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The Relationship Between a Fractal F^{α} −absolutely Continuous Function and a Fractal Bounded p −variation Function

Supriyadi Wibowo[1,*], Christiana Rini Indrati[2] Soeparmi[1] Cari[1]

[1] Universitas Sebelas Maret Surakarta, Surakarta, Indonesia
[2] Universitas Gadjah Mada, Yogyakarta, Indonesia
[*]Corresponding author. Email: supriyadi_w@staff.uns.ac.id

ABSTRACT
Wibowo et al. proposed the concept of the fractal F^{α} −absolutely continuous function and the fractal bounded p −variation function of order α where $0 < \alpha < 1$ and $1 < p < \infty$ on a fractal set F. In this paper, we study the relationship between those two functions. In particular, we represent the fractal bounded p −variation function is a fractal F^{α} −absolutely continuous on a fractal set F.

Keywords: *fractal bounded, fractal set.*

1. INTRODUCTION

Jordan introduced the concept of variation around 1880, while studying the convergence of Fourier series. Since then, many mathematicians have generalized this concept in various ways. There are many results about Banach space and embeddings of so-called Bounded Variation Spaces for one-variable functions, but there is no standard definition of variation for multivariable functions. The interested reader is encouraged to read [1] for some background information and preliminary findings on this subject.

Many researchers have developed bounded variation. As a generalization of the concept of bounded p −variation ($1 < p < \infty$), Castillo and Trousselot proposed a new concept of absolutely continuous and bounded (p, ϑ) −variation, in which ϑ is an increasing and continuous function. They discussed the relationship between the absolutely continuous function and the (p, ϑ) −bounded variation in that paper [2].

We can replace ϑ with integral staircase function because integral staircase functions are strictly increasing and continuous functions on a fractal set F. On a fractal set F, this function is also not differentiable [3].

The integral staircase function is a generalization of the Lebesgue-Cantor staircase function defined on a Cantor ternary set [4]. The integral staircase function is crucial in fractal calculus. Parvate and Gangal [5] and [6] developed fractal calculus. It is an algorithmic and straightforward generalization of the standard calculus, which is used in a variety of applications [7-13]

Wibowo et al. in [14] defined a new concept of fractal bounded variation, and we have got some relationship between fractal Lipschitz continuous function on a fractal set. In particular, we represent the characterization of the fractal Lipschitz continuous function related fractal bounded variation. On the other hand, Wibowo et al. in [15] introduce the concept of fractal F^{α} −absolutely continuous and obtain the algebraic properties of the function.

In this paper, we investigate the relationship between those two functions. We show that any fractal bounded p −variation function is F^{α} −absolutely continuous on the fractal F. The set $F \subset [a, b]$ is used throughout the paper to represent a fractal set.

2. PRELIMINARIES

We will review certain ideas in this area before moving on to the next section. The following are the definitions of subdivision, coarse-grained mass, and the integral staircase function.

Definition 2.1 [4] *A subdivision P of $[a,b], a<b$, is a finite set of points $\{a=x_0,x_1,\dots,x_n=b\}, x_i<x_{i+1}, i=0,1,\dots,n-1$. The interval $[x_i,x_{i+1}]$ for $i=0,1,\dots,n-1$ is called a component of the subdivision P. A subdivision P_1 of $[a,b]$ is said to be finer than a subdivision P_2 if $P_1\supset P_2$. In case $a=b$, the subdivision of $[a,b]$ is defined as the set $\{a\}$.*

Definition 2.2 [4] *Let $0<\alpha<1$ and let $f:[a,b]\longrightarrow\mathbb{R}$. Given $\delta>0$ and $a\le b$, the mass function $\gamma^\alpha(F,a,b)$* of *F is defined by*

$$\gamma^\alpha(F,a,b)=\lim_{\delta\to 0}\inf_{\{P_{[a,b]}:|P|\le\delta\}}\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^\alpha}{\Gamma(\alpha+1)}\theta(F,[x_i,x_{i+1}]),$$

where $|P|=\max_{0\le i\le n-1}(x_{i+1}-x_i)$, $\Gamma(.)$ is a gamma function and $\theta(.,.)$ is a flag function, i.e.,

$$\theta(F,[x_i,x_{i+1}])=\begin{cases}1, F\cap[x_i,x_{i+1}]\neq\emptyset\\0, F\cap[x_i,x_{i+1}]=\emptyset.\end{cases}$$

The integral staircase function S_F^α of order $\alpha\in(0,1)$ for a fractal set F is defined as follows by Definition 2.2.

Definition 2.3 [4] *Let a_0 be an arbitrary and fixed real number. The integral staircase function S_F^α of order $\alpha\in(0,1)$ for a fractal set F is defined by*

$$S_F^\alpha(x)=\begin{cases}\gamma^\alpha(F,a_0,x), x\ge a_0\\-\gamma^\alpha(F,a,a_0), x<a_0.\end{cases}$$

Theorem 2.4 [4] *Let $\alpha\in(0,1)$. Let F be a fractal subset of $[a,b]$. If $\gamma^\alpha(F,a,b)<\infty$, then for all $x,y\in(a,b)$ such that $x<y$, we have the following statements*

(i) $S_F^\alpha(x)$ *is increasing in x,*
(ii) *If $F\cap(x,y)=\emptyset$, then S_F^α is a constant on $[x,y]$,*
(iii) $S_F^\alpha(y)-S_F^\alpha(x)=\gamma^\alpha(F,x,y)$,
(iv) $S_F^\alpha(x)$ *is continuous on (a,b).*

The following are given a definition of a fractal F^α −absolutely continuous of order α and fractal bounded p −variation on a fractal set F for $0<\alpha<1$ and $1<p<\infty$.

Definition 2.5 [15] *Let $0<\alpha<1$. A function $f:[a,b]\longrightarrow\mathbb{R}$ is said to be F^α −absolutely continuous of order α, if for each $\varepsilon>0$, there exists $\delta>0$ such that whenever $\{[x_i,x_{i+1}]: x_i\in F, i=0,1,2,\dots,n-1\}$ is a finite collection of mutually disjoint subintervals of $[a,b]$ with*

$$\sum_{i=0}^{n-1}|S_F^\alpha(x_{i+1})-S_F^\alpha(x_i)|<\delta.$$

The set of all F^α −absolutely continuous functions on fractal set F denoted by $AC_F^\alpha[a,b]$.

Definition 3.4 [14] *Let $\alpha\in(0,1)$ and $1<p<\infty$. For a fractal set F and $f:[a,b]\longrightarrow\mathbb{R}$ be a function and a subdivision P,*

$$V_F^{\alpha,p}(f,P)=\sum_{i=0}^{n-1}\frac{|f(x_{i+1})-f(x_i)|^p}{|S_F^\alpha(x_{i+1})-S_F^\alpha(x_i)|^{p-1}}.$$

The fractal p −variation of function f over $[a,b]$, writen $V^{\alpha,p}(f,[a,b])$, is defined by

$$V_F^{\alpha,p}(f,[a,b])=\sup_P V_F^{\alpha,p}(f,P)$$

where the supremum is taken over all subdivision P of $[a,b]$.

A function f is said to be of fractal bounded p −variation on the fractal set F of order α, if $V_F^{\alpha,p}(f,[a,b])<\infty$. The set of all bounded p −variation on F of order α will be denoted by $BV_F^{\alpha,p}[a,b]$. The fractal p −variation function is defined as $v_F^{\alpha,p}(x)=V_F^{\alpha,p}(f,[a,x])$ for all $x\in F$.

3. RESULT AND DISCUSSION

In the following theorem, we prove that the fractal bounded p −variation is bounded.

Theorem 3.1 *Let* $0<\alpha<1$ and $1<p<\infty$. *If $f\in BV_F^{\alpha,p}[a,b]$ then f is bounded on $[a,b]$.*

Proof.

Let $x\in[a,b]$, then

$$\begin{aligned}|f(x)|-|f(a)| &\le |f(x)-f(a)|\\ &=\left(\frac{|f(x)-f(a)|}{|S_F^\alpha(x)-S_F^\alpha(a)|^{\frac{p-1}{p}}}\right)\left(|S_F^\alpha(x)-S_F^\alpha(a)|^{\frac{p-1}{p}}\right)\\ &\le\left(V_F^{\alpha,p}(f;[a,x])\right)^{\frac{1}{p}}\left(S_F^\alpha(x)+S_F^\alpha(a)\right)^{1-\frac{1}{p}}\\ &\le\left(V_F^{\alpha,p}(f;[a,b])\right)^{\frac{1}{p}}\left(2S_F^\alpha(b)\right)^{1-\frac{1}{p}}.\end{aligned}$$

We conclude that

$$|f(x)|\le|f(a)|+\left(V_F^{\alpha,p}(f;[a,b])\right)^{\frac{1}{p}}\left(2S_F^\alpha(b)\right)^{1-\frac{1}{p}}$$

for all $x\in[a,b]$.

Thus f is bounded on $[a,b]$.

Theorem 3.2 *Let* $0<\alpha<1$ and $1<p<\infty$. *If $f\in BV_F^{\alpha,p}[a,b]$ and $c\in(a,b)$ then*

$$V_F^{\alpha,p}(f;[a,b])=V_F^{\alpha,p}(f;[a,c])+V_F^{\alpha,p}(f;[c,b]).$$

Proof.

If $P=\{a=x_0,\dots,x_{k-1}\le c\le x_k,\dots,x_n=b\}$ is any partition of $[a,b]$ and $x_{k-1}\le c\le x_k$, then

$$\frac{|f(x_k)-f(x_{k-1})|^p}{|S_F^\alpha(x_k)-S_F^\alpha(x_k)|^{p-1}}\le\frac{|f(x_k)-f(c)|^p}{|S_F^\alpha(x_k)-S_F^\alpha(c)|^{p-1}}\frac{|f(c)-f(x_{k-1})|^p}{|S_F^\alpha(c)-S_F^\alpha(x_{k-1})|^{p-1}}.$$

So that

$$\sum_{i=0}^{n-1}\frac{|f(x_{i+1})-f(x_i)|^p}{|S_F^\alpha(x_{i+1})-S_F^\alpha(x_i)|^{p-1}}$$

$$\leq \sum_{i=0}^{k-2} \frac{|f(x_{i+1})-f(x_i)|^p}{|S_F^\alpha(x_{i+1})-S_F^\alpha(x_i)|^{p-1}} + \frac{|f(x_k)-f(c)|^p}{|S_F^\alpha(x_k)-S_F^\alpha(c)|^{p-1}} + \frac{|f(c)-f(x_{k-1})|^p}{|S_F^\alpha(c)-S_F^\alpha(x_{k-1})|^{p-1}} + \sum_{i=k}^{n-1} \frac{|f(x_{i+1})-f(x_i)|^p}{|S_F^\alpha(x_{i+1})-S_F^\alpha(x_i)|^{p-1}}$$
$$\leq V_F^{\alpha,p}(f;[a,c]) + V_F^{\alpha,p}(f;[c,b])$$

which implies that

$$V_F^{\alpha,p}(f;[a,b]) \leq V_F^{\alpha,p}(f;[a,c]) + V_F^{\alpha,p}(f;[c,b]). \quad (1)$$

To prove the other inequality, we let $\varepsilon > 0$ and select a partition $P = \{a = x_0, x_1, \dots, x_k = c\}$ of $[a,c]$ for which

$$\sum_{i=0}^{k-1} \frac{|f(x_{i+1})-f(x_i)|^p}{|S_F^\alpha(x_{i+1})-S_F^\alpha(x_i)|^{p-1}} \geq V_F^{\alpha,p}(f;[a,c]) - \frac{\varepsilon}{2}$$

and a partition $P' = \{c = x_k, x_{k+1}, \dots, x_n = b\}$ of $[c,b]$ for which

$$\sum_{i=k}^{n-1} \frac{|f(x_{i+1})-f(x_i)|^p}{|S_F^\alpha(x_{i+1})-S_F^\alpha(x_i)|^{p-1}} \geq V_F^{\alpha,p}(f;[a,c]) - \frac{\varepsilon}{2}.$$

Then for partition $P \cup P'$ of $[a,b]$ we have

$$\sum_{i=0}^{n-1} \frac{|f(x_{i+1})-f(x_i)|^p}{|S_F^\alpha(x_{i+1})-S_F^\alpha(x_i)|^{p-1}} = \sum_{i=0}^{k-2} \frac{|f(x_{i+1})-f(x_i)|^p}{|S_F^\alpha(x_{i+1})-S_F^\alpha(x_i)|^{p-1}} + \frac{|f(x_k)-f(c)|^p}{|S_F^\alpha(x_k)-S_F^\alpha(c)|^{p-1}} + \frac{|f(c)-f(x_{k-1})|^p}{|S_F^\alpha(c)-S_F^\alpha(x_{k-1})|^{p-1}} + \sum_{i=k}^{n-1} \frac{|f(x_{i+1})-f(x_i)|^p}{|S_F^\alpha(x_{i+1})-S_F^\alpha(x_i)|^{p-1}}$$
$$\geq V_R^{\alpha,p}(f;[a,c]) + V^{\alpha,p}(f;[c,b]) - \varepsilon.$$

This assures that

$$V_F^{\alpha,p}(f;[a,b]) \geq V_F^{\alpha,p}(f;[a,c]) + V_F^{\alpha,p}(f;[c,b]) - \varepsilon$$

for all $\varepsilon > 0$ and hence that

$$V_F^{\alpha,p}(f;[a,b]) \geq V_F^{\alpha,p}(f;[a,c]) + V_F^{\alpha,p}(f;[c,b]). \quad (2)$$

From inequality (1) and (2) we conclude that

$$V_F^{\alpha,p}(f;[a,b]) = V_F^{\alpha,p}(f;[a,c]) + V_F^{\alpha,p}(f;[c,b]).$$

The prove is complete.

Which shows that $f \in AC_F^\alpha[a,b]$.

(i) Let $f \in BV_F^{\alpha,p}[a,b]$. Let $P = \{x_0, x_1, \dots, x_n\}$ is any subdivision of $[a,b]$. Let $\varepsilon > 0$. On the interval $[x_i, x_{i+1}], k = 0,1,\dots,n-1$ we can find subdivision $P_i = \{x_i = t_{i,0}, \dots, t_{i,m_i-1} = x_{i+1}\}$ with $t_{i,l} < t_{i,l+1}$ for $l = 0,\dots,m_i-1$ such that

$$\sum_{l=0}^{m_i-1} \frac{|f(t_{i,l+1})-f(t_{i,l})|^p}{\left(S_F^\alpha(t_{i,l+1})-S_F^\alpha(t_{i,l})\right)^{p-1}} + \frac{\varepsilon}{n} \geq V_F^{\alpha,p}(f;[x_i,x_{i+1}]).$$

It follows that

$$\sum_{i=0}^{n-1} \left|v_F^{\alpha,p}(x_{i+1}) - v_F^{\alpha,p}(x_i)\right| = \sum_{i=0}^{n-1} V_F^{\alpha,p}(f;[x_i,x_{i+1}]) \leq \sum_{i=0}^{n-1} \left(\sum_{l=0}^{m_i-1} \frac{|f(t_{i,l+1})-f(t_{i,l})|^p}{\left(S_F^\alpha(t_{i,l+1})-S_F^\alpha(t_{i,l})\right)^{p-1}} + \frac{\varepsilon}{n}\right) < 2\varepsilon.$$

This shows that $v_F^{\alpha,p}$ is F^α −absolutely continuous on a fractal set F.

4. CONCLUSIONS

In this paper, we have discussed if f a fractal bounded p −variation function on a fractal set F then f and fractal p −variation $v_F^{\alpha,p}$ are F^α −fractal absolutely continuous on a fractal set F.

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