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Rainbow Connection Number of Shackle Graphs

M. Ali Hasan*, Risma Yulina Wulandari, A.N.M. Salman

Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung
*Corresponding author. Email: malihasan1996@gmail.com

ABSTRACT

Let G be a simple, finite and connected graph. For a natural number k, we define an edge coloring $c: E(G) \rightarrow \{1,2,\ldots,k\}$ where two adjacent edges can be colored the same. A $u-v$ path (a path connecting two vertices u and v in $V(G)$) is called a rainbow path if no two edges of path receive the same color. If there exists a $u-v$ rainbow path for any two distinct vertices in $V(G)$, then G is called rainbow connected. In this case, c is called a rainbow k −coloring. The rainbow connection number of G, denoted by $rc(G)$, is the smallest number k such that G has a rainbow k −coloring. In this paper, we obtain upper and lower bounds of rainbow connection number of shackle graph G for any graph G. Furthermore, we show that these bounds are sharp. Then, we get the exact value of rainbow connection number of shackle sun graph, friendship, cycle, complete graph with one edge removed, and fan graph with two certain spokes removed.

Keywords: *Rainbow coloring, Rainbow connection, Shackle graph.*

1. INTRODUCTION

All graphs in this paper are finite, undirected, and simple. Let $G=(V,E)$ be a simple connected finite graph with vertex set $V(G)$ and edge set $E(G)$. The distance between two vertices u and v, denoted by $d(u,v)$, is the number of edges in a shortest path connecting them. For a natural number k, we define an edge coloring $c: E(G) \rightarrow \{1,2,\ldots,k\}$ where two adjacent edges can be colored the same. A $u-v$ path, a path connecting two vertices u and v in $V(G)$, is called a rainbow path if there is no two edges with the same color in this path. If there is a $u-v$ *rainbow path* for any two distinct vertices in $V(G)$ under coloring c, then G is called a rainbow connected graph. In this case, c is called a rainbow k −coloring. The *rainbow connection number* of G, denoted by $rc(G)$, is the smallest number k such that G admits a rainbow k −coloring. This concept was introduced by Chartrand et al [1]. They studied the rainbow connection number of complete graph K_n, tree T_n, cycle C_n, wheel W_n, complete bipartite $K_{s,t}$, and complete k −partite $K_{n_1,n_2,\ldots,n_k}$. This topic continues to be developed. Many classes of graphs are studied, such as sunlet graph and its line [2], pencil graph [3], n −cross prism graph [4], and amalgamation graphs [5]. Other results can be found in [6].

In this paper, we investigate the rainbow connection number of shackle graphs. For simplifying, we define $[a,b]=\{x \in \mathbb{Z} : a \le x \le b\}$. Let m and n be two integer numbers at least 3 and G be a graph with order n. Suppose that two distinct vertices u and v in $V(G)$. A shackle of G, denoted by $shack(G,u,v,m)$, is a graph which is constructed by m copies of G, namely $G^1, G^2, \ldots, G^m$ such that in the i^{th} graph G (or G^i) the vertices u and v are given the superscript label, namely u^i and v^i where for each $i \in [1, m-1]$ the vertex v^i of G^i is attached to u^{i+1} of G^{i+1}. Thus, the graph $shack(G,u,v,m)$ has the property $v^i = u^{i+1}$ for $i \in [1, m-1]$. For further discussion, these vertices are called the common vertices. The rainbow connection number of shackle graphs has been determined for shackle tribun [7] and shackle antiprism [8]. In this paper, we investigate the lower and upper bounds of rainbow connection number of shackle graph G for any graph G. Furthermore, we will show the sharpness of these bounds.

2. MAIN RESULTS

The lower bound of rainbow connection number of shackle graphs is given in the following observation.

Observation 2.1 Let m and n be two integer numbers at least 3 and G be a graph with order n. Let u and v be two distinct vertices of G,

$$rc\big(shack(G,u,v,m)\big) \geq \max_{x\in V(G)} d(x,v) + (m-2)d(u,v) + \max_{y\in V(G)} d(u,y).$$

Proof. Certainly, there is a vertex x' in graph G^1 where the distance to the vertex v^1 is equal to $\max_{x\in V(G)} d(x,v)$. Meanwhile, in G^m there is a vertex y' such that $d(u^m,y') = \max_{y\in V(G)} d(u,y)$. Therefore, the distance of the shortest path P connecting x' and y' is $d(x',v^1) + (m-2)d(u,v) + d(u^m,y')$. If given any rainbow coloring, all edges in P path have to use the $d(x',y')$ distinct colors. ■

For the illustration of Observation 2.1, see Figure 1.

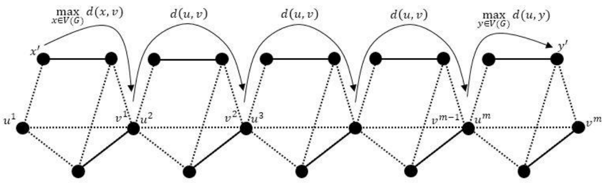

Figure 1 The illustration of Observation 2.1.

Obviously, every graph G has a rainbow k −coloring c. If such the coloring pattern is applied to all subgraphs G in $shack(G,u,v,m)$ and ensure all edges of $E(G^i)$ have distinct colors with all edges of $E(G^j)$ for distinct $i,j \in [1,m]$, then graph $shack(G,u,v,m)$ will be rainbow connected with $m\cdot rc(G)$ colors. Hence, we obtain the following result.

Theorem 2.2 Let m and n be integers at least 3 and G be a graph with order n. For any two vertices u and v in G,

$$rc\big(shack(G,u,v,m)\big) \leq m\cdot rc(G).$$

Proof. Let c be a rainbow $rc(G)$ −coloring of G. We define an edge coloring $c': E\big(shack(G,u,v,m)\big) \to [1, m\cdot rc(G)]$ with $c'(e) = c(e) + (i-1)rc(G)$ for each $e \in G^i$ and $i \in [1,m]$. Let $x,y \in V\big(shack(G,u,v,m)\big)$. If $x,y \in V(G^i)$ for some $i \in [1,m]$, then it is clearly that there exits an $x-y$ rainbow path. Thus, we assume that $x \in V(G^i)$ and $y \in V(G^j)$ for $i,j \in [1,m]$ with $i<j$. Observe that $c'\big(E(G^r)\big) \cap c'(E(G^s)) = \emptyset$ for all $r,s \in [1,m]$ with $r \neq s$. Therefore, there exist an $x-v^i$ rainbow path P_1, a v^i-u^{j-1} rainbow path P_2, and a $v^{j-1}-y$ rainbow path P_3. Thus, $P = P_1 \cup P_2 \cup P_3$ is an $x-y$ rainbow path. ■

For the illustration of Theorem 2.2, see Figure 2.

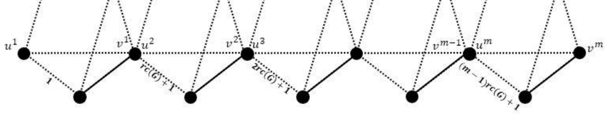

Figure 2 The illustration of Theorem 2.2.

Now, we prove the sharpness of the bounds in Observation 2.1 and Theorem 2.2. This result is given in the following corollary.

Corollary 2.3 Let m and n be two integers at least 3 and G be a graph with order n. Let u and v be two distinct vertices of G such that $d(u,v) = diam(G) = rc(G)$, then

$$rc\big(shack(G,u,v,m)\big) = m\cdot rc(G).$$

Proof. It follows by Theorem 2.2 that $rc\big(shack(G,u,v,m)\big) \leq m\cdot rc(G)$. For the lower bound, since $d(u,v) = diam(G)$, we have $\max_{x\in V(G)} d(x,v) = \max_{y\in V(G)} d(u,y) = d(u,v)$. Thus, by using Observation 2.1, we obtain that $rc\big(shack(G,u,v,m)\big) \geq m\cdot d(u,v) = m\cdot rc(G)$. ■

Next, we determine the exact value of rainbow connection number of shackle sun graph. A sun graph with order $2n$, denoted by S_n, is a graph constructed from a cycle graph C_n with order $n \geq 3$ by attaching one edge to each vertex in C_n. In the following theorem Rao K.S and Murali in [2] determined the rainbow connection number of S_n.

Theorem 2.4 [2] If $n \geq 3$,

$$rc(S_n) = \begin{cases} n, & \text{if } n \text{ is odd;} \\ \dfrac{3n-2}{2}, & \text{if } n \text{ is even.} \end{cases}$$

For n be even, we claim that the value of $rc(S_n)$ above should can be diminished, so we improve the result and state in the following Theorem.

Theorem 2.5 Let $n \geq 3$ and S_n be a sun graph with order $2n$,

$$rc(S_n) = \begin{cases} n, & \text{if } n \text{ is odd;} \\ n+1, & \text{if } n \text{ is even.} \end{cases}$$

Proof. For n be odd the result is obtained from Theorem 2.4. We assume that n is even. In this discussion, we define that for i and n two positive integers,

$$i \bmod^* n = \begin{cases} i \bmod n, & \text{if } i \neq kn \text{ for any } k \in \mathbb{N}; \\ n, & \text{if } i = kn \text{ for some } k \in \mathbb{N}. \end{cases}$$

Let $V(S_n) = \{u_i, v_i | i \in [1,n]\}$ such that $E(S_n) = \{u_iv_i, u_iu_{i+1} | i \in [1,n], i \bmod^* n\}$. Each index in label vertex is in $mod^*\, n$. Suppose that $n = 2k$ for some $k \in \mathbb{N}$. Define an edge coloring $c: E(S_n) \to [1, n+1]$ with

$$c(u_iv_i) = i, \text{for } i \in [1,2k];$$

$$c(u_iu_{i+1}) = \begin{cases} i-k, & \text{for } i \in [k+1, 2k]; \\ i+k+1, & \text{for } i \in [1,k]. \end{cases}$$

It will be showed that for each x and y in $V(S_n)$ there exists an $x-y$ rainbow path. Consider two vertices $x = v_i$ and $y = v_{i+k+1}$ for $i \in [1,k]$. Obviously, there is an $x-y$ rainbow path, i.e

$$x, u_i, u_{i+1}, \ldots, u_{i+k}, u_{i+k+1}, y$$

Consequently, for two vertices x and y where $(x, y) = (v_i, u_{i+k+1})$, (u_i, v_{i+k+1}) and (u_i, u_{i+k+1}) the rainbow path $x - y$ is contained in rainbow path $v_i - v_{i+k+1}$ for $i \in [1, k]$. For the other x and y pairs, the rainbow path is obtained by taking a shortest path connecting them. Thus, $rc(S_n) \le 2k + 1$.

Now, we will show that $2k$ colors are not enough to make S_n be rainbow connected. Suppose that $rc(S_n) = 2k$ for some coloring c'. Let $P_{i\to j}$ be a rainbow path of

$$v_i, u_i, u_{i+1}, \dots, u_{j-1}, u_j, v_j.$$

Note that all edges of cycle of S_n can be partitioned by

$$E_{i\to j} = \{u_i u_{i+1}, u_{i+1} u_{i+2} \dots, u_{j-1} u_j\} \text{ and}$$

$$E_{j\to i} = \{u_j u_{j+1}, u_{j+1} u_{j+2}, \dots, u_{i-1} u_i\}.$$

It is clear $E(C_n) = E_{i\to j} \cup E_{j\to i}$ and $E_{i\to j} \cap E_{j\to i} = \emptyset$. Therefore, for distinct $i, j \in [1,2k]$ where $d(u_i, u_j) \ge 2$, it must satisfy either $c'(u_i v_i), c'(u_j v_j) \in c'(E_{i\to j})$ or $c'(u_i v_i), c'(u_j v_j) \in c'(E_{j\to i})$. If we have $c'(u_i v_i) \in c'(E_{i\to j})$ and $c'(u_j v_j) \in c'(E_{j\to i})$ then there is no $P_{i\to j}$ rainbow path. The graph S_n has n bridges, so based on the fact in [9] it must be $rc(S_n) \ge n$. It means for each $i, j \in [1,2k]$ where $i \ne j$, then $c'(u_i v_i) \ne c'(u_j v_j)$. Without loss of generality (WLOG), color the edge $u_i v_i$ by i for $i \in [1,2k]$ and choose $c'(u_1 v_1), c'(u_{k+1} v_{k+1}) \in c'(E_{1\to k+1})$. Suppose that $c'(u_r u_{r+1}) = 1$ for some $r \in \{2,3, \dots, k-1\}$. It must be that $c'(u_1 u_2), c'(u_2 u_3), \dots, c'(u_r u_{r+1})$ belong to $\{c'(u_{r+2} u_{r+3}), c'(u_{r+3} u_{r+4}), \dots, c'(u_{2r} u_{2r+1})\}$. Those colors should be colored differently. Consequently, $1 \in c'(E_{1\to 2r})$ but $2r \in c'(E_{2r\to 1})$. There is no $P_{1\to 2r}$ rainbow path. In addtition, if $c'(u_1 u_2) = 1$, then there is no $P_{1\to 2}$ rainbow path. This condition forces $c(u_k u_{k+1}) = 1$. Hence, the edges $u_1 u_2, u_2 u_3, \dots, u_{k-1} u_k$ have colors $k+2, k+3, \dots, 2k$ sequentially. But, we obtain $k + 1 \notin c'(E_{1\to k+1})$, a contradiction. Thus, it must be $rc(S_n) \ge 2k + 1$. Combine with the upper bound $rc(S_n) \le 2k + 1$. It can be conclude $rc(S_n) = n + 1$ for n is even. ∎

Theorem 2.6 Let m and n be two integers at least 3 and S_n be a sun graph with order $2n$ where n is odd. For any two distinct vertices u and v in S_n,

$$rc\big(shack(S_n, u, v, m)\big) = mn \text{ for } n \text{ is odd, and}$$

$$rc\big(shack(S_n, u, v, m)\big) \in [mn, m(n+1)] \text{ for n is even.}$$

Proof. It follows by Theorem 2.2 and 2.6 that

$$rc\big(shack(S_n, u, v, m)\big) \le m \cdot\ rc(S_n).$$

For any u and v in S_n, the graph $shack(S_n, u, v, m)$ has $m \cdot n$ bridges. Based on the fact in [9] that is $rc(G) \ge b$ where b is the number of bridges of graph G, we obtain the lower bound

$$rc(shack(S_n, u, v, m)\ ) \ge mn.$$

Hence, $rc\big(shack(G, u, v, m)\big) = mn$ for n is odd and $mn \le rc\big(shack(G, u, v, m)\big) \le m(n+1)$ for n is even. ∎

For the illustration of Theorem 2.6, see Figure 3.

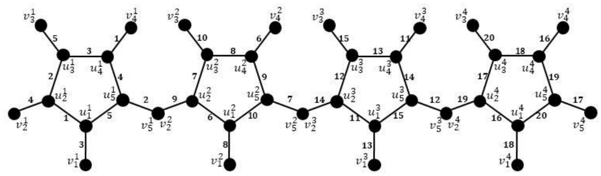

Figure 3 The rainbow 20-coloring of $shack(S_5, u, v, 4)$.

Then, we determine the rainbow connection number of shackle friendship graph. A friendship graph, denoted by F_n for $n \ge 3$, is a graph constructed by n graph C_3 that is connected by one common vertex. For furthermore discussion, we define the vertex and edge sets of F_n as

$$V(F_n) = \{t\} \cup \{x_i, y_i : i \in [1, n]\}$$

and

$$E(F_n) = \{t x_i, t y_i, x_i y_i : i \in [1, n]\},$$

respectively. This graph is isomorphic to amalgamation of K_3, so that the value of $rc(F_n)$ have been studied in [5]. The result is $rc(F_n) = 3$. The Theorem 2.7 explains that the graph $G \cong F_n$ satisfies this condition

$$\max_{x\in V(G)} d(x, v) + (m-2) d(u, v) + \max_{y\in V(G)} d(u, y) < rc\big(shack(G, u, v, m)\big) < m \cdot rc(G).$$

Theorem 2.7 For $n \ge 3$ let F_n be a friendship graph of order $2n + 1$. For any two distinct vertices u and v in F_n with $d(u, v) = 2$, then

$$rc\big(shack(F_n, u, v, m)\big) = 2m + 1.$$

Proof. The distance of u and v is 2. WLOG, let $u = x_1$ and $v = x_n$. The i^{th} graph F_n, denoted by F_n^i, has the vertex and edge sets as

$$V(F_n^i) = \{t^i\} \cup \{x_j^i, y_j^i : j \in [1, n]\}$$

and

$$E(F_n^i) = \{t^i x_j^i, t^i y_j^i : j \in [1, n]\},$$

respectively. It means that $x_n^i = x_1^{i+1}$ for $i \in [1, m-1]$.

We define an edge coloring $c' : E\big(shack(F_n, u, v, m)\big) \to [1,2m+1]$ by formula:

$$c'(t x_j^i) = c'(t y_1^i) = 2i - 1 \text{ for } i \in [1, m], j \in [1, n-1];$$

$$c'(t x_n^i) = c'(t y_n^i) = 2i \text{ for } i \in [1, m];$$

$$c'(x_j^i y_j^i) = 2m \text{ for } i \in [1, m], j \in [1, n] \text{ and;}$$

$$c'(t y_j^i) = 2m + 1 \text{ for } i \in [1, m], j \in [2, n-1].$$

It will be shown that for any two vertices x and y in $shack(F_n, u, v, m)$ there is an $x - y$ rainbow path. If $d(x, y) = 1$, then the rainbow path is the edge xy. Then, the other cases are showed in Table 1. Therefore, c' is a

Table 1. The $x - y$ rainbow paths in $shack(F_n, u, v, m)$

x	y	Condition	Rainbow path
x_j^i	x_k^i	$k \neq j \in [2, n-1], i \in [1, m]$	$x - t - y_k^i - y$
x_j^i	y_k^i	$k \neq j \in [2, n-1], i \in [1, m]$	$x - t - y$
y_j^i	x_k^i	$k \neq j \in [2, n-1], i \in [1, m]$	$x - t - x_k^i - y$
x_j^i	x_1^i, y_1^i	$j \in [2, n-1], i \in [1, m]$	$x - y_j^i - t - y$
y_j^i	x_1^i, y_1^i	$j \in [2, n-1], i \in [1, m]$	$x - t - y$
x_j^i, y_j^i	x_n^i, y_n^i	$j \in [1, n-1], i \in [1, m]$	$x - t - y$
t^i, x_j^i, y_j^i	t^l, x_k^l, y_k^l	$i < l, j, k, \in [1, n], i, l \in [1, m],$	$x - x_n^i - t^{i+1} - x_n^{i+1} - \cdots - x_1^l - y$

rainbow $(2m+1)$ −coloring of $shack(F_n, u, v, m)$. So, $rc\big(shack(F_n, u, v, m)\big) \leq 2m + 1$.

Suppose that $rc(shack(F\_n, u, v, m)) = 2m$ by using some rainbow $2m$ − coloring c''. Consider the vertices $x_1^1, x_2^m, y_2^m, x_n^m$ and y_n^m. We know $d(x_1^1, x_2^m) = d(x_1^1, y_2^m) = d(x_1^1, x_n^m) = d(x_1^1, y_n^m) = 2m$. It must be that those four shortest paths have to through the vertex t^m. In addition, the $x_1^1 - t^m$ shortest path is unique. WLOG, color all edges in $x_1^1 - t^m$ by $1, 2, \ldots, 2m-1$. Consequently, $c''(t^m x_2^m) = c''(t^m y_2^m) = c''(t^m x_n^m) = c''(t^m y_n^m) = 2m$. Therefore, there is no rainbow path connecting x_2^m to x_n^m. Contradiction. So, $rc\big(shack(F_n, u, v, m)\big) \geq 2m + 1$. We can conclude for any two distinct vertices u and v where $d(u, v) = 2$,

$$rc\big(shack(F_n, u, v, m)\big) = 2m + 1. \blacksquare$$

For the illustration of Theorem 2.7, see Figure 4.

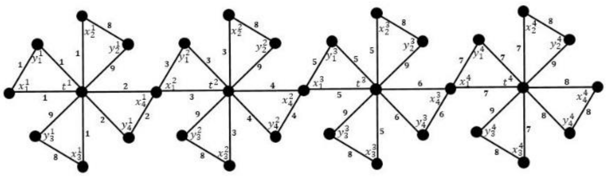

Figure 4 The rainbow 9-coloring of $shack(F_4, u, v, 4)$.

The given upper bound in Theorem 2.2 can be reduced by m. This condition is possible if all edges of each subgraph G^i and G^j in $shack(G, u, v, m)$ do not have to be colored differently. More details, it is explained in the following theorem.

Theorem 2.8 Let m be integer at least 3 and G be a graph with order $n \geq 4$. Let u and v be two distinct vertices in G. If there are two rainbow $rc(G)$ −coloring c_1 and c_2 on G such that for each x not u there exists an $x - u$ rainbow path that is not contained color $rc(G)$ under c_1, and for each y not v there exists a $y - v$ rainbow path that is not contained color $rc(G)$ under c_2, then

$$rc\big(shack(G, u, v, m)\big) \leq m(rc(G) - 1).$$

Proof. We define an edge coloring of $shack(G, u, v, m)$, $c': E\big(shack(G, u, v, m)\big) \to [1, m(rc(G) - 1)]$ by rule:

$$c'(e) = \begin{cases} c_1(e) + (i-1)(rc(G) - 1), \text{for } e \in E(G^i), \\ \quad i \in [1, m-1]; \\ c_2(e) + (m-1)(rc(G) - 1), \text{for } e \in E(G^m) \text{ and} \\ \quad c_2(e) \neq rc(G); \\ rc(G), \text{for } e \in E(G^m) \text{ and } c_2(e) = rc(G). \end{cases}$$

It will shown that for two distinct vertices x and y in $V(shack(G, u, v, m))$ there is an $x - y$ rainbow path. We divide into two cases.

Case 1. x and y in $V(G^i)$.

Note that for $i \in [1, m-1]$ the coloring c' on G^i and c_1 on G have the similar pattern, so do c' on G^m and c_2 on graph G. Therefore, for each x and y in $V(G^i)$ and $i \in [1, m]$ there is an $x - y$ rainbow path.

Case 2. $x \in V(G^i)$ and $y \in V(G^j)$ for distinct $i, j \in [1, m]$.

WLOG, assume that $i < j$. Based on case 1, there is an $x - v^i$ and $v^{j-1} - y$ rainbow path for each $x \in G^i$ and $y \in G^j$. Afterwards, based on c' coloring, we obtain a rainbow path connecting v^i and v^{j-1}, i.e. $v^i, \ldots, v^{i+1}, \ldots, v^{j-1}$. Assume P_1 is an $x - v^i$ rainbow path, P_2 is a $v^i - v^{j-1}$ rainbow path and P_3 is a $v^{j-1} - y$ rainbow path. Note that $c'(e_1) < c'(e_2) < c'(e_3)$ for each $e_i \in P_i, i = 1,2,3$. Thus, $P = P_1 \cup P_2 \cup P_3$ is an $x - y$ rainbow path.

Based on the two cases above, the coloring c' makes $shack(G, u, v, m)$ is rainbow connected with $m(rc(G) - 1)$ colors. Hence, $rc\big(shack(G, u, v, m)\big) \leq m(rc(G) - 1)$. $\blacksquare$

The upper bound on Theorem 2.8 is sharp. This can be shown in the particular case given the following Theorems 2.9 and 2.10.

Theorem 2.9 Let C_n be a cycle with order $n \geq 4$ and m be an integer with $m \geq 3$, then

$$rc(shack(C_n, u, v, m)) = m\left\lceil \frac{n}{2} \right\rceil$$

where $d(u,v) = \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. We divide into two cases.

Case 1. For n is even.

By using the result on [1], i.e. $rc(C_n) = \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor = diam(C_n)$ for n even and based on the Corollary 2.3, we obtain $rc\big(shack(C_n, u, v, m)\big) = \left\lfloor \frac{n}{2} \right\rfloor$.

Case 2. For n is odd.

Assume that $r = diam(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$. From [1], we know that $rc(C_n) = \left\lceil \frac{n}{2} \right\rceil = r + 1$. Define a coloring $c_1: E(C_n) \to [1, r+1]$ with $c_1(v_i v_{i+1}) = c_1(v_{r+i} v_{r+i+1}) = i$ for $i \in [1, r]$ and $c_1(v_n v_1) = r + 1$. It is easy to check that the coloring c_1 is a rainbow $(r+1)$ −coloring, and this is the rainbow coloring using the minimum number of colors.

Similarly, we obtain $rc(C_n) = r + 1$ for a rainbow $(r+1)$ −coloring c_2 on C_n with formula $c_2(v_i v_{i+1}) = r - c_1(v_i v_{i+1})$ for $i \in [1, r-1]$, $c_2(v_i v_{i+1}) = r + 1 - c_1(v_i v_{i+1})$ for $i \in [r+1, n-1]$, $c_2(v_n v_1) = r$ and $c_2(v_r v_{r+1}) = r + 1$. It is clear that for each vertex x in C_n, there is an $x - v_{r+1}$ rainbow path which not contain the color $r + 1$ under c_1 and for each vertex y in C_n there is an $y - v_n$ rainbow path which not contain the color $r + 1$ under c_2.

It follows by Theorem 2.8 that $rc\big(shack(C_n, u, v, m)\big) \le m \cdot r$ where $u = v_n$ and $v = v_{r+1}$ in C_n. We can calculate that

$$\max_{x \in V(C_n)} d(x, v_{r+1}) = \max_{y \in V(C_n)} d(v_n, y) = d(v_n, v_{r+1}) = r$$

so Observation 2.1 implies $rc\big(shack(C_n, u, v, m)\big) \ge r + (m-2)r + r = mr$. We can conclude, $rc(shack(C_n, u, v, m)) = m \left\lfloor \frac{n}{2} \right\rfloor$. ■

For the illustration of Theorem 2.9, see Figure 5.

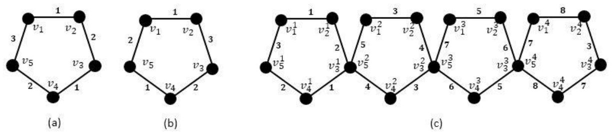

Figure 5 (a) The rainbow coloring c_1, (b) the rainbow coloring c_2 of C_5, and (c) the rainbow 8 −coloring of $shack(C_5, u, v, 4)$.

Theorem 2.10 Let K_n be a complete graph with order $n \ge 4$. Let w and x be two distinct vertices in K_n. If $K_n - e$ is a graph obtained by removing the edge $e = wx$ in K_n, then

$$rc\big(shack(K_n - e, u, v, m)\big) = \begin{cases} 2m, & \text{for } d(u,v) = 2 \\ m, & \text{if } d(u,v) = 1 \text{ and } u, v \notin \{w, x\} \\ m+1, & \text{otherwise.} \end{cases}$$

Proof. Let y and z be two vertices except w and x in $K_n - e$. Define an edge coloring $c_1: E(K_n - e) \to [1,2]$ with rules: $c_1(e') = 1$ for each $e' \ne yw$ in $K_n - e$ and $c_1(e') = 2$ for $e' = yw$. It is clear that w and x is the only pair of two vertices in which the distance is 2. There is an $w - x$ rainbow path, i.e. x, y, w. Meanwhile, the other two vertices pairs have distance one which evidently have a rainbow path. So, $rc(K_n - e) \le 2$. Since $d(w, x) = 2$, then $rc(K_n - e) \ge 2$. We obtain $rc(K_n - e) = 2$. Similarly, we can get $rc(K_n - e) = 2$ for the rainbow 2 −coloring $c_2: E(K_n - e) \to \{1,2\}$ with $c_2(e') = 2$ for the edge $e' = zx$ and $c_2(e') = 1$ for the others. There are three possibilities of three distinct common vertex u and v in $shack(K_n - e, u, v, m)$.

Case 1. For $d(u,v) = 2$.

Since $rc(K_n - e) = diam(K_n - e) = 2$, based on Corollary 2.3 we obtain $shack(K_n - e, u, v, m) = 2m$.

Case 2. For $d(u,v) = 1$ and $u, v \notin \{w, x\}$.

WLOG, choose $u = y$ and $v = z$. We have two colorings c_1 and c_2 satisfying the properties in Theorem 2.8, so we get the upper bound $rc\big(shack(K_n - e, u, v, m)\big) \le m \cdot (2-1) = m$. Now, using Observation 2.1 we obtain the lower bound $rc\big(shack(K_n - e, u, v, m)\big) \ge 1 + (m-2) \cdot 1 + 1 = m$. Hence, $rc(shack(K_n - e, u, v, m)) = m$ for $d(u,v) = 1$ and $u, v \notin \{w, x\}$.

Case 3. For others.

WLOG, choose $u = w$ and $v = z$. Define an edge coloring $c: E\big(shack(K_n - e, u, v, m)\big) \to [1, m+1]$ with formula $c(e) = c_1(e) + i - 1$ for $e \in (K_n - e)^i$, $i \in [1, m]$ where $(K_n - e)^i$ is the i^{th} graph $K_n - e$ of $shack(K_n - e, u, v, m)$. For each $a, b \in shack(K_n - e, u, v, m)$ there is an $a - b$ rainbow path of length $d(a, b)$. So, $rc\big(shack(K_n - u, v, m)\big) \le m + 1$. For the lower bound, using Observation 2.1 to get $rc\big(shack(K_n - e, u, v, m)\big) \ge 1 + (m-2) \cdot 1 + 2 = m + 1$. It can be conclude that in this case, $rc(shack(K_n - e, u, v, m)) = m + 1$. ■

For the illustration of Theorem 2.10, see Figure 6.

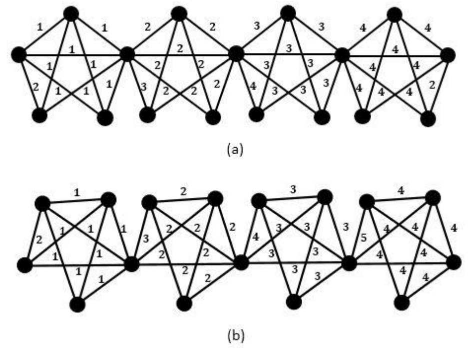

Figure 6 (a) The rainbow coloring in case 2 and (b) the rainbow coloring in case 3 of $shack(K_5 - e, u, v, 4)$.

Next, we show that there is a graph G that satisfies $rc\big(shack(G,u,v,m)\big) < m(rc(G)-1)$. Let a path P_n of $x_1, x_2, \dots, x_n$. A fan graph f_n is a graph which is constructed by a path P_n where each vertex of P_n is adjacent with one center vertex t. The graph f_n has vertex and edge sets as $V(f_n) = \{t\} \cup \{x_j | j \in [1,n]\}$ and edge set $E(f_n) = \{tx_j | j \in [1,n]\} \cup \{x_j x_{j+1} | j \in [1, n-1]\}$. Sometimes, the edge $x_j x_{j+1}$ and $x_j t$ are called the i^{th} rim and the i^{th} spoke of f_n, respectively.

Theorem 2.11 Let $G \cong f_n - \{e_1, e_2\}$ where the edge e_1 and e_2 are the 2^{nd} and the $(n-1)^{th}$ spokes of f_n, respectively. Let m and n be two integers where $m \geq 3$ and $n \geq 7$. If $u = x_2$ and $v = t$, then $rc(G) = 4$ and $rc(shack(G,u,v,m)) = 2m+2$.

Proof. First, we show that $rc(G) = 4$. Define edge coloring $c: E(G) \to [1,4]$ with:

$$c(G) = \begin{cases} 1, \text{for } e \in \{x_{n-2}x_{n-1}, tx_1\} \cup \{tx_i | i \in [1, n-2], i \text{ is even}\}; \\ 2, \text{for } e \in \{tx_i | i \in [3, n-2], i \text{ is odd}\} \cup \{x_2x_3, tx_{n-1}, tx_n\}; \\ 3, \text{for } e \in \{x_j x_{j+1} | j \in [1, n-3], j \text{ is odd}\}; \\ 4, \text{for } e \in \{x_j x_{j+1} | j \in [4, n-3], j \text{ is even}\} \cup \{x_{n-1}x_n\}. \end{cases}$$

We will show that for every two vertices x and y in G, there exists an $x-y$ rainbow path. For two vertices x and y with $d(x,y) = 1$, the rainbow path connecting them is the edge xy. Next, for $x = x_j$ and $y = x_k$ with $|j-k| > 1$, the rainbow path through t is $x - t - y$. So, c is the $4-$ rainbow coloring on G. Thus, we obtain $rc(G) \leq 4$. It is easy to check that $d(x_2, x_{n-1}) = 4$. For any rainbow coloring on G, path $x_2 - x_{n-1}$ should have at least 4 colors. It means $rc(G) \geq 4$. Hence, $rc(G) = 4$.

Now, we will show that $rc(shack(G,u,v,m)) = 2m+1$. Let G^i is the i^{th} graph G of $shack(G,u,v,m)$ with the vertex and edge sets as $V(G^i) = \{t^i, x_j^i | i \in [1,m], j \in [1,n]\}$ and $E(G^i) = \{t^i, x_j^i | i \in [1,m], j \in [1,n], j \neq 2, n-1\} \cup \{x_j^i x_{j+1}^i | i \in [1,m], j \in [1, n-1]\}$. Since $u = x_2$ and $v = t$, for each $i \in [1, m-1]$ it means the vertex t^i of G^i is attached to x_2^{i+1} of G^i. Define an edge coloring $c': E\big(shack(G,u,v,m)\big) \to [1, 2m+2]$ with $c'(e) = c(e) + 2i - 2$ for each $e \in E(G^i)$ and $i \in [1,m]$. For two distinct vertices x and y in subgraph G^i for some $i \in [1,n]$ they have the same coloring pattern, so there is an $x-y$ rainbow coloring. Furthermore, consider $x \in V(G^i)$ and $y \in V(G^j)$ for distinct $i,j \in [1,m]$. Based on previous statement, it is clear that there is an $x - t^i$ rainbow path for each $x \in G^i$ and $t^{j-1} - y$ for each $y \in G^j$. Afterwards, using the coloring c' we obtain a rainbow path connecting t^i and t^{j-1}, i.e. $t^i, \dots, t^{i+1}, \dots, t^{j-1}$. Assume that P_1 is an $x - t^i$ rainbow path, P_2 is a $t^i - t^{j-1}$ rainbow path and P_3 is a $t^{j-1} - y$ rainbow path. Note that $c'(e_1) < c'(e_2) < c'(e_3)$ for each $e_i \in P_i, i = 1,2,3$. Thus, $P = P_1 \cup P_2 \cup P_3$ is an $x-y$ rainbow path.

According to the two cases above, the coloring c' makes $shack(G,u,v,m)$ is rainbow connected using $2m+2$ colors. So, $rc\big(shack(G,u,v,m)\big) \leq 2m+2$. For the lower bound, use Observation 2.1 to obtain $rc\big(shack(G,u,v,m)\big) \geq \max_{x \in V(G)} d(x,u) + (m-2)d(u,v) + \max_{y \in V(G)} d(u,y) = 2 + (m-2)\cdot 2 + 4 = 2m+2$. We can conclude that $rc(shack(G,u,v,m)) = 2m+2$. ■

For the illustration of Theorem 2.11, see Figure 7.

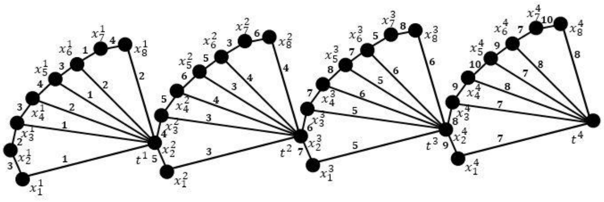

Figure 7 The rainbow $10-$coloring of $shack(f_8 - \{e_1, e_2\}, u, v, 4)$.

3. CONCLUSION

In this paper, we obtain the lower and upper bounds of rainbow connection number of shackle graphs. Also, we show that these bounds are sharp. Some shackle graphs we investigate have the exact value of rainbow connection number, i.e. shackle sun S_n for odd n, friendship F_n, cycle C_n, $K_n - e$ and $f_n - \{e_1, e_2\}$. In addition, we give an improvement the value of $rc(S_n)$ for even n in [7]. We show that $rc(S_n) = n+1$ for even n. However, the exact value of $rc\big(shack(S_n,u,v,m)\big)$ for even n is still open.

AUTHORS' CONTRIBUTIONS

The results in this paper contribute to the determination of the rainbow-connected number of graphs, especially in shackle graphs.

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