesting dynamics behavior to observe as shown in Figure 6 is the appearance of two different stability, known as the bi-stability phenomenon. This phenomenon occurs during $0.866893 < \alpha < 0.866894$ that the system (3) has a double stability. There are two stability in the solution of system (3) namely the interior equilibrium point (E_4) and the pest extinction equilibrium point (E_2) which are locally asymptotically stable, while the other interior equilibrium point are unstable. The phenomenon also occurs $0.658321 < \alpha < 0.72993019$. The solution of the interior equilibrium point (E_3) and the pest extinction equilibrium point (E_2) which are locally asymptotically stable.

c. Transcritical Bifurcation

The bifurcation diagram in Figure 6 illustrates the exchange between the equilibrium interior point (E_3) and the pest population extinction points (E_2). This phenomenon shows a Transcritical bifurcation. For α <0.658321, the pest population extinction points (E_2) are unstable. As it increases of the parameter α, a Branch Point (BP) apprears at $\alpha = 0.658321$. This BP indicates a Transcritical bifurcation, while a change in the equilibrium point of the pest population extinction points (E_2), which was initially unstable for $\alpha < 0.658321$, became stable for α >0.658321. On the other hand, there is a change in the coexist equilibrium point E_3, which was originally stable $\alpha > 0.658321$, became unstable for $\alpha < 0.658321$.

5. CONCLUSION

We have discussed a model describing pest-predator interactions taking into account the effects of fear on pest populations and food additives on predator populations. The predator growth model uses a modified Leslie-Gower model. We find that the pest-predator model has four equilibrium points. There is one equilibrium condition that is always unstable, namely the equilibrium point of extinction for all populations. The other equilibrium points are stable with certain conditions that have been proven. The results of the dynamic analysis have been confirmed through numerical simulations by continuing the parameter of the maximum rate of the predator predation, namely α. The numerical simulation results also show that the pest-predator interaction model describes rich dynamics with the occurrence of Hopf bifurcation, Transcritical bifurcations, Saddle-node bifurcations and double stability phenomena driven by the continuation of the maximum value parameter of Maximum rate of predator predation on pests.

AUTHORS' CONTRIBUTIONS

DS: Conceptualization, Data curation, Visualization, software (simulation and continuation numeric) and drafting manuscript: M and A; Formal analysis, Funding acquisition, and review. M.J; Methodology, data visualization and editing. All authors have read and approved the final manuscript.

REFERENCES

[1] N.A. Herawati and Sudarmaji, Helminths of the Rice-field Rat, Rattus argentiventer, In G.R Singleton (Ed.), Rat, Mice and People: Rodent Biology and Management, ACIAR Canberra, 2003.

[2] R. Kaliky, K. Yolanda, and Sudarmaji, Peran Keanekaragaman Hayati untuk Mendukung Indonesia sebagai Lumbung Pangan Dunia, *Seminar Nasional dalam Rangka Dies Natalis UNS ke 42 Tahun 2018*, 2(1)(2018) 23-31. Retrieved from: http://repository.lppm.unila.ac.id/6858/1/Makalah%20Seminar%20UNS_Tempe_Samsul%20Rizal%20UNILA_REVISI.pdf

[3] N.T. Haryadi, M. W. Jadmiko and T. Agustina., Pemanfaatan Burung Hantu untuk Mengendalikan Tikus di Kecamatan Semboro Kabupaten Jember (2016). Retrieved from: http://repository.unej.ac.id/handle/123456789/73295

[4] P. D. N. Srinivasu, B. S. R. V. Prasad and M. Venkatesulu, Biological Control Through Provision of Additional Food to Predators, Theoretical Population Biology, University of Missouri, Kansas City, USA,2007, pp 111-120. DOI: http://dx.doi.org/10.1016/j.tpb.2007.03.011

[5] Sen, et. al, Global Dynamics of an Additional Food Provided Predator-Prey System with Constant Harvest in Predators, Applied Mathematics and Computation 250 (2015) 193-211. DOI: http://dx.doi.org/10.1016/j.amc.2014.10.085

[6] B. S. R. V. Prasad, M. Banerjee and P. D. N. Srinivasu, Dynamics of Additional Food Provided Predator-Prey System with Mutually Interfering Predators, Mathematical Biosciences 246(1) (2013)176-190. DOI: http://dx.doi.org/10.1016/j.mbs.2013.08.013

[7] A. Suryanto, I. Darti and S. Anam, Stability analysis of pest-predator interaction model with infectious disease in prey, AIP Conference Proceedings 1937(1) (2017) p 020018. DOI: http://dx.doi.org/10.1063/1.5026090

[8] S.K. Sasmal, D.S. Mandal and J. Chattopadhyay, A Predator-pest Model with Allee Effect and Pest Culling and Addition Food Provision to the Predator-Application to Pest Control, Journal of Biological System 25(2) (2017) 295-326. DOI: https://doi.org/10.1142/S0218339017500152

[9] Z. Zhu, R. Wu, L. Lai and X. Yu, The Influence of fear effect to the Lotka-Volterra predator-prey System with Predator has other food Resource, Advances in Difference Equations 1 (2020) 237. https://doi.org/10.1186/s13662-020-02612-1.

[10] B. Sahoo, Predator-prey System with Seasonally Varying Additional Food to Predators, International Journal of Basic and Applied Sciences 1(4) (2012) 363-373. DOI: http://dx.doi.org/10.14419/ijbas.v1i4.205

[11] M. Aziz-Alaoui, M.D. Okiye, Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes, Applied Mathematics Letters 16 (7) (2003) 1069 – 1075. DOI: https://doi.org/10.1155/2010/813289

[12] S. Yu, Global asymptotic stability a modified Leslie-Gower with Beddington-DeAngelis functional response, Advances in Difference Equations 1 (2014) 1-14. DOI: http://dx.doi.org/10.1186/1687-1847-2014-84

[13] R.P. Gupta, P. Chandra, Bifurcation analysis of modified Leslie-Gower predator prey model with Michaelis-Menten type prey harvesting, Journal of Mathematical Analysis and Applications 398(1) (2013) 278 – 295. DOI: http://dx.doi.org/10.1016/j.jmaa.2012.08.057

[14] I. Darti, A. Suryanto, Dynamics preserving nonstandard Finite difference method for the modified Leslie-Gower predator-prey model with Holling type II functional responses. Far East Journal of Mathematical Sciences 99(5) (2016) 719 – 733, DOI: http://dx.doi.org/10.17654/MS099050719

[15] D.Savitri, A. Suryanto, W.M. Kusumawinahyu and Abadi, Dynamical Behavior of a Modified Leslie–Gower One Prey–Two Predators with Competition, Mathematics 8(5) (2020) 669. DOI: http://dx.doi.org/10.3390/math8050669

[16] A. Suryanto and I. Darti, Dynamics of Leslie-Gower Pest-Predator Model with Disease in Pest Including Pest-Harvesting and Optimal Implementation of Pesticide., International Journal of Mathematics and Mathematical Sciences (2019) 1-9. DOI: https://doi.org/10.1155/2019/5079171.

[17] S. Mondal, A. Maiti and G.P. Samanta, Effects of Fear and Additional Food in a Delayed Predator-Prey Model, Biophysical Reviews and Letter 13(4) (2018) 157-177 DOI: https://doi.org/10.1142/S1793048018500091.

[18] H.M. Ulfa, A. Suryanto and I. Darti, Dynamics of Leslie-Gower Predator-prey Model with additional food for Predator, International Journal of Pure and Applied Mathematics 115(2) (2017) 199-209.DOI: 10.12732/ijpam.v115i2.1

[19] N. Ali, M. Haque, E. Venturino and S Chakravart, Dynamics of a three species ratio-dependent food chain model with intra-specific competiton within the top predator, Computers in Biology and Medicine 85 63-74. DOI: http://dx.doi.org/10.1016/j.compbiomed.2017.04.007

[20] K. Sigmund, Kolmogorov and population dynamics. In Charpentier, E., Lesne, A., Nikolski, N.K., (Eds.), Kolmogorov's Heritage in Mathematics, Springer-Verlag, Berlin, 2007, pp. 177-187. DOI: https://doi.org/10.1007/978-3-540-36351-4_9