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Local Antimagic Vertex Coloring of Gear Graph

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ABSTRACT
Let $G = (V, E)$ be a graph that consist of a vertex set V and an edge set E. The local antimagic labeling f of a graph G with edge-set E is a bijection map from E to $\{1, 2, \ldots, |E|\}$ such that $w(u) \neq w(v)$, where $w(u) = \sum_{e \in E(u)} f(e)$ and $E(u)$ is the set of edges incident to u. In this labeling, every vertex v is assigned $w(v)$ as its color. The minimum number of colors in a local antimagic labelling, is called a local antimagic chromatic number and denoted by $\chi_{la}(G)$. This paper contribution is to determine the local antimagic chromatic number $\chi_{la}(G_n)$ of a gear graph. A gear graph is a graph obtained by inserting additional vertex between each pair of adjacent vertices on the circumference of the wheel graph W_n. The gear graph G_n has $2n + 1$ vertices and $3n$ edges.

Keywords: *Antimagic labeling, Local antimagic labeling, Local antimagic chromatic number, Gear graph.*

1. INTRODUCTION

Graphs are used in various fields, such as network topology modeling, database design, scheduling, traveling salesman problems, and so on [1]. Graph labeling is one of the interesting research topics in graph theory that assign an element of graph such as vertices and edges with the set of integers called labels [2].

Formally, the labeling of antimagic in a graph is defined as follows. Suppose $G = (V, E)$ be a simple undirected graph with a non-empty set of vertices V and a set of edges E. The number of vertices, denoted by $|V(G)|$, referred to the order of G and the number of edges, denoted by $|E(G)|$, called size of G [3]. If vertex v is an endpoint of edges e, then v is said to be incident on e and e is incident on v. Two vertices are adjacent if they are the endvertices of the same edge [4]. The local antimagic labeling f of a graph G with edge-set E is a bijection map from E to $\{1, 2, \ldots, |E|\}$ such that $w(u) \neq w(v)$, where $w(u) = \sum_{e \in E(u)} f(e)$ and $E(u)$ is the set of edges incident to u, for $u, v \in V(G)$. A graph G is called local antimagic if G has a local antimagic labeling [5]. The local antimagic chromatic number of G, denoted by $\chi_{la}(G)$, is the minimum number of colors that is needed that is induced by local antimagic labelings of G [6].

Arumugam, et al., [7] presented local antimagic chromatic number of several classes of graphs such as complete graph K_n for $n \geq 3, \chi_{la}(K_n) = n$; star graph $K_{1,n}$ for $n \geq 3, \chi_{la}(K_{1,n-1}) = n$; path graph P_n for $n \geq 3, \chi_{la}(P_n) = 3$; cycle graph C_n for $n \geq 3, \chi_{la}(C_n) = 3$; friendship graph F_n for $n \geq 2, \chi_{la}(F_n) = 3$; complete bipartite graph $K_{m,n}$ for $m, n \geq 2$, $\chi_{la}(K_{m,n}) = 2$ if only if $m \equiv n \ (mod\ 2)$ and wheel graph W_n with $n + 1$ order for $n \geq 3$, $\chi_{la}(W_n) = 4$, if $n \equiv 1,3 \ (mod\ 4)$, $\chi_{la}(W_n) = 3$, if $n \equiv 2 \ (mod\ 4)$, and $3 \leq \chi_{la}(W_n) \leq n$ if $n \equiv 0 \ (mod\ 4)$. Furthermore, Pratama, et al., [8] presented the local super antimagic total chromatic number of several classes of graph which is related with wheel graph, such as fan graph F_n for $n \geq 3$, $3 \leq \chi_{lsat}(F_n) \leq 4$; even gear graph G_n for $n \geq 4$, $2 \leq \chi_{lsat}(G_n) \leq 3$; odd sun flower graph SF_n for $n \geq 5$, $4 \leq \chi_{lsat}(SF_n) \leq 5$.

We study and determine the local antimagic chromatic numbers of a gear graph in this paper.

2. MAIN RESULT

We begin this section by presenting the gear graph as follow. Gear graph, denoted by G_n (Figure 1) is a graph obtained from wheel graph W_n by adding a vertex between each pair of adjacent rim vertices [9]. Clearly G_n has a set of vertices $V = \{c, v_i, u_i\ ; 1 \leq i \leq n\}$ and set of edges $E = \{cv_i, v_iu_i;\ 1 \le i \le n\} \cup \{u_iv_{i+1}\ ;\ 1 \le i \le n-1\} \cup \{u_nv_1\}$ such that $|V| = 2n+1$ and $|E| = 3n$.

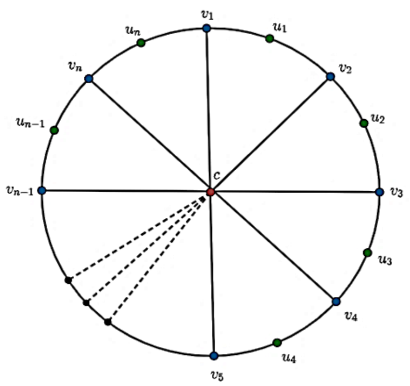

Figure 1 Gear Graph G_n.

2.1. Local Antimagic Chromatic Number of G_n for n odd

Lemma 2.1.1. gives the upper bound of $\chi_{la}(G_n)$ for $n \equiv 1\ (mod\ 2)$.

Lemma 2.1.1. $\mathcal{X}_{la}(G_n) \le 4$ for $n \equiv 1\ (mod\ 2)$ and $n \ge 3$.

Proof:

Case 1: for $n = 3$.

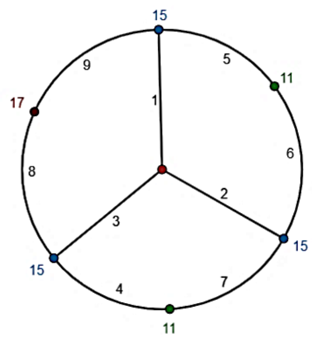

Figure 2 $\chi_{la}(G_3) = 4$.

Case 2: for $n \equiv 1\ (mod\ 2)$ and $n \ge 5$.

Label the edges cv_i, for $i = 1,2,\dots,n$, as follows:

$$f(cv_i) = \begin{cases} 1, & \text{for } i = 1, \\ n-i+2\,, & \text{for } i \text{ odd, } i \ne 1, \\ i+1, & \text{for } i \text{ even.} \end{cases}$$

Label the edges v_iu_i, for $i = 1,2,\dots,n$, as follows:

$$f(v_iu_i) = \begin{cases} \dfrac{3n+i}{2}\,, & i \text{ odd}, \\ \dfrac{6n-i}{2}, & i \text{ even}. \end{cases}$$

Label the edges u_iv_{i+1}, for $i = 1,2,\dots,n-1$, as follows:

$$f(u_iv_{i+1}) = \begin{cases} \dfrac{3n-i}{2}\,, & i \text{ odd}, \\ \dfrac{4n+i}{2}, & i \text{ even}. \end{cases}$$

Label the edges u_nv_1, as follows:

$$f(u_nv_1) = 3n.$$

In Figure 3, we have the labeling of the edges G_n for n odd.

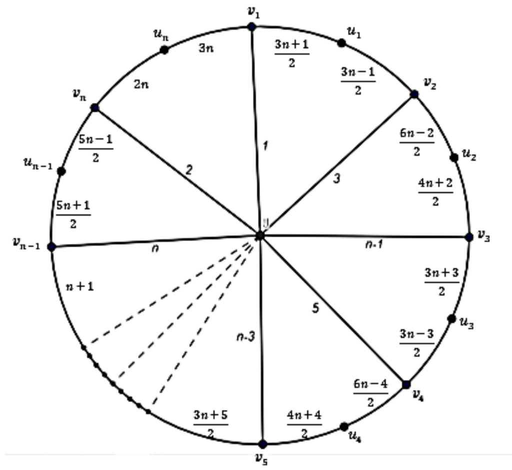

Figure 3 Labeling Edges G_n for n odd.

This labelling provides different weights for any two adjacent vertices, namely:

$$w(c) = \frac{1}{2}n(n+1),$$

$$w(v_i) = \frac{1}{2}(9n+3),$$

$$w(u_i) = \begin{cases} 3n, & \text{for } 1 \le i < n, i \text{ odd}, \\ 5n, & \text{for } 1 < i < n, i \text{ even}, \\ & i = n. \end{cases}$$

Thus, the labeling gives 4 different weights, that is, $\mathcal{X}_{la}(G_n) \le 4$, for $n \equiv 1\ (mod\ 2)$ and $n \ge 3$. ■

In Figure 4, we have example of labeling G_7. We now prove the lower bound of the local antimagic chromatic number of G_n for n is odd.

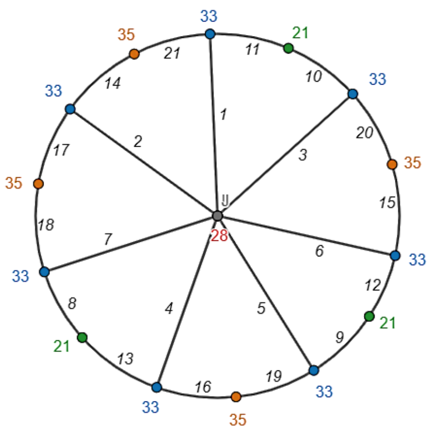

Figure 4 Labeling of Graph G_7.

Lemma 2.1.2. $\mathcal{X}_{la}(G_n) \geq 3$ for $n \equiv 1\ (mod\ 2)$ and $n \geq 3$.

Proof: Suppose that $\chi_{la}(G_n) = 2$ with $w(v_i) \neq w(u_i)$. Then there is a labeling cv_i for $1 \leq i \leq n$ such that $w(c) = w(u_i)$. The largest $w(v_i)$ cannot be more than the sum of the edge labels labeled n and two largest edge labels, that is, $w(v_i) = n + \frac{5n+1}{2} + (n+1) = \frac{9n+3}{2}$. While the weight of the central vertex must be at least the sum of the smallest n edge labels, that is, $w(c) = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$. It is clear that no $n \in \mathbb{N}$ satisfies the equation $9n + 3 = n(n+1)$, this is a contradiction. So, it must be $\mathcal{X}_{la}(G_n) \geq 3$ for $n \equiv 1 (mod\ 2)$. ■

Theorem 2.1. The local antimagic chromatic number of G_n for n odd and $n \geq 3$ satisfies $3 \leq \mathcal{X}_{la}(G_n) \leq 4$.

Proof: Based on Lemma 2.1.1. and Lemma 2.1.2., it is proven that $3 \leq \mathcal{X}_{la}(G_n) \leq 4$. ■

2.2. Local Antimagic Chromatic Number of G_n for n even

Lemma 2.2.1. gives the upper bound of $\chi_{la}(G_n)$ for $n \equiv 0\ (mod\ 2)$.

Lemma 2.2.1. $\mathcal{X}_{la}(G_n) \leq 4$ for $n \equiv 0\ (mod\ 2)$ and $n \geq 4$.

Proof: For $n \equiv 0\ (mod\ 2)$ and $n \geq 4$.

Label the edges cv_i, for $i = 1,2, \dots, n$, as follows:

$$f(cv_i) = \begin{cases} i, & 1 \leq i \leq \frac{n}{2}, i \text{ odd}, \\ n - i + 1, & 1 \leq i \leq \frac{n}{2}, i \text{ even}, \\ n, & i = \frac{n}{2} + 1, \\ i - 1, & \frac{n}{2} + 1 < i \leq n, i \text{ odd}, \\ n - i + 2, & \frac{n}{2} + 1 < i \leq n, i \text{ even}. \end{cases}$$

Label the edges $v_i u_i$, for $i = 1,2, \dots, n$, as follows:

$$f(v_i u_i) = \begin{cases} \frac{4n - i + 1}{2} & 1 \leq i \leq \frac{n}{2}, i \text{ odd}, \\ \frac{3n + i}{2} & 1 \leq i \leq \frac{n}{2}, i \text{ even}, \\ \frac{3n - i + 1}{2} & \frac{n}{2} < i \leq n, i \text{ odd}, \\ \frac{2n + i}{2} & \frac{n}{2} < i \leq n, i \text{ even}. \end{cases}$$

Label the edges $u_i v_{i+1}$, for $i = 1,2, \dots, n$-1, as follows:

$$f(u_i v_{i+1}) = \begin{cases} \frac{4n + i + 1}{2} & 1 \leq i \leq \frac{n}{2}, i \text{ odd}. \\ \frac{5n - i + 2}{2} & 1 \leq i \leq \frac{n}{2}, i \text{ even}, \\ \frac{5n + i + 1}{2} & \frac{n}{2} < i \leq n, i \text{ odd}, \\ \frac{6n - i + 2}{2} & \frac{n}{2} < i \leq n, i \text{ even}. \end{cases}$$

Label the edges $u_n v_1$, as follows:

$$f(u_n v_1) = \frac{5n + 2}{2}.$$

In Figure 5, we have labeling the edges G_n for n even.

This labelling formula gives us four different values of vertex weight as follows.

$$w(c) = \frac{1}{2}n(n+1),$$

$$w(v_i) = \begin{cases} \frac{9n+4}{2}, & 1 \le i \le \frac{n}{2}, i \text{ odd}, \\ & \frac{n}{2} \le i \le n, i \text{ even}, \\ \frac{9n+2}{2}, & 1 \le i \le \frac{n}{2}, i \text{ odd}, \\ & \frac{n}{2} \le i \le n, i \text{ even}, \end{cases}$$

$$w(u_i) = 4n + 1.$$

Thus, $\mathcal{X}_{la}(G_n) \le 4$, for $n \equiv 0 \ (mod\ 2)$ and $n \ge 4$. ■

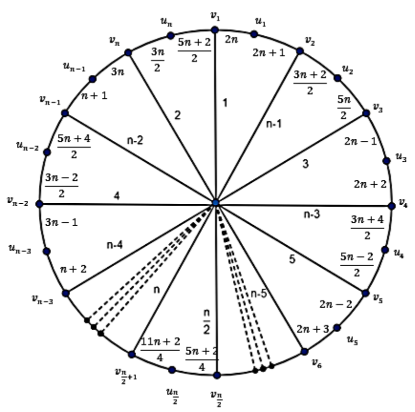

Figure 5 Labeling Edges G_n for n even.

In Figure 6, we have example of labeling G_8.

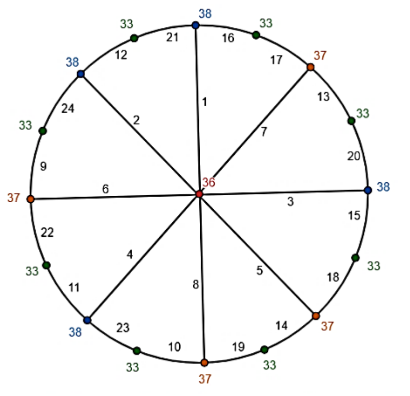

Figure 6. Labeling of G_8.

In Lemma 2.2.2, we show the lower bound of the local antimagic chromatic number of G_n for n is even.

Lemma 2.2.2. $\mathcal{X}_{la}(G_n) \ge 4$ for $n \equiv 0 \ (mod\ 2)$ and $n \ge 4$.

Proof: From Lemma 2.1.2. it has been proved that $\chi_{la}(G_n) \ge 3$ for n is odd. Suppose for n is even $\chi_{la}(G_n) = 3$. Similar to the proof of Lemma 2.1.2, the weights are $w(c) \ne w(v_i)$. Since vertices u_i and v_i are neighbors, they must be $w(u_i) \ne w(v_i)$. Suppose the weight of each vertex v_i is the same. Because every vertex v_i is adjacent to edge cv_i, v_iu_i, and $u_{i-1}v_i$ (or $u_nv_1, if\ i = 1$), a labeling of all edges G_n will be constructed so that each vertex v_i has the same weight. It is known that the sum of all edge labels G_n is $\sum_{i=1}^{3n} i = \frac{3n}{2}(3n+1)$, in this case there is no integer for even n that can evenly divide the sum of all edge labels G_n. This results in at least two vertices v_i has different weights. This is a contradiction, so it must be $\chi_{la}(G_n) \ge 4$ for $n \equiv 0 \ (mod\ 2)$. ■

Theorem 2.2. The local antimagic chromatic number of G_n for n even and $n \ge 4$ is $\mathcal{X}_{la}(G_n) = 4$.

Proof: Based on Lemma 2.2.1. and Lemma 2.2.2., it is proven that $\mathcal{X}_{la}(G_n) = 4$. ■

3. CONCLUSION

We proved the the local antimagic chromatic number of gear graph $\chi_{la}(G_n)$. We propose an open problem: What is the local antimagic chromatic number for other graph that is also related to wheel graph?

AUTHORS' CONTRIBUTIONS

Conceptualization, M.N.S., K.A.S.; proof methodology, M.N.S., K.A.S.; writing-original draft preparation, M.N.S, K.A.S.; writing-review and editing, M.N.S., K.A.S.; funding acquisition, K.A.S. All authors have read and agreed to the published version of the manuscript.

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