ATLANTIS PRESS

Advances in Computer Science Research, volume 96

Proceedings of the International Conference on Mathematics, Geometry, Statistics, and Computation (IC-MaGeStiC 2021)

On the Minimum Span of Cone, Tadpole, and Barbell Graphs

Hafif Komarullah, Ikhsanul Halikin*, Kiswara Agung Santoso

Graph, Combinatorics, and Algebra Research Group, Department of Mathematics, FMIPA, University of Jember
*Corresponding author. Email: ikhsan.fmipa@unej.ac.id

ABSTRACT

Let G be a simple and connected graph with p vertices and q edges. An $L(2,1)$-labelling on the graph G is a function $f: V(G) \to \{0, 1, \dots, k\}$ such that every two vertices with a distance one receive labels that differ by at least two, and every two vertices at distance two receive labels that differ by at least one. A number k is called as span of $L(2.1)$-labelling, if k is the largest vertex labels. The span of a graph G can be more than one, the minimum value of the span of a graph G is notated by $\lambda_{(2,1)}(G)$. In this paper, we determine the minimum span of cone, tadpole, and barbell graphs

Keywords: *$L(2,1)$-labelling, Minimum of span, Cone, Tadpole, and Barbell graphs.*

1. INTRODUCTION

Throughout this paper, all graphs are simple, connected, and undirected. Graph labelling is one of the research topic in graph theory. Graph labelling was first introduced in the mid-1960s. There are many kind research about graph labelling, one of them is $L(2,1)$-labelling. An $L(2,1)$-labelling is defined as a mapping from the set of vertices in graph to a set of non-negative integer such that the absolute value of difference between the vertex labels with distance one is two and the absolute value of difference between the vertex labels with distance two is one [1]. Formally, let $G = (V, E)$ be a graph and $d(u, v)$ notated the distance between vertex u and v. a function $f: V(G) \to \{0, 1, \dots, k\}$ is called $L(2,1)$-labelling if $|f(u) - f(v)| \geq 1$ for $d(u, v) = 2$ and $|f(u) - f(v)| \geq 2$ for $d(u, v) = 1$. The number k here is called as span of $L(2.1)$-labelling if k is the largest vertex labels. The span of a graph G can be more than one, the minimum value of the span of a graph G is notated by $\lambda_{(2,1)}(G)$ [2,3].

There are many research about the minimum span of a graph. Griggs and Yeh [3,4] in 1992 proved that $\lambda_{2,1}(S_{1,n}) = n + 1$, $\lambda_{2,1}(C_n) = 4$ and $\lambda_{2,1}(P_n) = 4$. The minimum span of fan graph (f_n) is $n + 1$ and wheel graph (W_n) is $n + 1$ [5]. Yuri *et al.* [6] in 2018 proved that the minimum span of Sierpinski graph $(S_{(n,m)})$ is 4 for $m = 2$ and $m = 3$.

In this paper, we will discuss about the minimum span of cone, tadpole, and barbell graphs. There are some properties about $L(2,1)$-labelling which will be used in this paper as follows.

Lemma 1.1 [7] If H is a subgraf of graph G, then $\lambda_{2,1}(H) \leq \lambda_{2,1}(G)$

Lemma 1.2 [3] Let $C_n$be a cycle graph with $n \geq 3$, then $\lambda_{2,1}(C_n) = 4$

2. MAIN RESULT

In this section we will discuss about the minimal span of cone, tadpole, and barbell graphs

2.1. Cone Graph

Cone graph is a graph formed from a join operation between graph cycle C_m and a graph $\overline{K}_n$. A cone graph is denoted by $C_{m,n}$. The set of vertices and edges on a cone graph will be notated as follows.

$$V(C_{m,n}) = \{u_i;\ i = 1, 2, \dots, m\} \cup \{v_j;\ j = 1, 2, \dots, n\}$$
$$E(C_{m,n}) = \{u_1 u_m, u_i u_{i+1}: i = 1, 2, \dots, m-1\} \cup \{u_i v_j: i = 1, 2, \dots, m;\ j = 1, 2, \dots, n\}.$$

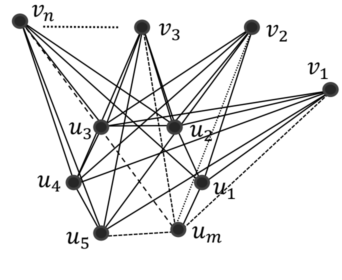

Figure 1 Cone graph $C_{m,n}$.

Theorem 2.1 Let $C_{m,n}$ be a cone graph with $m \geq 3$ and $n \geq 1$, then $\lambda_{2,1}(C_{m,n}) = n + 5$.

Proof. First, we will show that $\lambda_{2,1}(C_{m,n}) \geq n + 5$. Suppose that cone graph $C_{m,n}$ can be labelled only with labels $0,1,2,\dots,n+4$. Since cycle graph C_n is a subgraph of cone graph $C_{m,n}$, then based on the Lemma 1.2, the largest label of vertices u_i is 4. As the distance of vertex u_i and v_j is one, then the absolute value of the difference between the labels is at least two. Therefore, vertices v_j must be labelled by $6,7,\dots,n+4$. Since the distance of vertex v_j and v_{j+1} is two, then the absolute value of the difference between the labels is at least one. Thus, every label of v_j must be different. Since the number of vertices of $\{v_j\}$ is n, while the number of label used is $n-1$, then according to the pigeonhole principle, there are at least two vertices in v_j that has the same label. So, the cone graph cannot be labelled only with labels $1,2,\dots,n+4$. Therefore, we can conclude that $\lambda_{2,1}(C_{m,n}) \geq n + 5$.

Next, we will show that $\lambda_{2,1}(C_{m,n}) \leq n + 5$ by constructing the $L(2,1)$-labelling on a cone graph. Its labelling will be divided by three cases as follows.

1. for $m \equiv 0 \bmod 3$

$$f(u_i) = \begin{cases} 0, i \equiv 1 \bmod 3 \\ 2, i \equiv 2 \bmod 3 \\ 4, i \equiv 0 \bmod 3 \end{cases}$$

$$f(v_j) = j + 5; j = 1,2,\dots,n$$

Based on the condition of $L(2,1)$-labelling, the absolute value of the difference between the labels for every two vertices with distance one is at least two. This will be shown as follows.

a. The difference of vertex label u_i and u_{i+1} for $i \equiv 1 \bmod 3$
$|f(u_i) - f(u_{i+1})| = |0 - 2| \geq 2$

b. The difference of vertex label u_i and u_{i+1} for $i \equiv 2 \bmod 3$
$|f(u_i) - f(u_{i+1})| = |2 - 4| \geq 2$

c. The difference of vertex label u_i and u_{i+1} for $i \equiv 0 \bmod 3$
$|f(u_i) - f(u_{i+1})| = |4 - 0| \geq 2$

d. The difference of vertex label v_j and u_i for $j = 1,2,\dots,n$
$|f(v_j) - f(u_i)| = |j + 5 - 0| \geq 2, i \equiv 1 \; mod \; 3$
$|f(v_j) - f(u_i)| = |j + 5 - 2| \geq 2, i \equiv 2 \; mod \; 3$
$|f(v_j) - f(u_i)| = |j + 5 - 4| \geq 2, i \equiv 0 \; mod \; 3$

In addition, according to the rule of $L(2,1)$-labelling, the absolute value of the difference between the labels for every two vertices with distance two is at least one. This will be shown as follows.

a. The difference of vertex label u_i and u_{i+2} for $i \equiv 1 \; mod \; 3$
$|f(u_i) - f(u_{i+2})| = |0 - 4| \geq 1$

b. The difference of vertex label u_i and u_{i+2} for $i \equiv 2 \; mod \; 3$
$|f(u_i) - f(u_{i+2})| = |2 - 0| \geq 1$

c. The difference of vertex label u_i and u_{i+2} for $i \equiv 0 \bmod 3$
$|f(u_i) - f(u_{i+2})| = |4 - 2| \geq 1$

d. The difference of vertex label v_t and v_s for $t \neq s, s \geq t + 1$, dan $1 \leq t < s \leq j$
$|f(v_s) - f(v_t)| = |(s + 5) - (t + 5)| = |s - t|$
$= |(t + 1) - t| \geq 1$

We can conclude that for $m \equiv 0 \bmod 3$, the function f satisfied $L(2,1)$-labelling.

2. for $m \equiv 1 \bmod 3$

$$f(u_i) = \begin{cases} 0, i \equiv 1 \bmod 3 \; dan \; i \neq m \\ 2, i \equiv 2 \bmod 3 \; dan \; i \neq m - 2 \\ 4, i \equiv 0 \bmod 3 \; dan \; i \neq m - 1 \\ 3, i = m - 2 \\ 1, i = m - 1 \\ 4, i = m \end{cases}$$

$$f(v_j) = j + 5, j = 1,2,\dots,n$$

In the same way with the Case 1, we can prove that the function f satisfied $L(2,1)$-labelling for $m \equiv 1 \bmod 3$.

3. for $m \equiv 2 \bmod 3$

$$f(u_i) = \begin{cases} 0, i \equiv 1 \bmod 3 \; dan \; i \neq m - 1 \\ 2, i \equiv 2 \bmod 3 \; dan \; i \neq m \\ 4, i \equiv 0 \bmod 3 \\ 1, i = m - 1 \\ 3, i = m \end{cases}$$

$$f(v_j) = j + 5; j = 1,2,\dots,n$$

Again, it is easy to prove that the function f satisfied $L(2,1)$-labelling for $m \equiv 2 \bmod 3$. So, we have shown that $\lambda_{2,1}(C_{m,n}) \leq n + 5$. Therefore, we can conclude that $\lambda_{2,1}(C_{m,n}) = n + 5$. For example, an $L(2,1)$-labelling on a cone graph can be seen in Figure 2.

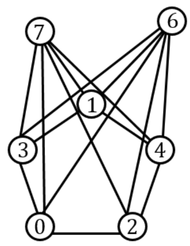

Figure 2 $L(2,1)$-labelling of cone graph $C_{5,2}$.

2.2. Tadpole Graph

Tadpole graph $(T_{m,n})$ is defined as a graph obtained by combining a vertex of cycle graph C_m with one of the leaf of path graph P_n. Suppose that the vertices and edges in the tadpole graph are notated as follows.

$V(T_{m,n}) = \{u_i; i = 1, 2, \dots, m\} \cup \{v_j; j = 1, 2, \dots, n\}$

$E(T_{m,n}) = \{u_1 u_m, u_i u_{i+1}; i = 1, 2, \dots, m - 1\} \cup \{u_1 v_1\} \cup \{v_i v_{i+1}; i = 1, 2, \dots, n - 1\}$

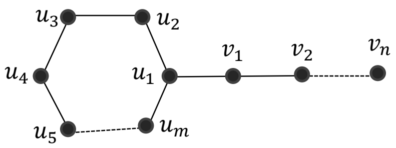

Figure 3 Tadpole graph $T_{m,n}$.

Theorem 2.2 Let $T_{m,n}$ be a tadpole graph with $m \geq 3$ and $n \geq 1$, then $\lambda_{2,1}(T_{m,n}) = 4$.
Proof. Let $T_{m,n}$ be a tadpole graph with $m \geq 3$ and $n \geq 1$, We will prove that $\lambda_{2,1}(T_{m,n}) \geq 4$. Since the cycle graph C_m is a subgraph of tadpole graph $T_{m,n}$, then based on the Lemma 1.2, we get $\lambda_{2,1}(T_{m,n}) \geq \lambda_{2,1}(C_m) = 4$. So, we have proved that $\lambda_{2,1}(T_{m,n}) \geq 4$.

Next, we will show that $\lambda_{2,1}(T_{m,n}) \leq 4$ by constructing $L(2,1)$-labelling on the tadpole graph $T_{m,n}$. We will consider four cases as follows.

1. for $m = 4$

$$f(u_i) = \begin{cases} 0, i = 1 \\ 3, i = 2 \\ 1, i = 3 \\ 4, i = 4 \end{cases}$$

$$f(v_j) = \begin{cases} 2, j \equiv 1 \bmod 3 \\ 4, j \equiv 2 \bmod 3 \\ 0, j \equiv 0 \bmod 3 \end{cases}$$

2. for $m \equiv 0 \bmod 3$

$$f(u_i) = \begin{cases} 0, i \equiv 1 \bmod 3 \\ 2, i \equiv 2 \bmod 3 \\ 4, i \equiv 0 \bmod 3 \end{cases}$$

$$f(v_j) = \begin{cases} 3, j \equiv 1 \bmod 4 \\ 1, j \equiv 2 \bmod 4 \\ 4, j \equiv 3 \bmod 4 \\ 0, j \equiv 0 \bmod 4 \end{cases}$$

3. for $m \equiv 1 \bmod 3$ and $m \neq 4$

$$f(u_i) = \begin{cases} 0, i \equiv 1 \bmod 3 \text{ and } i \neq m \\ 2, i \equiv 2 \bmod 3 \text{ and } i \neq m-2 \\ 4, i \equiv 0 \bmod 3 \text{ and } i \neq m-1 \\ 3, i = m-2 \\ 1, i = m-1 \\ 4, i = m \end{cases}$$

$$f(v_j) = \begin{cases} 3, j \equiv 1 \bmod 4 \\ 1, j \equiv 2 \bmod 4 \\ 4, j \equiv 3 \bmod 4 \\ 0, j \equiv 0 \bmod 4 \end{cases}$$

4. for $m \equiv 2 \bmod 3$

$$f(u_i) = \begin{cases} 0, i \equiv 1 \bmod 3 \; dan \; i \neq m-1 \\ 2, i \equiv 2 \bmod 3 \; dan \; i \neq m \\ 4, i \equiv 0 \bmod 3 \\ 1, i = m-1 \\ 3, i = m \end{cases}$$

$$f(v_j) = \begin{cases} 4, j \equiv 1 \bmod 3 \\ 2, j \equiv 2 \bmod 3 \\ 0, j \equiv 0 \bmod 3 \end{cases}$$

In the same way as Theorem 2.1, it is easy to prove that every two vertices with distance one receive labels that differ by at least two, and every two vertices at distance two receive labels that differ by at least one. So, we have proved that $\lambda_{2,1}(T_{m,n}) \leq 4$. Therefore, we can conclude that $\lambda_{2,1}(T_{m,n}) = 4$. As illustration of Theorem 2.2, we present an example of $L(2,1)$-labeling on a tadpole graph in Figure 4.

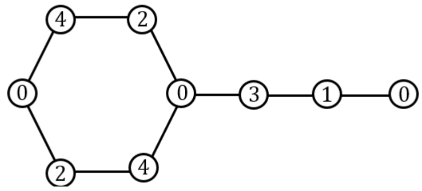

Figure 4 $L(2,1)$-labelling of tadpole graph $T_{6,3}$.

2.3. Barbell Graph

Barbell graph (B_n) is a graph obtained by connecting two complete graph K_n by an edge. Suppose that the vertices and edges in the barbell graph are notated as follows.
$V(B_n) = \{u_i; i = 1, 2, \ldots, n\} \cup \{v_j; j = 1, 2, \ldots, n\}$
$E(B_n) = \{u_s u_t, s \neq t\} \cup \{u_1 v_1\} \cup \{v_s v_t; s \neq t\}$

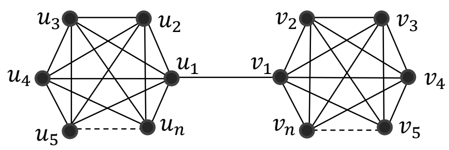

Figure 5 Barbell graph B_n.

Theorem 2.3 Let B_n be a barbell graph with $n \geq 3$, then $\lambda_{2,1}(B_n) = 2n - 1$.
Proof. First, we will prove that $\lambda_{2,1}(B_n) \geq 2n - 1$. Suppose that barbell graph B_n can be labelled by $0,1,2, \ldots, 2n-2$. Since barbell graph formed by two complete graphs and every two vertices are adjacent, then based on the rule of $L(2,1)$-labelling, the absolute value of the different between the labels is at least two. Therefore, every complete graph in a barbell graph must be labelled by even number or odd number only. This shows that the barbell graph requires n even labels and n odd labels. But, there are just n even labels and $n-1$ odd labels. Therefore, we need one odd label except $0,1,2, \ldots, 2n-2$. So, it is obvious that $\lambda_{2,1}(B_n) \geq 2n - 1$.

Next, we will prove that $\lambda_{2,1}(B_n) \leq 2n - 1$ by constructing the $L(2,1)$-labelling on a barbell graph. Let $f : V(B_n) \rightarrow \{0,1,2, \ldots, 2n-1\}$ be a function as follows.
$f(u_i) = 2i - 2, i = 1,2, \ldots, n$
$f(v_j) = 2(n - j) + 1, j = 1,2, \ldots, n$
It is easy to prove that $\lambda_{2,1}(B_n) \leq 2n - 1$. Therefore, we can conclude that $\lambda_{2,1}(B_n) = 2n - 1$. In the figure 6, we give an example of $L(2,1)$-labelling on barbell graph with $n = 4$.

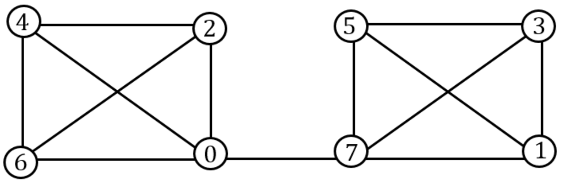

Figure 6 $L(2,1)$-labelling of barbell graph B_4.

ACKNOWLEDGMENTS

This research has been supported by "Hibah Kelompok Riset Graf, Combinatorika, dan Aljabar (GCA), Universitas Jember, Tahun Anggaran 2021, Nomor 14737/UN25/LT/2021"

REFERENCES

[1] P. Bella, D. Král, B. Mohar, K. Quittnerová, Labeling planar graphs with a condition at distance two, European Journal of Combinatorics, vol. 28, Issue 8, 2007, Pages 2201-2239. DOI: https://doi.org/10.1016/j.ejc.2007.04.019

[2] Z. Shao, R.K. Yeh, D. Zhang, The (2, 1)-Labeling on Graphs and Frequency Assigment Problem. Applied Mathematics Letters, vol. 21, 2008, pp.37-41. DOI: https://doi.org/10.1016/j.aml.2006.08.029

[3] J. Griggs, R. Yeh, Labelling graph with a condition at distance 2, SIAM J. Discrete Math, vol. 5, 1992, pp. 586-595. DOI: https://doi.org/10.1137/0405048

[4] R.K. Yeh, A survey on labeling graphs with a condition at distance two, Discrete Mathematics, vol. 306, Issue 12, 2006, pp. 1217-1231. DOI: https://doi.org/10.1016/j.disc.2005.11.029

[5] S. Fatimah, I.W. Sudarsana, S. Musdalifah, Pelabelan L(2,1) Pada Operasi Beberapa Kelas Graf (in Indonesian), Jurnal Ilmiah Matematika dan Terapan, 2016, 13(2), pp. 73-84.

[6] C. S. Yuri, Susiana, Pelabelan L(2,1) pada Graf Sierpinski $S(n,k)$ (in Indonesian), Jurnal Sains Indonesia, 2018, 42(1), pp. 22-24. DOI: https://doi.org/10.24114/ jsi.v42i1.12244

[7] A. Lum, Upper Bound on L(2,1)-labelling Number of Graphs with Maximum Degree Δ, 2007, https://www.whitman.edu/documents/academics/mathematics/lumaa.pdf