ATLANTIS PRESS
Advances in Computer Science Research, volume 96
Proceedings of the International Conference on Mathematics, Geometry, Statistics, and Computation (IC-MaGeStiC 2021)

Further Result of H-Supermagic Labeling for Comb Product of Graphs

Ganesha Lapenangga P.*, Aryanto, Meksianis Z. Ndii

University of Nusa Cendana
**Corresponding author. Email: ganesha.lapenangga@staf.undana.ac.id*

ABSTRACT
Let $G = (V, E)$ and $H = (V', E')$ be a connected graph. H-magic labeling of graph G is a bijective function $f: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that for every subgraph H' of G isomorphic to H, $\sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e) = k$. Moreover, it is H-supermagic labeling if $f(V) = \{1, 2, \ldots, |V|\}$. A graph G having such labeling called H-supermagic graph. Next, we introduce comb product of graph. Suppose G and H are two connected graph and o is vertex in H. A comb product between G and H, denoted by $G \rhd_o H$, is a graph obtained by taking a copy of graph G and $|V(G)|$ copies of graph H, then identifying the i-th copy of graph H at vertex o to i-th vertex of graph G. In this paper, we construct $H_1 \rhd_o H_2$-supermagic labeling of graph $G \rhd_o H_2$ where G is H_1-supermagic graph.

Keywords: *Comb product, H-supermagic labeling, H-magic.*

1. INTRODUCTION

A graph G, denoted by $G = (V, E)$, is an ordered set consist of a non-vacuous set V and a set $E \subseteq [V]^2$. We use $V(G)$ and $E(G)$ respectively as a notation of vertex and edge set of graph G [1]. One of a branch theory in graph is graph labeling. There are so many types of labeling of graph, we can see it all in [2], but our focus in this paper related to H-magic labeling. This theory was introduced by Gutiérrez and Lladó [3]. Let us talk about formal definition of H-(super)magic graph and a requirement of it. Let G and H be a connected graph. A graph G admits H-magic if for every edge $e \in E(G)$, there exist a subgraph H' in G isomorphic to H such that $e \in E(H')$ and there is a bijective function $f: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ which for every subgraph isomorphic to H, the sum of every vertex and edge label is equal to a number, named k. This number is called the magic number. Futhermore, it is said to be H-supermagic if $f(V) = \{1, 2, \ldots, |V(G)|\}$ [2]. In [3], Gutiérrez and Lladó constructed a star $K_{1,h}$-magic labeling of a bipartite graph $K_{m,n}$. They also give a path P_h-magic labeling of graph P_n and cycle C_n. Llado and Moragas also proved that graph W_n for odd $n \geq 5$ is C_3-supermagic [4]. Ngurah and others give a view results related to C_n-supermagic graphs and P_h-supermagic graph. They also provided the C_3 and C_4-supermagic graphs such as book B_n, ladder $P_n \times P_2$, fan $P_n + K_1$, triangle ladder TL_n, and grid $P_m \times P_n$ for some positive integers m and n [5]. Roswitha and others also continued to work on cycle supermagic labeling. They provided cycle supermagic labeling of generalized jahangir graph $(J_{k,s})$, wheel graph for even n, and also complete bipartite graph $K_{2,n}$[6]. Chithra and others introduced the C_m-E-supermagic labeling of generalized book $B_{n,m}$, Grid, $G_{n,m}$, and friendship graph GF_n^m[7]. Sadariria and Susanti also proved H-supermagic labeling of some graphs with cycle, such as corona product between star and cycle, also book and cycle [8]. We try to observe some problems in H-supermagic labeling. By doing that, we find a problem related to comb product of graph that was introduced by Saputro and others. Suppose G and H be a graph and o is the vertex in H. A comb product between G and H, denoted by $G \rhd_o H$, is a graf obtained by taking a copy of graph G and $|V(G)|$ copies of graph H then identifying the i-th copy of graph H at vertex o to the i-th vertex of graph G [9]. Putra constructed $P_2 \rhd_o G$-supermagic labeling for comb product of path (P_n) and cycle (C_n) with any graph G [10]. Following of the result, we construct the $H_1 \rhd_o H_2$-supermagic labeling of graph $G \rhd_o H_2$ provided that G is H_1-supermagic graph.

2. MAIN RESULT

Let us give the vertex and edge set of graph $G \rhd_o H_2$. Suppose that,

$$|V(G)| = a, |E(G)| = b$$
$$|V(H_1)| = y, |E(H_1)| = z \quad (1)$$
$$|V(H_2)| = c, |E(H_2)| = d$$

then, the vertex and edge set is

$V(G \rhd_o H_2) = \{v_i \mid 1 \le i \le a\} \cup \{v_{i,j} \mid 1 \le i \le a; 1 \le j \le c-1\}$

$E(G \rhd_o H_2) = \{e_l \mid 1 \le l \le b\} \cup \{e_{i,h} \mid 1 \le i \le a; 1 \le h \le d\}$.

For Z is a set of integer, we write $\sum Z$ to describe sum of every integer in Z and $[a,b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$. Let us now give the theorem about $H_1 \rhd_o H_2$ magic labeling of graph $G \rhd_o H_2$.

Theorem 2.1. *Suppose that G is H_1-supermagic graph and H_2 is connected graph such that the number of subgraph H_1 in G is equal to the number of subgraph $H_1 \rhd_o H_2$ in $G \rhd_o H_2$. If $|V(G)|$ odd, then $G \rhd_o H_2$ is $H_1 \rhd_o H_2$-supermagic graph.*

Proof. If H_2 is trivial graph [1], then $G \rhd_o H_2 = G$ and $H_1 \rhd_o H_2 = H_1$, so it is clear by the premise. Next, suppose not. Since G is H_1-supermagic graph, then there exist bijective function $f: V(G) \cup E(G) \to \{1, 2, \dots, |V(G)| + |E(G)|\}$ such that $\sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e) = k$ for all subgraph H isomorphic to H_1 and $f(V) = \{1, 2, \dots, |V(G)|\}$. Assume that $s(f) = \{f(v) \mid v \in V(H)\} \cup \{f(e) \mid e \in E(H)\}$ is an arbitrary set of vertex and edge label of subgraph H isomorphic to H_1. We get that $\sum s(f) = k$. Then, define a function g as a label to graph $G \rhd_o H_2$. We divide it in two cases.

Case 1. $c + d$ even

Note that since H_2 is not a trivial graph, then $c \ge 3$ for odd c and $c \ge 4$ for even c. Then, define a function

$$g(v_i) = f(v_i)\,; 1 \le i \le a$$

$$g(v_{i,1}) = \begin{cases} \frac{(3a+3)}{2} - i, & i \in [1, \frac{a+1}{2}] \\ \frac{(5a+3)}{2} - i, & i \in [\frac{a+3}{2}, a] \end{cases} \quad (2)$$

$$g(v_{i,2}) = \begin{cases} 2a + 2i - 1, & i \in [1, \frac{a+1}{2}] \\ a + 2i - 1, & i \in [\frac{a+3}{2}, a] \end{cases} \quad (3)$$

$$g(v_{i,j}) = \begin{cases} ja + a + 1 - i; & i \in [1, a], odd\ j \in [3, c-1] \\ ja + i; & i \in [1, a], even\ j \in [3, c-1] \end{cases} \quad (4)$$

$$g(e_{i,h}) = \begin{cases} (c+h)a + 1 - i; & i \in [1, a],\ odd\ h \in [1, d] \\ (c+h-1)a + i; & i \in [1, a]\ \ even\ h \in [1, d] \end{cases} \quad (5)$$

$$g(e_l) = (c + d - 1)a + f(e_l)\,; 1 \le l \le b \quad (6)$$

It is clear that the function is a bijective. Note that if we sum all label of vertex and edge from an arbitrary subgraph H_2, we get that

$$\sum_{v \in V(H_2) \setminus \{o\}} g(v) + \sum_{e \in E(H_2)} g(e) = \frac{ac^2 + ad^2 + c + d - a - 1}{2} + acd$$

Let $s(g) = \{g(v) \mid v \in V(H_1 \rhd_o H_2)\} \cup \{g(e) \mid e \in E(H_1 \rhd_o H_2)\}$. According to the function defined at Equation (2)-(6),

$$\sum s(g) = k + acz + adz - az + acdy + \frac{ac^2y + ad^2y + cy + dy - ay - y}{2}$$

It means that the sum is equal for any subgraph isomorphic to $H_1 \rhd_o H_2$.

Case 2. $c + d$ odd

Define a function

$$g(v_i) = f(v_i)\,; 1 \le i \le a$$

$$g(v_{i,j}) = \begin{cases} ja + a + 1 - i; & i \in [1, a], odd\ j \in [1, c-1] \\ ja + i; & i \in [1, a], even\ j \in [1, c-1] \end{cases}$$

$$g(e_{i,h}) = \begin{cases} ca + (h-1)a + i; & i \in [1, a], odd\ h \in [1, d] \\ ca + ha + 1 - i; & i \in [1, a], even\ h \in [1, d] \end{cases}$$

$$g(e_l) = (c + d - 1)a + f(e_l); 1 \le l \le b$$

Then, by the same method we apply in *Case 1*, we get that

$$\sum s(g) = k + acz + adz - az + acdy + \frac{ac^2y + ad^2y + cy + dy - ay - y}{2}$$

From the both cases above, we get that $g(V(G \rhd_o H_2)) = \{1, 2, \dots, ac\}$ where $ac = |V(G \rhd_o H_2)|$, g is a bijective function, and for all subgraph of $G \rhd_o H_2$ isomorphic to $H_1 \rhd_o H_2$, the sum of every vertex and edge label is constant. So, we conclude that g is $H_1 \rhd_o H_2$-supermagic labeling of graph $G \rhd_o H_2$ where $|V(G)|$ is odd. ■

Note that this is not just hold for odd $|V(G)|$, but it also hold for even $|V(G)|$ and odd $|V(H_2)| + |E(H_2)|$. The proof is similar to the Theorem 2.1, by taking a function defined in *Case 2*, then apply to the proof. By the short explanation, we derive it into this corollary.

Corollary 2.2. *Suppose that G is H_1-supermagic graph and H_2 is connected graph such that the number of subgraph H_1 in G is equal to the number of subgraph $H_1 \rhd_o H_2$ in $G \rhd_o H_2$. If $|V(G)|$ is even and $|V(H_2)| + |E(H_2)|$ odd, then $G \rhd_o H_2$ is $H_1 \rhd_o H_2$ -supermagic graph.* ∎

Here is an example. In [6], Roswitha and others give a C_3-supermagic labeling of graph W_4 with magic number $k = 36$. By using the Theorem 2.1 and adding comb product to W_4 with star $K_{1,3}$, we can create $C_3 \rhd_o K_{1,3}$-supermagic labeling of graph $W_4 \rhd_o K_{1,3}$. We provide it in Figure 1.

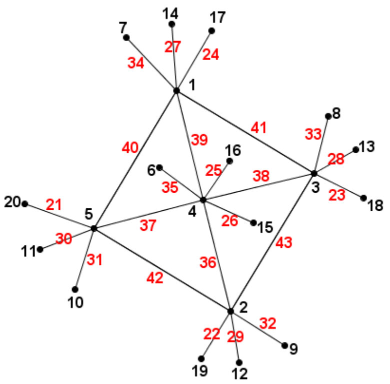

Figure 1 The $C_3 \rhd_o K_{1,3}$-supermagic labeling of graph $W_4 \rhd_o K_{1,3}$.

Note that the magic number is $k^* = 495$. We can get that by substituting $a = 5, c = 4, d = y = z = 3, k = 36$ to $k^* = k + acz + adz - az + acdy + \frac{ac^2y+ad^2y+cy+dy-ay-y}{2}$. We also give an open problem for even $|V(G)|$ and even $|V(H_2)| + |E(H_2)|$.

Open Problem 1. *For even $|V(G)|$ and even $|V(H_2)| + |E(H_2)|$, where the number of subgraph H_1 in G is equal to the number of subgraph $H_1 \rhd_o H_2$ in $G \rhd_o H_2$, determine the $H_1 \rhd_o H_2$ -supermagic labeling of $G \rhd_o H_2$.*

3. CONCLUSION

For any graph G, H_1, H_2, odd $|V(G)|$, the graph $G \rhd_o H_2$ is $H_1 \rhd_o H_2$-supermagic with constant $k^* = k + acz + adz - az + acdy + \frac{ac^2y+ad^2y+cy+dy-ay-y}{2}$ for a, c, d in Equation (1) and k is the magic number of H_1 -supermagic labeling of graph G. Moreover, for even $|V(G)|$ and odd $|V(H_2)| + |E(H_2)|$, the graph $G \rhd_o H_2$ is $H_1 \rhd_o H_2$-supermagic with same constant above.

ACKNOWLEDGMENTS

The first author was funded by "Hibah Fakultas" Research Grant No. 003/UN15.15.2.PPK/SPP/FST/IV/2021 from Faculty of Science and Technology University of Nusa Cendana.

REFERENCES

[1] R. Diestel, Graph Theory (5th Ed.), Berlin, Springer, 2017.

[2] J. A. Galian, A dynamic survey of graph labelling, Electronic Journal of combinatorics, 2018, 23rd Ed., Dynamic Surveys, #DS6.

[3] A. Gutiérrez, A. Lladó, Magic coverings, J. Combin. Math. Combin. Comput., Vol. 55, 2005, pp. 43-56.

[4] A. Lladó, J. Moragas, Cycle-magic graphs, Discrete Math., vol. 307, 2007, pp. 2925-2933. DOI: https://doi.org/10.1016/j.disc.2007.03.007.

[5] A. A. G Ngurah, A. N. M Salman, L. Susilowati, H-supermagic labelings of graphs, Discrete Mathematics, 2010, pp. 1293–1300. DOI: https://doi.org/10.1016/j.disc.2009.12.011.

[6] M. Roswitha, E. T. Baskoro, T. K. Maryati, N. A. Kurdhi & I. Susanti, Further results on cycle-supermagic labeling, AKCE International Journal of Graphs and Combinatorics, Vol. 10(2), 2013, pp. 211-220, DOI: https://doi.org/10.1080/09728600.2013.12088737.

[7] C. Chithra, G. Marimuthu, G. Kumar. C_m - E - supermagic labeling of graphs. AKCE International Journal of Graphs and Combinatorics, Vol. 17(1), 2019, pp. 510-518. DOI: https://doi.org/10.1016/j.akcej.2019.03.016.

[8] H. Sadariria, Y. Susanti, H-supermagic labeling on edge coronation of some graphs with a cycle, AIP Conference Proceedings, AIP Publishing LLC, 2192, 2019, pp. 040014, DOI: https://doi.org/10.1063/1.5139140.

[9] S. Wido Saputro, N. Mardiana, I. A. Purwasih, The metric Dimension of Comb Product Graphs, MATEMATIK VESNIK Vol. 69(4), 2017, pp. 248-258.

[10] G. L. Putra, The $P_2 \rhd_o G$-supermagic labeling of comb product of graphs, Tensor: Pure and Applied Mathematics Journal, Vol. 2(1), 2021, pp. 13-18.