On Ramsey Minimal Graphs for a 3-Matching Versus a Path on Five Vertices

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ABSTRACT

Let G, H, and F be simple graphs. The notation F → (G, H) means that any red-blue coloring of all edges of F contains a red copy of G or a blue copy of H. The graph F satisfying this property is called a Ramsey (G, H)-graph. A Ramsey (G, H)-graph is called minimal if for each edge e ∈ E(F), there exists a red-blue coloring of F − e such that F − e contains neither a red copy of G nor a blue copy of H. In this paper, we construct some Ramsey (3K₂, P₅)-minimal graphs by subdivision (5 times) of one cycle edge of a Ramsey (2K₂, P₃)-minimal graph. Next, we also prove that for any integer m ≥ 3, the set R(mK₂, P₅) contains no connected graphs with circumference 3.

Keywords: Ramsey minimal graph, 3-matching, Path.

1. INTRODUCTION

Given simple graphs G and H, any red-blue coloring of the edges of F is called a (G, H)-coloring if it has neither red copy of G nor blue copy of H. The notation F → (G, H) means that in any red-blue coloring of F there exists a red copy of G or a blue copy of H as a subgraph. A graph F is said to be a Ramsey (G, H)-minimal if F → (G, H) but for any e ∈ E(F) there exists a (G, H)-coloring on graph F − e. The set of all Ramsey (G, H)-minimal graphs is denoted by R(G, H). Burr, Erdős, Faudree, and Schelp [1] proved that if H is an arbitrary graph then R(mK₂, H) is a finite set. One of the challenging problems in Ramsey Theory is to characterize all graphs in the set R(mK₂, H) for a given graph H. As usual, Kₘ, Cₙ, and Pₙ denote a complete graph, a cycle, and a path on n vertices, respectively. For any connected graph G, and m ≥ 2, the notation mG means a disjoint union of m copies of a graph G. A t-matching, denoted by tK₂, is a graph with t components where every component is a graph K₂.

In general, it is difficult to characterize all graphs belonging to R(mK₂, H). However, for some particular graph H, this set R(mK₂, H) has been known. For instance, Burr, Erdős, Faudree, and Schelp [1] showed that R(2K₂, 2K₂) = {C₅, 3K₂} and R(2K₂, K₃) = {K₆, 2K₂, G₁}, where G₁ is a graph having the vertex-set V(G₁) = {c, uᵢ, vᵢ, w₁| i = 1, 2} and the edge-set E(G₁) = {cuᵢ, cvᵢ, cw₁| i = 1, 2} ⋃ {u₁u₂, v₁v₂, w₁w₂} ⋃ {u₁v₁, u₂w₁, v₁w₁}. Burr et al. [2] showed that R(2K₂, P₃) = {C₈, C₉, 2P₃}. Baskoro and Yulianti [3] proved that R(2K₂, P₅) = {C₉, C₁₀, C₁₁, 2P₅, C₄⁺}, where C₄⁺ is a graph formed by a cycle on 4 vertices C₄ and two pendant vertices so that two vertices of degree 3 in the cycle C₄ are adjacent. Furthermore, they [3] also proved that R(2K₂, P₃) = {C₉, C₁₀, C₁₁, 2P₅} ∪ {A₁| i ∈ [1, 7]}, where A₁'s are the graphs depicted in Figure 1. Wijaya, Baskoro, Assiyatun, and Suprijanto [4] showed that the cycle C₄ belongs to R(mK₂, P₅) if and only if s ∈ [mn − n + 1 ≤ s ≤ mn − 1]. Other results on characterizing all Ramsey minimal graphs for the pair of a matching versus a path can be seen in [5 – 8].

Figure 1 Some Ramsey (2K₂, P₅)-minimal graphs.
In [1], Burr, Erdős, Faudree, and Schelp gave a family of $\binom{n+1}{2}$ non-isomorphic graphs in $R(2K_2, K_n)$ for $n \geq 4$. These graphs are constructed from a complete graph $K_{n+1}$. In the same paper, Burr, Erdős, Faudree, and Schelp also gave a family of $(n - 2)$ non-isomorphic graphs belonging to $R(2K_2, K_{1,n})$. Motivated by them, Wijaya, Baskoro, Assiyatun, and Suprijanto [9] constructed some graphs in $R(mK_2, P_3)$ by subdivision (3 times) on any non-pendant edge of a connected graph in $R((m - 1)K_2, P_3)$. Furthermore, Wijaya, Baskoro, Assiyatun, and Suprijanto [10] constructed a family of Ramsey $(m - 1)K_2, P_3)$ minimal graph by the subdivision process on any cycle-edge (4 times).

In this paper, we focus on constructing Ramsey $(3K_2, P_3)$ minimal graphs for 3-matching versus a path with five vertices. We also prove that there is no graph with circumference 3 belonging to $R(mK_2, P_3)$ for any integer $m \geq 3$. A circumference of a graph is the length of the longest cycle in that graph.

The following two lemmas provide the necessary and sufficient conditions for any graph in $R(3K_2, H)$ for any graph $H$.

**Lemma 1.1** [9, 10] For any fixed graph $H$, the graph $F \rightarrow (3K_2, H)$ holds if and only if the following four conditions are satisfied: (i) $F - \{u, v\} \supseteq H$ for each $u, v \in V(F)$, (ii) $F - \{u\} - E(K_3) \supseteq H$ for each $u \in V(F)$ and a triangle $K_3$ in $F$, (iii) $F - E(K_3) \supseteq H$ for every two triangles in $F$, (iv) $F - E(S_3) \supseteq H$ for every induced subgraph with 5 vertices $S$ in $F$.

**Lemma 1.2** [9, 10] Let $H$ be a simple graph. Suppose $F$ is a Ramsey $(3K_2, H)$-graph. $F$ is said to be minimal if for each $e \in E(F)$ satisfies $(F - e) \not\rightarrow (3K_2, H)$, that is, (i) $(F - e) - \{u, v\} \not\supseteq H$ for each $u, v \in E(F)$, (ii) $F - \{u\} - E(K_3) \not\supseteq H$ for each $u \in V(F)$ and a triangle $K_3$ in $F$, (iii) $F - E(2K_3) \not\supseteq H$ for every two triangles in $F$, (iv) $F - E(S_3) \not\supseteq H$ for every induced subgraph with 5 vertices $S$ in $F$.

Any graph satisfying all conditions stated in Lemmas 1 and 2 is a Ramsey $(3K_2, H)$-minimal graph. The condition stated in Lemma 1.2 is called the minimality property of a graph in $R(3K_2, H)$.

Next theorem is one of the important properties of a Ramsey $(mK_2, H)$-minimal graph.

**Theorem 1.3** [9] Let $H$ be a graph and $m > 1$ be an integer. If $F \in (mK_2, H)$, then for any $v \in V(F)$ and $K_3 \subseteq F$, both graphs $F - \{v\}$ and $F - E(K_3)$ contain a Ramsey $(m - 1)K_2, H)$-minimal graph.

**2. MAIN RESULTS**

In this section, we give some graphs belonging to $R(3K_2, P_3)$. We construct these graphs by the subdivision process on any cycle edge of a connected graph in $R(2K_2, P_3)$ depicted in Figure 1. Before doing this, first we show that a graph $F_1$, depicted in Figure 2, is a Ramsey $(3K_2, P_3)$-minimal graph. The vertex set of a graph $F_1$ is $V(F_1) = \{v_1, v_2, ..., v_{11}\}$ and the edge set of a graph $F_1$ is $E(F_1) = \{v_iv_{i+1} | i = 1, 2, ..., 10\} \cup \{v_1v_3, v_3v_{10}\}$.

![Figure 2](20x805 to 90x833)

**Figure 2** A graph $F_1$ and some red-blue colorings of $F_1$ so that $F_1$ contains no red $3K_2$ but it contains a blue $P_3$.

**Proposition 2.1** Let $F_3$ be a graph on 11 vertices and 12 edges as depicted in Figure 2. The graph $F_1$ is a Ramsey $(3K_2, P_3)$-minimal graph.

**Proof.** First, we prove that for any red-blue coloring of $F_1$ there exists a red $3K_2$ or a blue $P_3$ in $F_1$. We can see that $F_1 - \{v_i, v_j\}$ always contains a path $P_3$ for any $1 \leq i, j \leq 11$. It can be verified that $F_1 - E(S_3) \supseteq H$ for every induced subgraph with 5 vertices $S$ in $F_1$. Since $F_1$ has no triangle then by Lemma 1.1, $F_1 \rightarrow (3K_2, P_3)$.

Next, we prove the minimality property of $F_1$. For any edge $e$ we will show that $(F_1 - e) \not\rightarrow (3K_2, P_3)$. If $e$ is one of dashed edges in Figure 2, then each red-blue coloring in Figure 2 provides a $(3K_2, P_3)$ coloring on $F_1 - e$, namely a coloring that have neither red $3K_2$ nor blue $P_3$. Therefore $F_1 \not\in R(3K_2, P_3)$.

Next, we construct some Ramsey $(3K_2, P_3)$-minimal graphs from previous known Ramsey $(2K_2, P_3)$-minimal graphs by subdivision process. Consider each of Ramsey $(2K_2, P_3)$-minimal graphs in Figure 1. By the subdivision (5 times) on any of its cycle-edges we produce Ramsey $(3K_2, P_3)$-minimal graphs in Figure 3. In total, we obtain 12 non-isomorphic graphs belonging to $R(3K_2, P_3)$. Two non-isomorphic graphs $F_4$ and $F_5$ are obtained from the subdivision of $A_1$. Two non-isomorphic graphs $F_6$ and $F_7$ are formed from $A_2$. Two non-isomorphic graphs $F_{10}$ and $F_{11}$ are obtained from the graph $A_6$. Last, two non-isomorphic graphs $F_{12}$ and $F_{13}$ are formed from $A_7$. In the following theorem, we will
prove that these graphs are Ramsey \((3K_2, P_5)\)-minimal graphs.

**Theorem 2.2** All the graphs \(F_2, F_3, \ldots, F_{13}\) in Figure 3 are Ramsey \((3K_2, P_5)\)-minimal graphs.

![Figure 3](image)

**Figure 3** Some graphs belong to \(R(3K_2, P_5)\).

**Proof.** Let \(F\) be any graph in Figure 3. It is easy to see that \(F\) satisfies all the conditions in Lemma 1.1. Then, \(F \rightarrow (3K_2, P_5)\) holds. Now, we will show the minimality property of \(F\). Let \(e\) be any edge in \(F\). If \(e\) is one of dashed edges, then a \((3K_2, P_5)\)-coloring on \(F - e\) is provided in Figures 4 and 5 for all cases of \(F\) and \(e\). ■

![Figure 4](image)

**Figure 4** The \((3K_2, P_5)\)-colorings on \(F_i - e\) if \(e\) is one of dashed edges and for \(i \in [2, 7]\).

Actually, there are two non-isomorphic graphs obtained by the subdivision (5 vertices) on any cycle edge of \(A_5\) (see Figure 1). One of these two graphs is \(F_{10}\) and the other is obtained by subdivision (5 vertices) on the edge incident with a vertex of degree 4 and a vertex of degree 3. The last graph is a Ramsey-\((3K_2, P_5)\) graph but not minimal since it contains a graph \(F_i \in R(3K_2, P_5)\) (in Figure 2).

![Figure 5](image)

**Figure 5** The \((3K_2, P_5)\)-colorings on \(F_i - e\) for \(i \in [8, 13]\) if \(e\) is one of dashed edges.

In the following theorem, we will give a property of graphs belonging to \(R(mK_2, P_5)\).

**Theorem 2.3** There is no Ramsey \((mK_2, P_5)\)-minimal graph with circumference 3 for any integer \(m \geq 2\).

**Proof.** We will prove the theorem by induction on \(m\). If \(m = 2\) then it has been shown that there is no \((2K_2, P_5)\)-minimal graph with circumference 3 (see [3]).

Assume that there is no \((tK_2, P_5)\)-minimal graph with circumference 3 for any positive integer \(t \leq m - 1\). We will show that there is no \((mK_2, P_5)\)-minimal graph with circumference 3. Suppose to the contrary that there exists a graph \(F\) which is a Ramsey \((mK_2, P_5)\)-minimal graph with circumference 3. Then, \(F\) must be a unicyclic graph. Let \(C\) be the cycle in \(F\) with \(V(C) = \{u_1, u_2, u_3\}\). According to Theorem 1.3, \(F - \{u_i\}\) for every \(i \in [1, 3]\) contains a graph \(G \in R((m - 1)K_2, P_5)\). By assumption, the set \(R((m - 1)K_2, P_5)\) has no graph with circumference 3. So, \(G\) must be isomorphic to \((m - 1)P_5\). It forces that \(F - E(C)\) is a graph \(P_{n_1} \cup P_{n_2} \cup P_{n_3}\) where \(n_1 + n_2 + n_3 \geq 5m = 15\). It implies that \(F\) contains a graph \(mP_5\). Hence, \(F\) is not minimal. Otherwise, without loss of generality, we consider \(n_2 \geq n_3 \geq n_1 + n_3 = 5m - 1 \geq 14\) and assume \(u_1 \in V(P_{n_1}), u_2 \in V(P_{n_2})\), and \(u_3 \in V(P_{n_3})\). Suppose w.l.o.g. \(n_1 \geq n_2 \geq n_3\) and \(V(P_{n_1}) = \{v_1, v_{n_1 - 1}, v_{n_1 - 2}, \ldots, v_2, v_1\}\) where \(v_1\) is the pendant vertex of a path \(P_{n_1}\) and \(E(P_{n_1}) = \{v_1, v_2, v_{n_1 - 2}, v_{n_1 - 1}\}\).
\[\{u_i v_{n_i-1}, v_i v_{i+1}\} i \in [1, n_1 - 2]\). Clearly \(n_1 \geq 5\). If
\(n_1 > 5\), we set the vertex \(v_5 \in V(P_{n_1})\), then we obtain that \(F - \{v_5\}\) does not contain a graph \((m - 1)P_5\), which would contradict Theorem 1.3. In the case of \(n_1 = 5\) we have \(n_2 = 5\) and \(n_3 = 4\). We obtain \(F - \{u_1\} \nsubseteq 2P_5\), a contradiction with Theorem 1.3. Thus, the proof is complete.

3. CONCLUSION

In this paper, we discuss on the construction of Ramsey \((3K_2, P_5)\)-minimal graphs. By the subdivision of any cycle edge of 7 Ramsey \((2K_2, P_5)\)-minimal graphs (in Figure 1) we obtain 13 non-isomorphic Ramsey \((3K_2, P_5)\)-minimal graphs. We also show that there is no Ramsey \((mK_2, P_5)\)-minimal graph circumference 3 for any integer \(m \geq 2\).

For a future work, we pose some open problems below.

Open Problem 1. Characterize all graphs belonging to \(R(3K_2, P_5)\) by excluding all graphs resulted in this paper.

Open Problem 2. Are there any connected graphs with circumference 4 or 5 belonging to \(R(3K_2, P_5)\)?

Open Problem 3. Is it true that the subdivision (5 times) on any cycle-edge of a connected Ramsey \(((m - 1)K_2, P_5)\)-minimal graph always produces a connected Ramsey \((mK_2, P_5)\)-minimal graph?

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