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On Ramsey Minimal Graphs for a 3-Matching Versus a Path on Five Vertices

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ABSTRACT

Let G, H, and F be simple graphs. The notation $F \longrightarrow (G, H)$ means that any red-blue coloring of all edges of F contains a red copy of G or a blue copy of H. The graph F satisfying this property is called a Ramsey (G, H)-graph. A Ramsey (G, H)-graph is called minimal if for each edge $e \in E(F)$, there exists a red-blue coloring of $F - e$ such that $F - e$ contains neither a red copy of G nor a blue copy of H. In this paper, we construct some Ramsey $(3K_2, P_5)$-minimal graphs by subdivision (5 times) of one cycle edge of a Ramsey $(2K_2, P_5)$-minimal graph. Next, we also prove that for any integer $m \geq 3$, the set $R(mK_2, P_5)$ contains no connected graphs with circumference 3.

Keywords: *Ramsey minimal graph,* 3*-matching, Path.*

1. INTRODUCTION

Given simple graphs G and H, any red-blue coloring of the edges of F is called a (G, H)*-coloring* if it has neither red copy of G nor blue copy of H. The notation $F \rightarrow (G, H)$ means that in any red-blue coloring of F there exists a red copy of G or a blue copy of H as a subgraph. A graph F is said to be a *Ramsey* (G, H)*-minimal* if $F \rightarrow (G, H)$ but for any $e \in E(F)$ there exists a (G, H)-coloring on graph $F - e$. The set of all Ramsey (G, H)-minimal graphs is denoted by $R(G, H)$. Burr, Erdős, Faudree, and Schelp [1] proved that if H is an arbitrary graph then $R(mK_2, H)$ is a finite set. One of challenging problems in Ramsey Theory is to characterize all graphs in the set $R(mK_2, H)$ for a given graph H. As usual, K_n, C_n, and P_n denote a complete graph, a cycle, and a path on n vertices, respectively. For any connected graph G, and $m \geq 2$, the notation mG means a disjoint union of m copies of a graph G. A t-*matching*, denoted by tK_2, is a graph with t components where every component is a graph K_2.

In general, it is difficult to characterize all graphs belonging to $R(mK_2, H)$. However, for some particular graph H, this set $R(mK_2, H)$ has been known. For instance, Burr, Erdős, Faudree, and Schelp [1] showed that $R(2K_2, 2K_2) = \{C_5, 3K_2\}$ and $R(2K_2, K_3) = \{K_5, 2K_3, G_1\}$, where G_1 is a graph having the vertex-set $V(G_1) = \{c, u_i, v_i, w_i \mid i = 1,2\}$ and the edge-set $E(G_1) = \{cu_i, cv_i, cw_i \mid i = 1,2\} \cup \{u_1u_2, v_1v_2, w_1w_2\} \cup \{u_1v_1, u_1w_1, v_1w_1\}$. Burr *et al.* [2] showed that $R(2K_2, P_3) = \{C_4, C_5, 2P_3\}$. Baskoro and Yulianti [3] proved that $R(2K_2, P_4) = \{C_5, C_6, C_7, 2P_4, C_4^+\}$, where C_4^+ is a graph formed by a cycle on 4 vertices C_4 and two pendants vertices so that two vertices of degree 3 in the cycle C_4 are adjacent. Furthermore, they [3] also proved that $R(2K_2, P_5) = \{C_6, C_7, C_8, C_9, 2P_5\} \cup \{A_i \mid i \in [1,7]\}$, where A_is are the graphs depicted in Figure 1. Wijaya, Baskoro, Assiyatun, and Suprijanto [4] showed that the cycle C_s belongs to $R(mK_2, P_n)$ if and only if $s \in [mn - n + 1 \leq s \leq mn - 1]$. Other results on characterizing all Ramsey minimal graphs for the pair of a matching versus a path can be seen in [5 – 8].

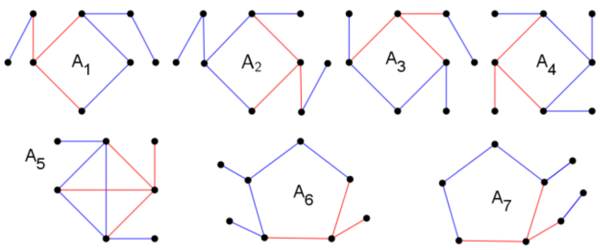

Figure 1 Some Ramsey $(2K_2, P_5)$-minimal graphs.

In [1], Burr, Erdős, Faudree, and Schelp gave a family of $\frac{(n+1)}{2}$ non-isomorphic graphs in $R(2K_2, K_n)$ for $n \geq 4$. These graphs are constructed from a complete graph K_{n+1}. In the same paper, Burr, Erdős, Faudree, and Schelp also gave a family of $(n-2)$ non-isomorphic graphs belonging to $R(2K_2, K_{1,n})$. Motivated by them, Wijaya, Baskoro, Assiyatun, and Suprijanto [9] constructed some graphs in $R(mK_2, P_3)$ by subdivision (3 times) on any non-pendant edge of a connected graph in $R((m-1)K_2, P_3)$. Furthermore, Wijaya, Baskoro, Assiyatun, and Suprijanto [10] constructed a family of Ramsey (mK_2, P_4) minimal graphs from any Ramsey $((m-1)K_2, P_4)$ minimal graph by the subdivision process on any cycle-edge (4 times).

In this paper, we focus on constructing Ramsey $(3K_2, P_5)$ minimal graphs for 3-matching versus a path with five vertices. We also prove that there is no graph with circumference 3 belonging to $R(mK_2, P_5)$ for any integer $m \geq 3$. A *circumference* of a graph is the length of the longest cycle in that graph.

The following two lemmas provide the necessary and sufficient conditions for any graph in $R(3K_2, H)$ for any graph H.

Lemma 1.1 [9, 10] For any fixed graph H, the graph $F \longrightarrow (3K_2, H)$ holds if and only if the following four conditions are satisfied: (i) $F - \{u, v\} \supseteq H$ for each $u, v \in V(F)$, (ii) $F - \{u\} - E(K_3) \supseteq H$ for each $u \in V(F)$ and a triangle K_3 in F, (iii) $F - E(2K_3) \supseteq H$ for every two triangles in F, (iv) $F - E(S_5) \supseteq H$ for every induced subgraph with 5 vertices S in F. ∎

Lemma 1.2 [9, 10] Let H be a simple graph. Suppose F is a Ramsey $(3K_2, H)$-graph. F is said to be *minimal* if for each $e \in E(F)$ satisfies $(F - e) \nrightarrow (3K_2, H)$, that is, (i) $(F - e) - \{u, v\} \not\supseteq H$ for each $u, v \in V(F)$, (ii) $F - \{u\} - E(K_3) \not\supseteq H$ for each $u \in V(F)$ and a triangle K_3 in F, (iii) $F - E(2K_3) \not\supseteq H$ for every two triangles in F, (iv) $F - E(S_5) \not\supseteq H$ for every induced subgraph with 5 vertices S in F. ∎

Any graph satisfying all conditions stated in Lemmas 1 and 2 is a Ramsey $(3K_2, H)$-minimal graph. The condition stated in Lemma 1.2 is called the *minimality property* of a graph in $R(3K_2, H)$.

Next theorem is one of the important properties of a Ramsey (mK_2, H)-minimal graph.

Theorem 1.3 [9] Let H be a graph and $m > 1$ be an integer. If $F \in (mK_2, H)$, then for any $v \in V(F)$ and $K_3 \subseteq F$, both graphs $F - \{v\}$ and $F - E(K_3)$ contain a Ramsey $((m-1)K_2, H)$-minimal graph. ∎

2. MAIN RESULTS

In this section, we give some graphs belonging to $R(3K_2, P_5)$. We construct these graphs by the subdivision process on any cycle edge of a connected graph in $R(2K_2, P_5)$ depicted in Figure 1. Before doing this, first we show that a graph F_1, depicted in Figure 2, is a Ramsey $(3K_2, P_5)$-minimal graph. The vertex set of a graph F_1 is $V(F_1) = \{v_1, v_2, \ldots, v_{11}\}$ and the edge set of a graph F_1 is $E(F_1) = \{v_i v_{i+1} \mid i = 1, 2, \ldots, 10\} \cup \{v_2 v_9, v_3 v_{10}\}$.

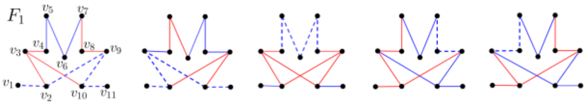

Figure 2 A graph F_1 and some red-blue colorings of F_1 so that F_1 contains no red $3K_2$ but it contains a blue P_5.

Proposition 2.1 Let F_1 be a graph on 11 vertices and 12 edges as depicted in Figure 2. The graph F_1 is a Ramsey $(3K_2, P_5)$-minimal graph.

Proof. First, we prove that for any red-blue coloring of F_1 there exists a red $3K_2$ or a blue P_5 in F_1. We can see that $F_1 - \{v_i, v_j\}$ always contains a path P_5 for any $1 \leq i, j \leq 11$. It can be verified that $F_1 - E(S_5) \supseteq H$ for every induced subgraph with 5 vertices S in F_1. Since F_1 has no triangle then by Lemma 1.1, $F_1 \rightarrow (3K_2, P_5)$. Next, we prove the minimality property of F_1. For any edge e we will show that $(F_1 - e) \nrightarrow (3K_2, P_5)$. If e is one of dashed edges in Figure 2, then each red-blue coloring in Figure 2 provides a $(3K_2, P_5)$ coloring on $F_1 - e$, namely a coloring that have neither red $3K_2$ nor blue P_5. Therefore $F_1 \in R(3K_2, P_5)$. ∎

Next, we construct some Ramsey $(3K_2, P_5)$-minimal graphs from previous known Ramsey $(2K_2, P_5)$-minimal graphs by subdivision process. Consider each of Ramsey $(2K_2, P_5)$-minimal graphs in Figure 1. By the subdivision (5 times) on any of its cycle-edges we produce Ramsey $(3K_2, P_5)$-minimal graphs in Figure 3. In total, we obtain 12 non-isomorphic graphs belonging to $R(3K_2, P_5)$. Two non-isomorphic graphs F_2 and F_3 are obtained from the subdivision of A_1. Two non-isomorphic graphs F_4 and F_5 are formed from A_2. Two non-isomorphic graphs F_6 and F_7 are obtained from A_3. One graph called F_8 is obtained from the graph A_4. One graph F_9 is formed from A_5. Two non-isomorphic graphs F_{10} and F_{11} are obtained from the graph A_6. Last, two non-isomorphic graphs F_{12} and F_{13} are formed from A_7. In the following theorem, we will prove that these graphs are Ramsey $(3K_2, P_5)$-minimal graphs.

Theorem 2.2 All the graphs $F_2, F_3, \dots, F_{13}$ in Figure 3 are Ramsey $(3K_2, P_5)$-minimal graphs.

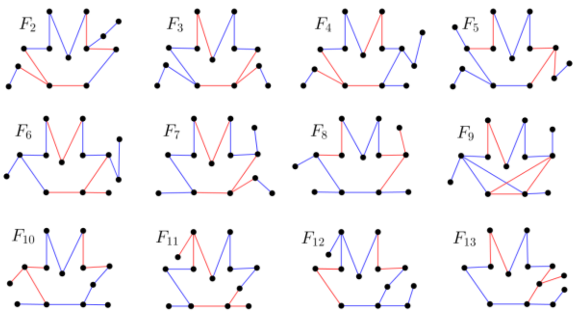

Figure 3 Some graphs belong to $R(3K_2, P_5)$.

Proof. Let F be any graph in Figure 3. It is easy to see that F satisfies all the conditions in Lemma 1.1. Then, $F \longrightarrow (3K_2, P_5)$ holds. Now, we will show the minimality property of F. Let e be any edge in F. If e is one of dashed edges, then a $(3K_2, P_5)$-coloring on $F - e$ is provided in Figures 4 and 5 for all cases of F and e. ∎

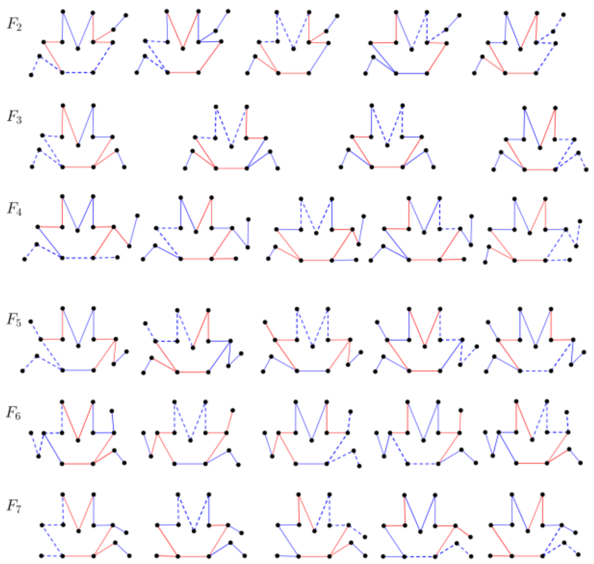

Figure 4 The $(3K_2, P_5)$-colorings on $F_i - e$ if e is one of dashed edges and for $i \in [2, 7]$.

Actually, there are two non-isomorphic graphs obtained by the subdivision (5 vertices) on any cycle edge of A_5 (see Figure 1). One of these two graphs is F_{10} and the other is obtained by subdivision (5 vertices) on the edge incident with a vertex of degree 4 and a vertex of degree 3. The last graph is a Ramsey-$(3K_2, P_5)$ graph but not minimal since it contains a graph $F_1 \in R(3K_2, P_5)$ (in Figure 2).

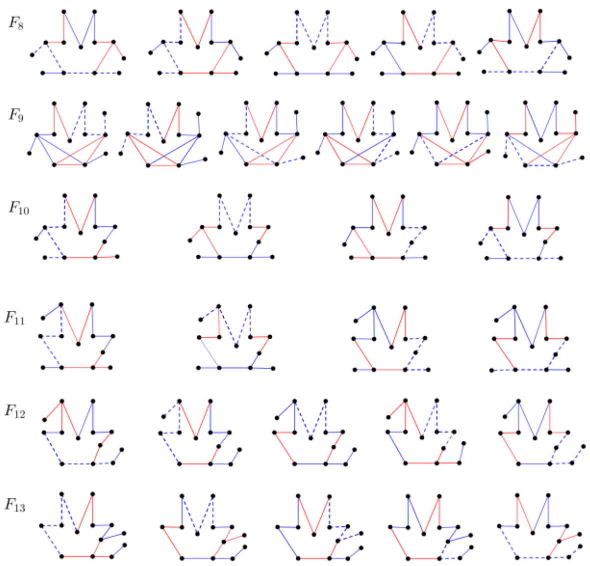

Figure 5 The $(3K_2, P_5)$-colorings on $F_i - e$ for $i \in [8, 13]$ if e is one of dashed edges.

In the following theorem, we will give a property of graphs belonging to $R(mK_2, P_5)$.

Theorem 2.3 There is no Ramsey (mK_2, P_5)-minimal graph with circumference 3 for any integer $m \geq 2$.

Proof. We will prove the theorem by induction on m. If $m = 2$ then it has been shown that there is no $(2K_2, P_5)$-minimal graph with circumference 3 (see [3]).

Assume that there is no (tK_2, P_5)-minimal graph with circumference 3 for any positive integer $t \leq m - 1$. We will show that there is no (mK_2, P_5)-minimal graph with circumference 3. Suppose to the contrary that there exists a graph F which is a Ramsey (mK_2, P_5)-minimal graph with circumference 3. Then, F must be a unicyclic graph. Let C be the cycle in F with $V(C) = \{u_1, u_2, u_3\}$. According to Theorem 1.3, $F - \{u_i\}$ for every $i \in [1, 3]$ contains a graph $G \in R\big((m-1)K_2, P_5\big)$. By assumption, the set $R\big((m-1)K_2, P_5\big)$ has no graph with circumference 3. So, G must be isomorphic to $(m-1)P_5$. It forces that $F - E(C)$ is a graph $P_{n_1} \cup P_{n_2} \cup P_{n_3}$ where $n_1 + n_2 + n_3 \geq 5m = 15$. It implies that F contains a graph mP_5. Hence, F is not minimal. Otherwise, without loss of generality, we consider $n_1 + n_2 + n_3 = 5m - 1 \geq 14$ and assume $u_1 \in V(P_{n_1})$, $u_2 \in V(P_{n_2})$, and $u_3 \in V(P_{n_3})$. Suppose w.l.o.g. $n_1 \geq n_2 \geq n_3$ and $V(P_{n_1}) = \{u_1, v_{n_1-1}, v_{n_1-2}, \dots, v_2, v_1\}$ where v_1 is the pendant vertex of a path P_{n_1} and $E(P_{n_1}) =$

$\{u_1v_{n_1-1}, v_iv_{i+1} | \, i \in [1, n_1-2]\}$. Clearly $n_1 \geq 5$. If $n_1 > 5$, we set the vertex $v_5 \in V(P_{n_1})$, then we obtain that $F - \{v_5\}$ does not contain a graph $(m-1)P_5$, which would contradict Theorem 1.3. In the case of $n_1 = 5$ we have $n_2 = 5$ and $n_3 = 4$. We obtain $F - \{u_1\} \not\supseteq 2P_5$, a contradiction with Theorem 1.3. Thus, the proof is complete. ■

3. CONCLUSION

In this paper, we discuss on the construction of Ramsey $(3K_2, P_5)$-minimal graphs. By the subdivision of any cycle edge of 7 Ramsey $(2K_2, P_5)$-minimal graphs (in Figure 1) we obtain 13 non-isomorphic Ramsey $(3K_2, P_5)$-minimal graphs. We also show that there is no Ramsey (mK_2, P_5)-minimal graph circumference 3 for any integer $m \geq 2$.

For a future work, we pose some open problems below.

Open Problem 1. Characterize all graphs belonging to $R(3K_2, P_5)$ by excluding all graphs resulted in this paper.

Open Problem 2. Are there any connected graphs with circumference 4 or 5 belonging to $R(3K_2, P_5)$?

Open Problem 3. Is it true that the subdivision (5 times) on any cycle-edge of a connected Ramsey $((m-1)K_2, P_5)$-minimal graph always produces a connected Ramsey (mK_2, P_5)- minimal graph?

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