A Minimum Coprime Number *for Amalgamation of Wheel*

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ABSTRACT
Let G be a simple graph of order n. A *coprime labeling* of a graph G is a vertex labeling of G with distinct positive integers from the set $\{1, 2, \ldots, k\}$ for some $k \geq n$ such that any adjacent labels are relatively prime. The minimum value of k for which G has a coprime labelling, denoted as $\mathfrak{pr}(G)$, is called the *minimum coprime number* of G. A coprime labeling of G with the largest label being $\mathfrak{pr}(G)$ is said a *minimum coprime labeling* of G. In this paper, we give the exact value of the minimum coprime number for amalgamations of wheel W_n when n is odd positive integer.

Keywords: *Minimum coprime labeling, Minimum coprime number, Amalgamation of wheel.*

1. INTRODUCTION

Let G be a simple graph with the vertex-set $V(G)$ and the edge-set $E(G)$. A *coprime labeling* of a graph G is an injective function $f: V(G) \to \{1,2, \ldots, k\}$ so that the labels of any two adjacent vertices are relatively prime. Clearly that $k \geq n$, where n is the number of vertices of a graph G. If $k = n$, then the function f is called a *prime labeling* of G. A graph that admits a prime labelling is called a *prime*. However, it does not make sense to refer to a graph as coprime, since all graphs have a coprime labeling (for instance, use the first n prime integers as the labels) (see [1]). Therefore, the problem of a coprime labeling is to find the minimum value of k namely a *minimum coprime number* of G and denoted as $\mathfrak{pr}(G)$. A coprime labeling of G with the largest label being $\mathfrak{pr}(G)$ is said a *minimum coprime labeling* of G.

The concept of a prime labeling originated with Entringer and was first introduced in a paper by Tout, Dabbouvy, and Howalla [2]. Around 1980, Entringer gave conjecture that all trees are prime graphs. Among the classes of trees known as prime are paths, stars, spiders, olive trees, palm trees, binomial trees, all trees of order up to 50, banana trees, and all caterpillars with maximum degree at most 5 (see [2, 3, 4, 5, 6, 7]).

Deretsky, Lee, and Mitchem [8] proved that all cycles C_n (i.e., a 2-regular graph with n vertices) are prime. Lee, Wui, and Yeh [9] proved that wheel W_n (i.e., a cycle C_n with one central vertex adjacent to n vertices in C_n) is prime if and only if n is even; a complete graph K_n (i.e., an $(n-1)$-regular graph with n vertices) is not prime for $n \geq 4$. The other results about prime labeling can be seen in [10, 11, 12, 13, 14, 15, 16] and completely the survey about this labeling in Galian [17].

Asplund and Fox [18] obtained the exact value of the minimum coprime number of complete graph K_n and odd wheel W_n (i.e., wheel W_n with odd n) namely $\mathfrak{pr}(K_n) = p_{n-1}$, where p_{n-1} is the first $(n-1)$ primes; and $\mathfrak{pr}(W_n) = n + 2$ for any odd integer $n \geq 3$. In another paper, Asplund and Fox [19] gave the minimun coprime number of Generalized Petersen and Prism Graphs. Lee [20] determine the minimum coprime number for a few well-studied classes of graphs, including the coronas of complete graphs with empty graphs and the joins of two paths.

Herein, we discuss about the minimum coprime labeling for amalgamation of wheel. Amalgamation of t copies of G at the fixed vertex $v_0 \in V(G)$, denoted by $Amal(G, v_0, t)$, is the graph obtained from t copies of G by identifying t copies of G at the fixed vertex v_0. Lee, Wui, and Yeh [11] have shown that $Amal(G, v_0, t)$ has a prime labeling when G is a path, a cycle, or an even wheel. They also showed that the amalgamation of odd wheel is not prime. Therefore, in this paper we give the the exact value of the minimum coprime number of

$Amal(W_n, v_0, t)$ when n is odd and v_0 is the central vertex of wheel W_n, namely the vertex of degree n in W_n.

2. MAIN RESULTS

In this section we will discuss prime and coprime labeling for amalgamation of wheel W_n for any odd positive integer. Before doing that, we discuss about two integers said to be relatively prime. We know that $gcd(a,1) = 1$ for any integer a. Any two consecutive integers also has the greatest common divisor one, namely $gcd(a, a+1) = 1$. Two lemmas below useful to prove that two integers are relatively prime.

Theorem 2.1 [21] Two integers a and b are said to be *relatively prime*, if there exist two integers x and y such that $ax + by = 1$. ∎

Lemma 2.2 Let a be odd positive integer. If a positive integer r does not have odd factor other than one, then $\gcd(a, a+r) = 1$.

Proof. Suppose that $\gcd(a, a+r) = k$. Then $a = kx$ and $a + r = ky$. So $k(y-x) = r$. Both a and $a+r$ are odd. So, k must be odd. Since r does not have odd factor other than one, we get $k = 1$. Therefore $\gcd(a, a+r) = 1$. ∎

Let v_0 be the central vertex of wheel W_n. Suppose the vertex-set and edge-set of an $Amal(W_n, v_0, t)$ are $V(Amal(W_n, v_0, t)) = \{v_0\} \cup \{v_{ij} | i \in [1,t], j \in [1,n]\}$, where degree of v_0 and v_{ij} is $d(v_0) = nt$ and $d(v_{ij}) = 3$, respectively, and $E(Amal(W_n, v_0, t)) = \{v_0 v_{ij},\ v_{i1} v_{in} \mid i \in [1,t],\ j \in [1,n]\} \cup \{v_{ij} v_{ij+1} \mid i \in [1,t], j \in [1, n-1]\}$, respectively. An $Amal(W_n, v_0, t)$ has $(nt+1)$ vertices. The lower bound of the minimum coprime number for amalgamation of the odd wheel is given in Lemma below.

Lemma 2.3 Let v_0 be the central vertex of wheel W_n. For each integer $t > 1$ and odd integer $n \geq 1$, $\mathfrak{pr}(Amal(W_n, v_0, t)) \geq (n+1)t + 1$.

Proof. Let $n, t \geq 1$ be integers where n is odd. We know that an $Amal(W_n, v_0, t)$ has $(nt+1)$ vertices, where the one vertex, called the central vertex v_0 adjacent to all vertices in $Amal(W_n, v_0, t)$. There are t cycles with length n where every cycle needs $\frac{n-1}{2}$ even labels and $\frac{n+1}{2}$ odd labels. So, a graph $Amal(W_n, v_0, t)$ needs $\left(\left(\frac{n+1}{2}\right)t + 1\right)$ odd labels. Clearly that there are not enough odd labels in the set $\{1,2,\dots,nt+1\}$. It means that an $Amal(W_n, v_0, t)$ cannot be labeled by $1, 2, \dots, nt+1$ such that every two adjacent vertices have the relatively prime labels. Since there are $\left(\left(\frac{n+1}{2}\right)t + 1\right)$ odd labels in the set $\{1, 2, \dots, (n+1)t+1\}$, hence $\mathfrak{pr}(Amal(W_5, v_0, t)) \geq (n+1)t + 1$. ∎

To obtain the exactly of the minimum coprime number for amalgamation of odd wheel W_n, we consider two cases, namely for either $n = 1\ (mod\ 4)$ or $n = 3\ (mod\ 4)$.

Theorem 2.4 Let v_0 be the central vertex of wheel W_n. For each integer $t > 1$ and $n = 1\ (mod\ 4)$, the minimum coprime number for amalgamation of odd wheel W_n is $\mathfrak{pr}(Amal(W_n, v_0, t)) = (n+1)t + 1$.

Proof. Let n and t be positive integers where $n = 1\ (mod\ 4)$. By Lemma 2.3, we have $\mathfrak{pr}(Amal(W_n, v_0, t)) \geq (n+1)t + 1$.

We now show that $\mathfrak{pr}(Amal(W_n, v_0, t)) \leq (n+1)t + 1$ by defined a coprime labeling of a graph $Amal(W_n, v_0, t)$ as below. We define $f: V(Amal(W_n, v_0, t)) \longrightarrow \{1,2,\dots,(n+1)t+1\}$ where $f(v_0) = 1$ and for $i = 1, 2, \dots, t$,

$$f(v_{ij}) = \begin{cases} (n+1)i - (n-j), \text{ for } 1 \leq j \leq \frac{n+1}{2}, \\ (n+1)i + 1, \quad \text{for } j = \frac{n+3}{2}, \\ (n+1)i + \left(\frac{n+3}{2} - j\right), \text{ for } \frac{n+5}{2} \leq j \leq n. \end{cases}$$

Next, we show that the greatest common divisor of every labels of two adjacent vertices are one. We consider three cases below.

- Suppose that $\gcd\left(f\left(v_{i\left(\frac{n+1}{2}\right)}\right), f\left(v_{i\left(\frac{n+3}{2}\right)}\right)\right) = \gcd\left((n+1)i - \left(\frac{n-1}{2}\right), (n+1)i + 1\right) = k$. Since $n = 1\ (mod\ 4)$, then $(n+1)i - \left(\frac{n-1}{2}\right) = 1\left(mod\left(\frac{n+1}{2}\right)\right)$ is even, while $(n+1)i + 1 = 1\left(mod\left(\frac{n+1}{2}\right)\right)$ is odd. So $k \neq \frac{n+1}{2}$ must be odd. Let $(n+1)i - \left(\frac{n-1}{2}\right) = kx$ and $(n+1)i + 1 = ky$. Then $k(y-x) = \frac{n+1}{2}$. So it must be $k = 1$. Therefore labels of $v_{i\left(\frac{n+1}{2}\right)}$ and $v_{i\left(\frac{n+3}{2}\right)}$ are relatively prime.
- Now, consider the labels of vertex $v_{i\left(\frac{n+3}{2}\right)}$ and $v_{i\left(\frac{n+5}{2}\right)}$. Suppose that $\gcd\left(f\left(v_{i\left(\frac{n+3}{2}\right)}\right), f\left(v_{i\left(\frac{n+3}{2}\right)}\right)\right) = \gcd\big((n+1)i + 1, (n+1)i - 1\big) = k$. Since n is odd, $(n+1)i + 1$ is odd. By applying Lemma 2.2, we get $\gcd\big((n+1)i + 1, (n+1)i - 1\big) = 1$.
- Last, we consider the vertex v_{i1} and v_{in}. We suppose that $\gcd(f(v_{i1}), f(v_{in})) = \gcd\left((n+1)i - (n-1), (n+1)i + \left(\frac{3-n}{2}\right)\right) = k$. Since $n = 1\ (mod\ 4)$, $(n+1)i - (n-1)$ is even, and $(n+1)i + \left(\frac{3-n}{2}\right)$ is odd, but both are in $2\left(mod\left(\frac{n+1}{2}\right)\right)$. Hence $k \neq \frac{n+1}{2}$ must be odd. We next consider $(n+1)i - (n-1) =$

kx and $(n+1)i + \left(\frac{3-n}{2}\right) = ky$. So, $k(y-x) = \frac{n+1}{2}$. Therefore $k = 1$. It means v_{i1} and v_{in} have relatively prime labels.

We have shown that every label of two adjacent vertices is relatively prime. Therefore, the function f is a minimum coprime labeling with the largest label being $\mathfrak{pr}\left(Amal(W_n, v_0, t)\right) = (n+1)t + 1$. ■

For example, a minimum coprime labeling of $Amal(W_5, v_0, 4)$ can be seen in Figure 1, where $\mathfrak{pr}\left(Amal(W_5, v_0, 4)\right) = 25$.

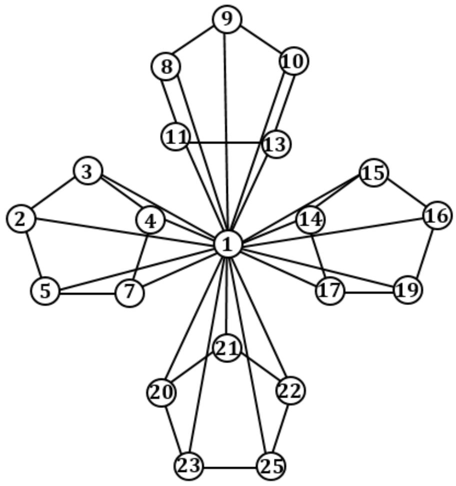

Figure 1 A coprime labelling of $Amal(W_5, v_0, 4)$

We observe now an $Amal(W_n, v_0, t)$ for $n = 3(mod\ 4)$. For this case, we have not got the minimum coprime labeling in general. Therefore, we give it for some positive integer n.

Proposition 2.5 Let v_0 be the central vertex of wheel W_3. The minimum coprime number for amalgamation of W_3 is $\mathfrak{pr}\left(Amal(W_3, v_0, t)\right) = 4t + 1$, for any positive integer t.

Proof. According to Lemma 2.3, we left prove that $\mathfrak{pr}\left(Amal(W_3, v_0, t)\right) \geq 4t + 1$, by defining a labeling on the $Amal(W_3, v_0, t)$. Define $f: V\left(Amal(W_3, v_0, t)\right) \rightarrow \{1, 2, \dots, 4t+1\}$ where $f(v_0) = 1$ and for $i = 1, 2, \dots, t$,

$$f(v_{ij}) = \begin{cases} 2i, & \text{for } j = 1, \\ 4i - 1, & \text{for } j = 2, \\ 4i + 1, & \text{for } j = 3. \end{cases}$$

Now, we show that the greatest common divisor of every two labels of adjacent vertices is one.

- First, $\gcd\left(f(v_{i1}), f(v_{i2})\right) = (2i, 4i-1) = 1$, since there exists $x = 2$ and $y = -1$ so that $(2i)x + (4i-1)y = 1$ and applying Theorem 2.1.
- Suppose $\gcd\left(f(v_{i2}), f(v_{i3})\right) = (4i-1, 4i+1) = k$. Since $4i-1$ is odd, and $|(4i-1) - (4i+1)| = 2$, by Lemma 2.1, $\gcd(4i-1, 4i+1) = 1$.
- Suppose $\gcd\left(f(v_{i1}), f(v_{i3})\right) = \gcd(2i, 4i+1) = k$. There exists $x = -2$ and $y = 1$ so that $(2i)x + (4i+1)y = 1$. By Theorem 2.1, we get $\gcd(2i, 4i+1) = 1$.

Thus every label of adjacent vertices is relatively prime. Hence the function f is a minimum coprime labeling and $\mathfrak{pr}\left(Amal(W_3, v_0, t)\right) = 4t + 1$. ■

Proposition 2.6 Let v_0 be the central vertex of wheel W_7. For a positive integer $t \leq 47$, an $Amal(W_7, v_0, t)$ has a minimum coprime labeling with the largest label being $8t + 1$.

Proof. Let t be positive integer and $t \leq 47$. Define $f: V\left(Amal(W_7, v_0, t)\right) \longrightarrow \{1, 2, \dots, 8t+1\}$, where $f(v_0) = 1$ and for $i = 1, 2, \dots, 47$, we consider two cases, namely

For $i \neq 6\ (mod\ 7)$,

$$f(v_{ij}) = \begin{cases} 8i + j - 7, \text{for } j = 1, 2, \dots, 6, \\ 8i + 1, \quad \text{for } j = 7, \end{cases}$$

while for $i = 6\ (mod\ 7)$,

$$f(v_{ij}) = \begin{cases} 8\left(\frac{i+1}{7}\right), & \text{for } j = 1, \\ 8i + j - 7, & \text{for } j = 2, 3, \dots, 6, \\ 8i + 1, & \text{for } j = 7. \end{cases}$$

Now, we show that the greatest common divisor of every label of two adjacent vertices are one.

- By Lemma 2.2, we obtain $\gcd\left(f(v_{i6}), f(v_{i7})\right) = (8i-1, 8i+1) = 1$.
- Now, we consider the vertex v_{i1} and v_{i7}. For $i = 6\ (mod\ 7)$, namely $i = 6, 13, 20, 27, 34, 41$, $\gcd\left(f(v_{i1}), f(v_{i7})\right) = \gcd\left(8\left(\frac{i+1}{7}\right), 8i+1\right) = 1$. For $i \neq 6\ (mod\ 7)$, suppose for a contradiction, $\gcd\left(f(v_{i1}), f(v_{i7})\right) = \gcd(8i-6, 8i+1) = k \neq 1$. We know that $8i - 6 = 2\ (mod\ 4)$ is even and $8i + 1 = 1\ (mod\ 4)$ is odd. So k must be odd. Let $8i - 6 = kx$ and $8i + 1 = ky$. We get $k(y-x) = 7$. Therefore $k = 7$. Consequently, $i = 6\ (mod\ 7)$, a contradiction. Thus $\gcd(8i-6, 8i+1) = 1$.
- For $i = 6\ (mod\ 7)$, namely $i = 6, 13, 20, 27, 34, 41$, we can easily count that $\gcd\left(f(v_{i1}), f(v_{i2})\right) = \gcd\left(8\left(\frac{i+1}{7}\right), 8i-5\right) = 1$.

Thus, every label of two adjacent vertices is relatively prime. Thus any integer $t \leq 47$, the $Amal(W_7, v_0, t)$ has a minimum coprime labeling with the largest label being $\mathfrak{pr}\left(Amal(W_7, v_0, t)\right) = 8t + 1$. ■

For an illustration, a minimum coprime labeling for $Amal(W_3, v_0, 3)$ and $Amal(W_7, v_0, 6)$ as depicted in Figure 2 and Figure 3, respectively.

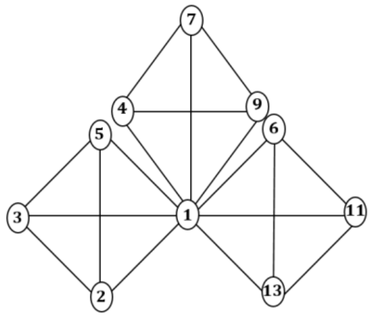

Figure 2. A coprime labeling of $Amal(W_3, v_0, 3)$

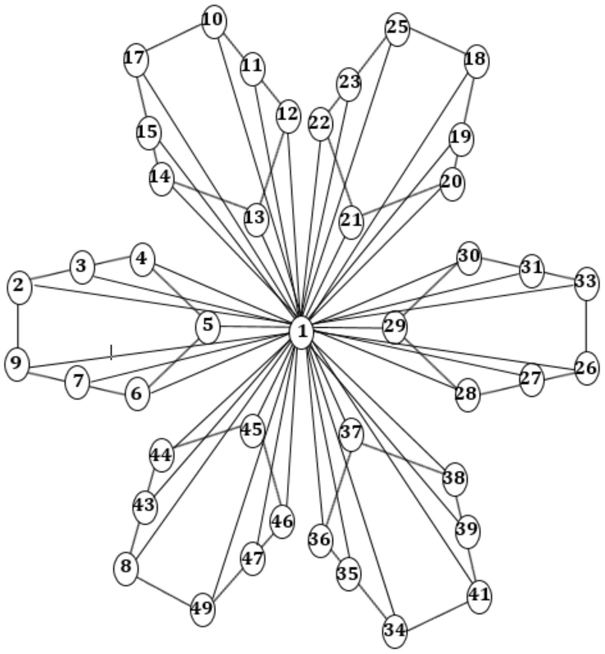

Figure 3. A coprime labeling of $Amal(W_7, v_0, 6)$

Proposution 2.7 Let v_0 be the central vertex of wheel W_{11}. For each integer $t > 1$, the minimum coprime number for amalgamation of t copies of W_{11} is $\mathfrak{pr}(Amal(W_{11}, v_0, t)) = 12t + 1$.

Proof. According to Lemma 2.1, we left prove that $\mathfrak{pr}(Amal(W_{11}, v_0, t)) \geq 12t + 1$, by defining a labeling on the $Amal(W_{11}, v_0, t)$. Define $f: V(Amal(W_{11}, v_0, t)) \to \{1, 2, \dots, 12t + 1\}$ where

$$f(v_0) = 1$$

$$f(v_{ij}) = \begin{cases} 12i - (11 - j), & \text{for } j = 1,2,\dots,9, \\ 12i + 1, & \text{for } j = 10, \\ 12i - 1, & \text{for } j = 11. \end{cases}$$

Now, we show that the greatest common divisor of every label of any two adjacent vertices is one.

- Suppose that $\gcd(f(v_{i9}), f(v_{i10})) = \gcd(12i - 2, 12i + 1) = k$. We know that $12i - 2$ is even, $12i + 1$ is odd, but both of them are in $1(mod\ 3)$. So $k \neq 3$. Let $12i - 2 = kx$ and $12i + 1 = ky$. Then $k(y - x) = 3 = 1 \cdot 3$. Hence $k = 1$. Therefore labels of v_{i9} and v_{i10} are relatively prime.
- By applying Lemma 2.2, we obtain $\gcd(f(v_{i10}), f(v_{i11})) = \gcd(12i + 1, 12i - 1) = 1$.
- Last, suppose that $\gcd(f(v_{i1}), f(v_{i11})) = \gcd(12i - 10, 12i - 1) = k$. We know that $12i - 10$ is even, and $12i - 1$ is odd, but both of them are in $2\ (mod\ 3)$. So, $k \neq 3$. Suppose $12i - 10 = kx$ and $12i - 1 = ky$. Then $k(y - x) = 9 = 1 \cdot 9$. Since neither $12i - 10$ nor $12i - 1$ is not in $0\ (mod\ 9)$, then it must be $k = 1$. Thus $f(v_{i1})$ and $f(v_{i11})$ are relatively prime, that is $\gcd(12i - 10, 12i - 1) = 1$.

Due to any two adjacent vertices having the relatively prime labels, the function f is a minimum coprime labeling with the largest label being $\mathfrak{pr}(Amal(W_{11}, v_0, t)) = 12t + 1$. ■

For an illustration, a coprime labeling for amalgamation of 3 copies of W_{11} can be seen in Figure 4.

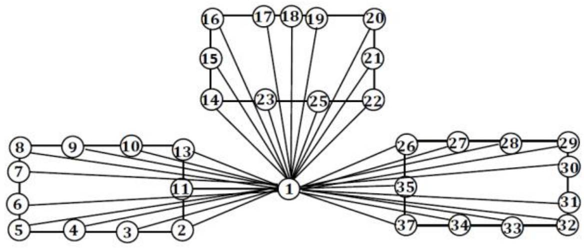

Figure 4. A coprime labeling of $\boldsymbol{Amal(W_{11}, v_0, 3)}$

3. CONCLUDING REMARKS

We conclude this paper by providing several open questions regarding minimum coprime numbers.

Question 1. Let v_0 be the vertex of degree 7 in W_7. Can the minimum coprime labeling be defined for amalgamation of wheel W_7, $Amal(W_7, v_0, t)$, for any positive integer t?

Question 2. Let v_0 be the vertex of degree n in W_n, and t be a positive integer. Can the minimum coprime labeling be defined generally for amalgamation of odd wheel W_n, $Amal(W_n, v_0, t)$ when $n = 3\ (mod\ 4)$?

Question 3. Can the minimum coprime number be determined for amalgamation of complete graph, $Amal(K_n, v, t)$?

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