Distinguishing Number of the Generalized Theta Graph

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ABSTRACT

A generalized theta graph is a graph constructed from two distinct vertices by joining them with l (>=3) internally disjoint paths of lengths greater than one. The distinguishing number $D(G)$ of a graph G is the least integer d such that G has a vertex labelling with d labels that is preserved only by a trivial automorphism. The partition dimension of a graph G is the least k such that $V(G)$ can be k-partitioned such that the representations of all vertices are distinct with respect to that partition. In this paper, we establish a relation between the distinguishing number and the partition dimension of a graph. We also determine the distinguishing number for the generalized theta graph.

Keywords: *Distinguishing number, Partition dimension, Generalized theta graph.*

1. INTRODUCTION

The distinguishing number of a graph was introduced by Albertson & Collins [1] in 1996. In a graph $G(V,E)$, a labelling $\varphi : V(G) \to \{1, 2, \ldots, r\}$ is said to be *r-distinguishing labelling* if all of the vertex labels are preserved only by the identity automorphism. The *distinguishing number* of a graph G, denoted *by* $D(G)$, is defined as the least integer k such that G has k-distinguishing labelling.

The *partition dimension* of a graph was introduced by Chartrand *et al.* [2, 3] to tackle the problem of finding the metric dimension for a graph. For any vertex v and w in a connected graph $G(V,E)$ and a subset S of V, the *distance* $d(v,w)$ between two vertices vertex v and w is the length of a shortest path connecting both vertices, and the distance $d(v,S)$ between v and S is defined as $d(v,S) = \min\{d(v,x)\,|x \in S\}$. Let $\pi = (S_1, S_2, \ldots, S_k)$ be an ordered k-partition of $V(G)$. The *representation* $r(v\,|\,\pi)$, of a vertex v with respect to π is defined as $r(v\,|\pi) = \big(d(v,S_1), d(v,S_2), \ldots, d(v,S_k)\big)$. The k-partition π is a *resolving partition* of G if the representations of all vertices are distinct. The *partition dimension* of the graph G is the minimum k for which there is a resolving k-partition of G and denoted as $pd(G)$.

The distinguishing number of some well-known graphs are as follows. For a cycle C_n on n vertices, we have that $D(C_n) = 2$, for any $n \geq 6$, since there is a minimum distinguishing labelling c on a cycle C_n such that $c(v_1) = 1$, $c(v_2) = 1$, $c(v_3) = 2$, $c(v_4) = 1$, and $c(v_k) = 2$, for $5 \leq k \leq n$. But, if $n = 3,4,5$, then $D(C_n) = 3$ [1]. For a path P_n and a complete graph K_n, $D(P_n) = 2$ and $D(K_n) = n$, for any $n \geq 2$ [1]. For the generalized Petersen graph $P(n,k)$, $D(P(n,k)) = 2$, for any $(n,k) \neq (4,1), (5,2)$. Otherwise, $D\big(P(n,k)\big) = 3$ [4]. If G is a hypercube Q_n, then $D(Q_n) = 2$, for any $n \geq 4$ [5]. For the results of the partition dimension of some standard graphs, see for instance [3, 6 – 10]. The distinguishing number and partition dimension of some well-known graphs are summarized in **Table *1***.

In [11, 12],; for any integers $s_i \geq 1$ for each i, the *generalized theta graph* $\Theta(s_1, s_2, \ldots, s_l)$ is a graph consists of a pair of end vertices, say x and y, joined by l *internally disjoint paths* of lengths $s_i + 1$ for each $i \in [1, l]$. If such internally disjoint paths connecting the vertices x and y consist of n_i disjoint paths of length $s_i + 1$ (including x and y), for $i = 1,2, \ldots, k$, then the generalized theta graph is denoted by $\Theta({s_1}^{n_1}, {s_2}^{n_2}, \ldots, {s_k}^{n_k})$. If $s_i = s$, for all i, the generalized theta graph is called *uniform*, and it is denoted by $\Theta(s^l)$.

Table 1. Distinguishing number and partition dimension of some well-known graphs

Graph	Distinguishing Number	Partition Dimension
Path P_n	$D(P_n) = 2, n \geq 2$ [1]	$pd(P_n) = 2, n \geq 2$ [2]
Cycle C_n	$D(C_n) = 2, n \geq 6$ [1]	$pd(C_n) = \begin{cases} 4, n = even \\ 3, n = odd \end{cases}$ [13]
Wheel W_n	$D(W_n) = \begin{cases} 2 \ \ , n \geq 7 \\ 3, n = 5,6 \end{cases}$ [14]	$\lceil \sqrt[3]{2n} \rceil \leq pd(W_n) \leq 2\lceil \sqrt{n} \rceil + 1,$ $for\ all\ n \geq 4$ [6]
Generalized Petersen $GP(n,k)$	$D(GP(n,k)) = \begin{cases} 3, (n,k) = (4,1) or (5,2) \\ 2, \quad otherwise \end{cases}$ [4]	$pd(GP(n,k)) \leq 4, n = 2k+1$ [10]
Hypercube Q_n	$D(Q_n) = \begin{cases} 2 \ \ , n \geq 4 \\ 3, n = 3 \end{cases}$ [5]	Not known yet
Star $K_{1,n-1}$	$D(K_{1,n-1}) = n-1, n \geq 3$ [1]	$pd(K_{1,n-1}) = n-1, n \geq 3$ [3]
Complete Graph K_n	$D(K_n) = n$ [1]	$pd(K_n) = n$ [3]
Complete Bipartite $K_{n,n}$	$D(K_{n,n}) = n+1$ [15]	$pd(K_{n,n}) = n+1$ [16]

For example, in Figure 1(a) we have the generalized theta graph $\Theta(1^3, 3^4, 4^2)$ and Figure 1(b) gives $\Theta(2^4)$.

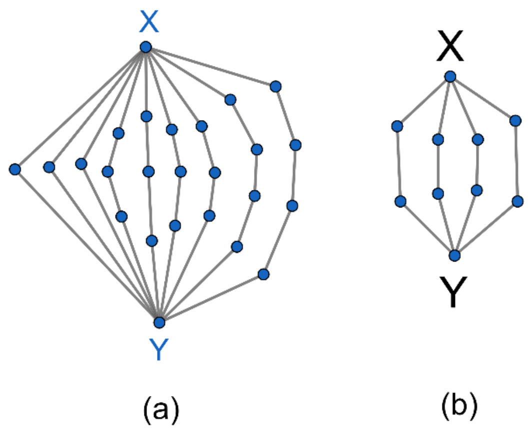

Figure 1 *(a)* $\Theta(1^3, 3^4, 4^2)$ *and (b)* $\Theta(2^4)$.

Lemma 1.1 Let $G = \Theta(s_1^{n_1}, s_2^{n_2}, \ldots, s_k^{n_k})$, $Odd(G)$ denoted the number of longitudes with odd number of internal vertices in each, and $Even(G)$ denoted the number of longitudes with odd number of internal vertices in each. Then $pd(G) = m$ if and only iff

a) $m^2 - 4m + 6 \leq Odd(G) \leq m^2 - 2m + 2,$

b) $m^2 - 5m + 9 \leq Odd(G) \leq m^2 - 3m + 4.$

In this paper, we are going to derive a relation between the distinguishing number and the partition dimension of any graph. Then, we explore the distinguishing number for a particular class of graphs called the generalized theta graph.

2. RESULTS

First, we are going to establish a general relation between the distinguishing number and the partition dimension of any graph. We will show that $D(G) \leq pd(G)$ for any graph G. Then, we will explore the graph G attaining the equality sign and finding the gap between these two parameters for the generalized theta graph.

Theorem 2.1 For any graph G, $D(G) \leq pd(G)$.

Proof. Let $G(V,E)$ be a graph. Let π be a resolving partition of G with $\pi = (S_1, S_2, \ldots, S_k), k \in \mathbb{N}$. Let t be a k-labelling of G induces such partition. We will show that t is a distinguishing labelling of G. Let x, y be any two distinct vertices in S_i for some $i \in [1,k]$. Since π is a resolving partition then $r(x|\pi) \neq r(y|\pi)$. This means that there exists S_j such that $d(x, S_j) \neq d(y, S_j)$. This implies that there is no nontrivial automorphism mapping x and y, preserving vertex-labellings. Therefore, t is a distinguishing labelling of G. Thus, $D(G) \leq pd(G)$. ❑

It is known that if G is a path P_n, a star $K_{1,n-1}$, a complete graph K_n or a complete bipartite graph $K_{n,n}$, then $D(G) = pd(G)$. However, it is very hard in general to characterize all graphs G satisfying $D(G) = pd(G)$. How big is the gap between $D(G)$ and $pd(G)$ for any graph G? this question is also very interesting.

In Theorem 2.2, we will present the upper bound of $pd(G)$ if G is a generalized theta graph. This bound was derived by Mohan. In Theorem 2.3, we will determine the distinguishing number of the generalized theta graph. The generalized theta graph is an example of a graph that has a considerable large gap between its distinguishing number and its partition dimension.

Theorem 2.2 [12] Let G be the generalized theta graph $\Theta(s_1{}^{n_1}, s_2{}^{n_2}, \dots, s_k{}^{n_k})$. Then, $pd(G) \leq \sum_{i=1}^{k} n_i$.

Theorem 2.3 Let G be the generalized theta graph $\Theta(s_1{}^{n_1}, s_2{}^{n_2}, \dots, s_k{}^{n_k})$. Let r be the least positive integer such that $n_i \leq r^{s_i}$, for all i. Then, $D(G) = max\{r, 2\}$.

Proof. Let $G \cong \Theta(s_1{}^{n_1}, s_2{}^{n_2}, \dots, s_k{}^{n_k})$, and x, y be the north and south poles of G, respectively. For any $i \in [1, k]$, we define $\mathcal{P} = \{P_{s_i}^1, P_{s_i}^2, \dots, P_{s_i}^{n_i}\}$ as the n_i paths on s_i vertices connecting x and y in the generalized theta graph G. For instance, if $s_2 = 4$ and $n_2 = 5$, then $\mathcal{P} = \{P_4^1, P_4^2, \dots, P_4^5\}$ is the 5 internal paths on 4 vertices connecting x and y. Let r be the least positive integer such that $n_i \leq r^{s_i}$, for all i. Let $q = max\{r, 2\}$.

For any $i \in [1, n_i]$, we define

$A_i = \{(x_1, x_2, \dots, x_{s_i}) | x_j \in [1, q], j \in [1, s_i]\}$. For instance, if $q = 3$ and $s_2 = 4$, then $A_2 = \{(x_1, x_2, x_3, x_4) \mid x_j \in [1,3], j \in [1,4]\}$.

If $r = 1$ define a q-labelling f on G such that $f(x) = 1, f(y) = 2$, and $f(v) = 1$, for any other vertices v in G. If $r \neq 1$, define a q-labelling f on G such that $f(x) = 1, f(y) = 2$, and for $i \in [1, k]$, and $t \in [1, n_i]$,

$f\left(V\left(P_{s_i}^t\right)\right) = a_i$, where $a_i \in A_i$, and

$f\left(V\left(P_{s_i}^t\right)\right) \neq f\left(V\left(P_{s_i}^l\right)\right)$ for $t \neq l$.

Note that $f(\{v_1, v_2, \dots, v_n\}) = (w_1, w_2, \dots, w_n)$ if $f(v_i) = w_i, \forall i$.

Since r is the least integer satisfies that $n_i \leq r^{s_i}$, for all $i \in [1, k]$, then f is a distinguishing labelling of G with minimum number of labelling. Therefore, $D(G) = max\{r, 2\}$. ❑

Corollary 2.4 Let $G = \Theta(s_1, s_2, \dots, s_l)$, $l > 2$, and $s_1 < s_2 < \dots < s_l$. Then, $D(G) = 2$.

Proof. *Let* $G = \Theta(s_1, s_2, \dots, s_l), l > 2$, and $s_1 < s_2 < \dots < s_l$. Since $r = 1$ and by **Theorem 2.3** we have that $D(G) = 2$. ❑

For example, the distinguishing labellings of the generalized theta graphs $\Theta(2^5, 3^4, 4^2)$ and $\Theta(2,4,7)$ are given in **Figure 2**. By Theorem 2.2 and Lemma 1.1, , the partition dimension of $\Theta(2^5, 3^4, 4^2)$ is between 4 and 11. By Theorem 2.3, its distinguishing number is 3.

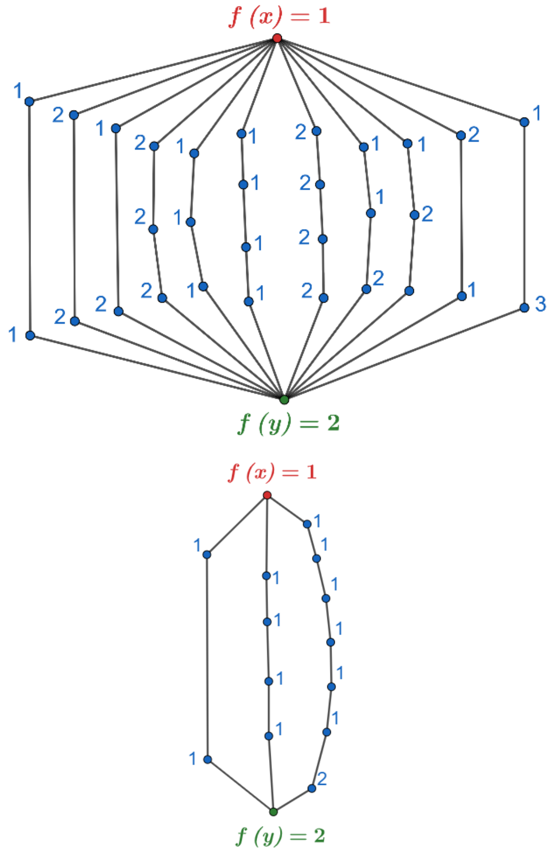

Figure 2. *The distinguishing labellings of* $\Theta(2^5, 3^4, 4^2)$ *and* $\Theta(2,4,7)$

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