Local Antimagic Vertex Coloring of Corona Product Graphs $P_n \circ P_k$

Setiawan, Kiki Ariyanti Sugeng*

Department of Mathematics, Faculty of Mathematics and Sciences
Universitas Indonesia, Depok 16424, Indonesia
*Corresponding author. *Email: kiki@sci.ui.ac.id*

ABSTRACT

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A bijection map $f: E \rightarrow \{1,2, \ldots, |E|\}$ is called a local antimagic labeling if, for any two adjacent vertices u and v, they have different vertex sums, i.e. $w(u) \neq w(v)$, where the vertex sum $w(u) = \Sigma_{e \in E(u)} f(e)$, and $E(u)$ is the set of edges incident to u. Thus, any local antimagic labeling induces a proper vertex coloring of G where the vertex v is assigned the color (vertex sum) $w(v)$. Let G and H be two graphs. The Corona product $G \circ H$ is obtained by taking one copy of G along with $|V(G)|$ copies of H, and via putting extra edges making the i^th^ vertex of G adjacent to every vertex of the i^th^ copy of H, where $1 \leq i \leq |V(G)|$. The local antimagic chromatic number, denoted $\mathcal{X}_{la}(G)$, is the minimum number of colors taken over all colorings induced by local antimagic labelings of G. In this paper, we present the local antimagic chromatic number $\mathcal{X}_{la}(P_n \circ P_k)$ for the corona product of path P_n and P_k where k is a small number.

Keywords: *Antimagic labeling, Local antimagic labeling, Local antimagic chromatic number, Corona product graph, Path.*

1. INTRODUCTION

Graph theory is a part of discrete mathematics that is used as a device to describe and state a problem to be more easily understood and solved. A graph $G = (V, E)$ with $V(G)$ and $E(G)$ is respectively set of vertices and set of edges on graph G. Number of vertices of G denoted as $|V(G)| = p$ and number of the edges of G are denoted as $|E(G)| = q$ [1].

Labeling on a graph is a mapping from each graph element to a positive number, which it is called a label. If the function domain is vertex set (or edge set), so the labeling is called vertex labeling (or edge labeling) [2].

Let G be a simple undirected graph so that it does not have an isolated vertex. Let $f : E \rightarrow \{1, 2, \ldots, |E|\}$ be a bijective mapping. Weight of each vertex $u \in V(G)$, $w(u) = \sum_{e \in E(u)} f(e)$, in which $E(u)$ is an edge set that incident to vertex u. If $w(u) \neq w(v)$ for every two different vertices u and $v \in V(G)$, then f is called antimagic labeling from G. Graph G is called as antimagic if G has antimagic labeling [3].

Furthermore, f is called local antimagic labeling if $w(u) \neq w(v)$ only for every two adjacent vertices, u and $v \in V(G)$. Thus, it is clear that a local antimagic labeling must be an antmagic and induce proper vertex coloring of G where the vertex v is assigned the color (vertex sum) $w(v)$ [3].

Two graphs can be operated by various operations, such as joint operation$(G + H)$, Cartesian Product $(G \times H)$, Corona Product $(G \circ H)$, Tensor Product, etc. In this paper we discuss a corona product of two paths, $P_n \circ P_k$ with k is limited to k = 1,3, 5. The corona product of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G along with |V (G)| copies of H, and via putting extra edges making the *i*^th^ vertex of G adjacent to every vertex of the *i*^th^ copy of H, where $1 \leq$ i $\leq$ |V(G)| [4].

In the paper [4], Arumugam et al. presented the local antimagic chromatic number $\mathcal{X}_{la}(G \circ \overline{K_m})$ for the corona product of a graph G with the null graph $\overline{K_m}$ on m $\geq$ 1 vertices, when G is a path P_n, a cycle C_n, and a complete graph K_n, as follows:

- $\mathcal{X}_{la}(P_n \circ K_1) = n + 2$, for $n \geq 4$.
- $\mathcal{X}_{la}(P_n \circ K_m) = mn + 2$,for $m, n \geq 2$
- $\mathcal{X}_{la}(C_n \circ K_1) = n + 2$, for $n \geq 4$.

- $\mathcal{X}_{la}(C_n \circ K_m) = mn + 3$, f or $n \geq 5$, *dan* $3 \leq m \leq 6$
- $\mathcal{X}_{la}(K_n \circ K_1) = 2n - 1$, for $n \geq 3$.
  $\mathcal{X}_{la}(K_n \circ K_m) = mn + 2$, for $m \geq 2, n \geq 3$,

In this paper, we present the local antimagic chromatic number for corona product of graph $P_n \circ P_k$ with k is odd and $k \leq 5$.

2. RESULT AND DISCUSSION

2.1 Local Antimagic Chromatic Number on graph $P_n \circ P_1$

Graph $P_n \circ P_1$ (Figure 1) is a graph that has a vertex set $V = \{u_i\, ; 1 \leq i \leq n\} \cup \{v_i\, ; 1 \leq i \leq n\}$ and edge set $E = \{u_i u_{i+1}\, ; 1 \leq i \leq n-1\} \cup \{u_i v_i\, ;\ 1 \leq i \leq n\}$ and $|V| = 2n$ and $|E| = 2n - 1$.

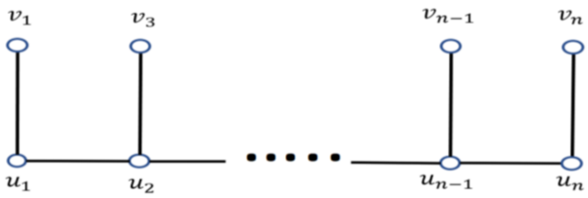

Figure 1 Graph $P_n \circ P_1$.

The local antimagic vertex coloring on corona product of graph $P_n \circ P_1$ is the same as the $P_n \circ K_1$ graph that has been studied by [3]. The Illustration of the coloring can be seen in Figure 2, and the theorem is as follows.

Theorem 2.1 $\mathcal{X}_{la}(P_n \circ P_1) = \mathcal{X}_{la}(P_n \circ K_1) = n + 2,\ for\ n \geq 4$

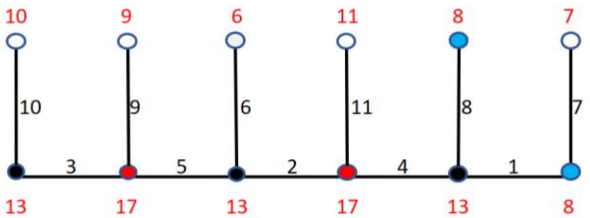

Figure 2 Coloring Graf $P_6 \circ P_1$.

2.2 Local Antimagic Chromatic Number on graph $P_n \circ P_3$

Graph $P_n \circ P_3$ (Figure 3) is a graph that has vertex set $V = \{u_i\, ; 1 \leq i \leq n\} \cup \{v_j^i\, ; 1 \leq i \leq n\, , 1 \leq j \leq 3\}$ and edge set $E = \{u_i u_{i+1}\, ; 1 \leq i \leq n-1\} \cup \{u_i v_j^i\, ;\ 1 \leq i \leq n\, , 1 \leq j \leq 3\} \cup \{v_j^i v_{j+1}^i\, ;\ 1 \leq i \leq n\, , 1 \leq j \leq 2\}$ and $|V| = 4n$ and $|E| = 6n - 1$.

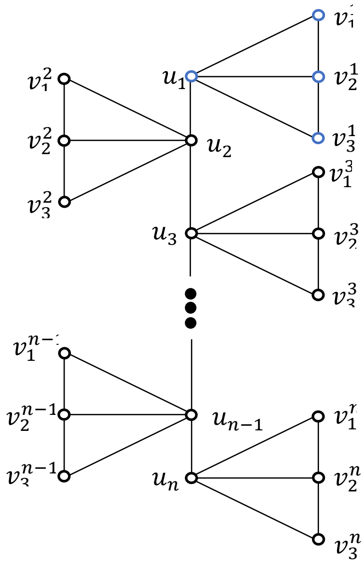

Figure 3 Graph $P_n \circ P_3$.

Lemma 2.2.1 $\mathcal{X}_{la}(P_n \circ P_3) \leq 6$ for $n \geq 4$

Proof:

Case 1 : $n \equiv 0\ (mod\ 2)$

1. For case $1 \leq i \leq n-1$, of course odd i lives in $1 \leq i \leq n-1$ and even i lives in $2 \leq i \leq n-2$, then label the edges $u_i u_{i+1}$, as follows:

$$f(u_i u_{i+1}) = \begin{cases} 4n + i + 1\, , i \text{ odd}, \\ 5n + i, \qquad i \text{ even}. \end{cases}$$

2. For case $1 \leq i \leq n$, of course odd i lives in $1 \leq i \leq n-1$ and even i lives in $2 \leq i \leq n$, then:

   a. Label the edges $u_i v_1^i$,, as follows:

$$f(u_i v_1^i) = \begin{cases} 4n - \frac{i-1}{2}\, , i \text{ odd}, \\ \frac{7n-i}{2} + 1, i \text{ even}. \end{cases}$$

   b. Label the edges $u_i v_2^i$, as follows:

$$f(u_i v_2^i) = \begin{cases} 6n - i\, , \qquad i \text{ odd}, \\ 5n - i + 1, i \text{ even}. \end{cases}$$

   c. Label the edges $u_i v_3^i$, as follows:

$$f(u_i v_3^i) = \begin{cases} 3n - \dfrac{i-1}{2}\, , i \text{ odd}, \\ \dfrac{5n-i}{2} + 1, i \text{ even}. \end{cases}$$

   d. Label the edges $v_1^i v_2^i$, as follows:

$$f(v_1^i v_2^i) = \begin{cases} \dfrac{i+1}{2}\, , i \text{ odd}, \\ \dfrac{n+i}{2}\, , i \text{ even}. \end{cases}$$

   e. Label the edges $v_2^i v_3^i$, as follows:

$$f(v_2^i v_3^i) = \begin{cases} n + \dfrac{i+1}{2}, i \text{ odd}, \\ n + \dfrac{n+i}{2}, i \text{ even}. \end{cases}$$

The illustration of the edge labeling is shown in Figure 4.

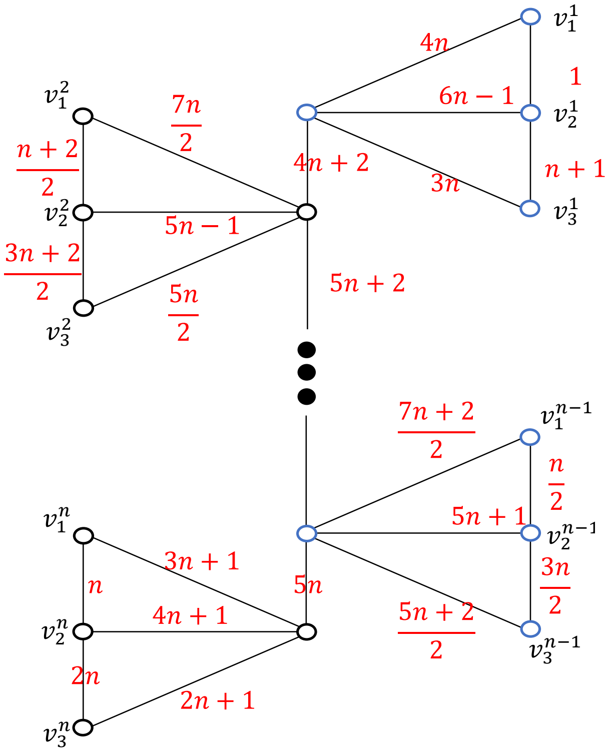

Figure 4 The edge labeling of $P_n \circ P_3$ for $n \equiv 0 \ (mod\ 2)$

The weight of each vertex is the sum of the label of edges incident to that vertex, as follow:

1. $w(v_1^i) = 4n+1, \quad 1 \le i \le n,$
2. $w(v_2^i) = 7n+1, \quad 1 \le i \le n,$
3. $w(v_3^i) = 4n+1, \quad 1 \le i \le n,$
4. $w(u_i) = \begin{cases} 20n+3, & 2 \le i \le n, i \text{ even}, \\ 22n+1, & 1 \le i \le n-1, i \text{ odd}, \end{cases}$
5. $w(u_1) = 17n+1,$
6. $w(u_n) = 14n+3.$

Therefore, it is obtained, $\mathcal{X}_{la}(P_n \circ P_3) \le 6$, for $n \equiv 0 \ (mod\ 2), n \ge 4.$

Case 2 : $n \equiv 1 \ (mod\ 2)$

1. For case $1 \le i \le n-1$, of course odd i lives in $1 \le i \le n-2$ and even i lives in $2 \le i \le n-1$, then label the edges $u_i u_{i+1}$, as follows:

$$f(u_i u_{i+1}) = \begin{cases} 4n+i+1, i \text{ odd}, \\ 5n+i-1, i \text{ even}. \end{cases}$$

2. For case $1 \le i \le n$, of course odd i lives in $1 \le i \le n$ and even i lives in $2 \le i \le n-1$, then:

   a. Label the edges $u_i v_1^i$, as follows:

$$f(u_i v_1^i) = \begin{cases} 4n - \dfrac{i-1}{2}, & i \text{ odd}, \\ \dfrac{7n-i-1}{2} + 1, i \text{ even}. \end{cases}$$

   b. Label the edges $u_i v_2^i$, as follows:

$$f(u_i v_2^i) = \begin{cases} 6n-i, i \text{ odd}, \\ 5n-i, i \text{ even}. \end{cases}$$

   c. Label the edges $u_i v_3^i$, as follows:

$$f(u_i v_3^i) = \begin{cases} 3n - \dfrac{i-1}{2}, & i \text{ odd}, \\ \dfrac{5n-i-1}{2} + 1, i \text{ even}. \end{cases}$$

   d. Label the edges $v_1^i v_2^i$, as follows:

$$f(v_1^i v_2^i) = \begin{cases} \dfrac{i+1}{2}, & i \text{ odd}, \\ \dfrac{n+i+1}{2}, i \text{ even}. \end{cases}$$

   e. Label the edges $v_2^i v_3^i$, as follows:

$$f(v_2^i v_3^i) = \begin{cases} n + \dfrac{i+1}{2}, & i \text{ odd}, \\ n + \dfrac{n+i+1}{2}, i \text{ even}. \end{cases}$$

The weight of each vertex is the sum of the label of edges incident to that vertex, as follow:

1. $w(v_1^i) = 4n+1, 1 \le i \le n,$
2. $w(v_2^i) = 7n+1, 1 \le i \le n,$
3. $w(v_3^i) = 4n+1, 1 \le i \le n,$
4. $w(u_i) = \begin{cases} 20n, & 1 \le i \le n, i \text{ odd}, \\ 22n, & 2 \le i \le n-1, i \text{ even}, \end{cases}$
5. $w(u_1) = 17n+1,$
6. $w(u_n) = 17n-1.$ ■

Lemma 2.2.2 $\mathcal{X}_{la}(P_n \circ P_3) \ge 6, for\ n \ge 4.$

Proof: Let $\mathcal{X}(G)$ be the usual chromatic number of a graph G. For any graph G, $\mathcal{X}_{la}(G) \ge \mathcal{X}(G)$. [5] If F_n is fan graph, then $\mathcal{X}(F_n) = 3$. [6]

Let $V(P_n \circ P_3) = \{u_i\,; 1 \le i \le n\} \cup \{v_j^i\,; 1 \le i \le n, 1 \le j \le 3\}$ and $E = \{u_i u_{i+1}\,; 1 \le i \le n-1\} \cup \{u_i v_j^i\,;\ 1 \le i \le n,\ 1 \le j \le 3\} \cup \{v_j^i v_{j+1}^i;\ 1 \le i \le n, 1 \le j \le 2\}$. It will be shown that $\mathcal{X}_{la}(P_n \circ P_3) \ge 6$ i.e., by showing that $\mathcal{X}_{la}(P_n \circ P_3) \ne 5$.

The graph $P_n \circ P_3$ has the form of n fan graph(F_n) that each center of the fan is a vertex of a path P_n. According to [2], $\mathcal{X}(F_3) = 3$, i.e. $w(v_1^i) = w(v_3^i) \neq w(v_2^i) \neq w(u_i)$ or needed 2 colors on vertex v_j^i, and 1 color on vertex u_i, for i=1,2,3 and j=1,2,…,n.

Suppose $\mathcal{X}_{la}(P_n \circ P_3) = 5$. Since $\{v_j^i ; 1 \le i \le n, 1 \le j \le 3\}$ has 2 colors, then $\{u_i u_{i+1} ; 1 \le i \le n-1\}$ has 3 colors, namely $w(u_i) = w(v_2^{i+1})$ (for i odd) $w(u_i) = w(v_2^{i+1})$(for i even) and $w(u_n)$ [3].

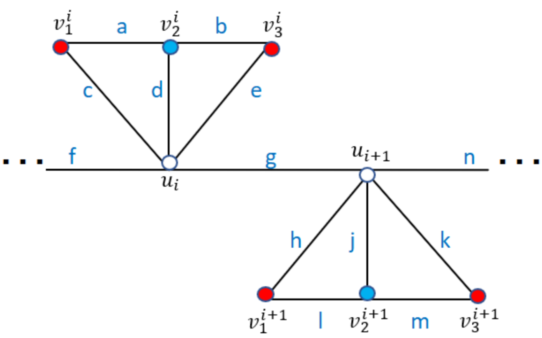

Figure 5 Coloring $w(u_i)$ *and* $w(v_2^{i+1})$.

See the illustration in Figure 5. Then for a certain value of i, we obtain the weight of vertices u_i and v_2^{i+1} [7], as follows :

$w(u_i) = c + d + e + f + g$, for i odd

$$w(u_{i+1}) = g + h + j + k + n$$
$$w(u_i) - w(u_{i+1}) = c + d + e + f - h - j - k - n$$
$$= c + d + e + f - h - (w(v_2^{i+1}) - l - m) - k - n$$
$$w(u_i) + w(v_2^{i+1}) = \alpha,\ for\ \alpha = w(u_{i+1}) + c + d + e + f - h + l + m - k - n$$

Thus, $w(u_i) \neq w(v_2^{i+1})$, contradiction. Similar for $w(u_i)$, for i even. So that $\mathcal{X}_{la}(P_n \circ P_3) \neq 5$. ■

Theorem 2.2 $\mathcal{X}_{la}(P_n \circ P_3) = 6$, for $n \ge 4$

Proof:

Based on Lemma 2.2.1 and Lemma 2.2.2, it is proven that $\mathcal{X}_{la}(P_n \circ P_3) = 6, for\ n \ge 4$.

■An example of the local antimagic coloring of graph $P_6 \circ P_3$, is shown in Figure 6.

2.3 *Local Antimagic Chromatic Number on graph* $P_n \circ P_5$

Graph $P_n \circ P_5$ (Figure 9) is a graph that has vertex set $V = \{u_i ; 1 \le i \le n\} \cup \{v_j^i ; 1 \le i \le n, 1 \le j \le 5\}$ and edge set $E = \{u_i u_{i+1} ; 1 \le i \le n-1\} \cup \{u_i v_j^i ; 1 \le i \le n, 1 \le j \le 5\} \cup \{v_j^i v_{j+1}^i ; 1 \le i \le n, 1 \le j \le 4\}$ and $|V| = 6n$ and $|E| = 10n - 1$.

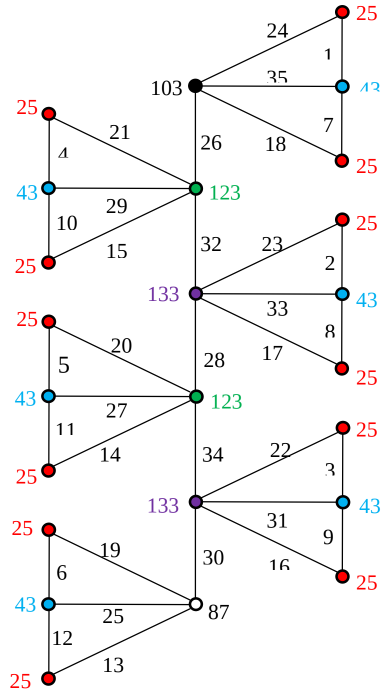

Figure 6 *Graph* $P_6 \circ P_3$.

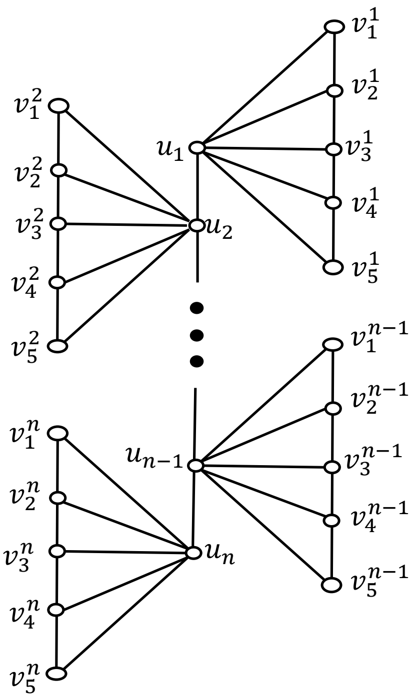

Figure 7 *Graph* $P_n \circ P_5$.

Lemma 2.3.1 $\mathcal{X}_{la}(P_n \circ P_5) \le 7$ for $n \ge 5$

Proof:

Case 1: $\boldsymbol{n \equiv 1\ (mod\ 2)}$

1. For case $1 \le i \le n$, of course odd i lives in $1 \le i \le n$ and even i lives in $2 \le i \le n-1$, then:

a. Label the edges $v_1^i v_2^i$, as follows:

$$f(v_1^i v_2^i) = \begin{cases} \frac{i+1}{2}, & i \text{ odd}, \\ \frac{n}{2} + \frac{i+1}{2}, & i \text{ even}. \end{cases}$$

b. Label the edges $v_2^i v_3^i$, as follows:

$$f(v_2^i v_3^i) = \begin{cases} 2n + \frac{i+1}{2}, & i \text{ odd}, \\ \frac{5n}{2} + \frac{i+1}{2}, & i \text{ even}. \end{cases}$$

c. Label the edges $v_3^i v_4^i, 1 \le i \le n$, as follows:

$$f(v_3^i v_4^i) = \begin{cases} 3n + \frac{i+1}{2}, & i \text{ odd}, \\ \frac{7n}{2} + \frac{i+1}{2}, & i \text{ even}. \end{cases}$$

d. Label the edges $v_4^i v_5^i$, as follows:

$$f(v_4^i v_5^i) = \begin{cases} n + \frac{i+1}{2}, & i \text{ odd}, \\ \frac{3n}{2} + \frac{i+1}{2}, & i \text{ even}. \end{cases}$$

e. Label the edges $u_i v_1^i$, as follows:

$$f(u_i v_1^i) = \begin{cases} 10n - \frac{i+1}{2}, & i \text{ odd}, \\ \frac{19n}{2} - \frac{i+1}{2}, & i \text{ even}. \end{cases}$$

f. Label the edges $u_i v_2^i$, as follows:

$$f(u_i v_2^i) = \begin{cases} 8n - i, & i \text{ odd}, \\ 7n - i, & i \text{ even}. \end{cases}$$

g. Label the edges $u_i v_3^i$, as follows:

$$f(u_i v_3^i) = \begin{cases} 6n - i + 1, & i \text{ odd}, \\ 5n - i + 1, & i \text{ even}. \end{cases}$$

h. Label the edges $u_i v_4^i$, as follows:

$$f(u_i v_4^i) = \begin{cases} 6n - i, & i \text{ odd}, \\ 5n - i, & i \text{ even}. \end{cases}$$

i. Label the edges $u_i v_5^i$, as follows:

$$f(u_i v_5^i) = \begin{cases} 9n - \frac{i+1}{2}, & i \text{ odd}, \\ \frac{17n}{2} - \frac{i+1}{2}, & i \text{ even}. \end{cases}$$

2. For case $1 \le i \le n-1$, of course odd i lives in $1 \le i \le n-2$ and even i lives in $2 \le i \le n-1$, then label the edges $u_i u_{i+1}$ is $f(u_i u_{i+1}) = 6n + 2i$.

The weight of each vertex is the sum of the label of edges incident to that vertex, as follow:

1. $w(v_1^i) = w(v_5^i) = 10n, \ 1 \le i \le n,$
2. $w(v_2^i) = w(v_4^i) = 10n + 1, \ 1 \le i \le n,$
3. $w(v_3^i) = 11n + 2, \ 1 \le i \le n,$
4. $w(u_1) = 45n - 2,$
5. $w(u_i) = \begin{cases} 47n - 2, \ 2 \le i \le n-1, i \text{ even}, \\ 51n - 2, \ 3 \le i \le n-2, \ i \text{ odd}, \end{cases}$
6. $w(u_n) = 43n - 2.$ ■

Case 2: $n \equiv 0 \ (mod\ 2)$

1. For case $1 \le i \le n$, of course odd i lives in $1 \le i \le n-1$ and even i lives in $2 \le i \le n$, then:

a. Label the edges $v_1^i v_2^i$, as follows:

$$f(v_1^i v_2^i) = \begin{cases} \frac{i+1}{2}, & i \text{ odd}, \\ \frac{n+i}{2}, & i \text{ even}. \end{cases}$$

b. Label the edges $v_2^i v_3^i$, as follows:

$$f(v_2^i v_3^i) = \begin{cases} 2n + \frac{i+1}{2}, & i \text{ odd}, \\ \frac{5n+i}{2}, & i \text{ even}. \end{cases}$$

c. Label the edges $v_3^i v_4^i$, as follows:

$$f(v_3^i v_4^i) = \begin{cases} 3n + \frac{i+1}{2}, & i \text{ odd}, \\ \frac{7n+i}{2}, & i \text{ even}. \end{cases}$$

d. Label the edges $v_4^i v_5^i$, as follows:

$$f(v_4^i v_5^i) = \begin{cases} n + \frac{i+1}{2}, & i \text{ odd}, \\ \frac{3n+i}{2}, & i \text{ even}. \end{cases}$$

e. Label the edges $u_i v_1^i$, as follows:

$$f(u_i v_1^i) = \begin{cases} 10n - \frac{i+1}{2}, & i \text{ odd}, \\ \frac{19n-i}{2}, & i \text{ even}. \end{cases}$$

f. Label the edges $u_i v_2^i$, as follows:

$$f(u_i v_2^i) = \begin{cases} 8n - i, & i \text{ odd}, \\ 7n - i + 1, & i \text{ even}. \end{cases}$$

g. Label the edges $u_i v_3^i$, as follows:

$$f(u_i v_3^i) = \begin{cases} 6n - i + 1, & i \text{ odd}, \\ 5n - i + 2, & i \text{ even}. \end{cases}$$

h. Label the edges $u_iv_4^i$, as follows:

$$f(u_iv_4^i) = \begin{cases} 6n - i, & i \text{ odd}, \\ 5n - i + 1, & i \text{ even}. \end{cases}$$

i. Label the edges $u_iv_5^i$, as follows:

$$f(u_iv_5^i) = \begin{cases} 9n - \dfrac{i+1}{2}, & i \text{ odd}, \\ \dfrac{17n - i}{2}, & i \text{ even}. \end{cases}$$

2. For case $1 \le i \le n-1$, of course odd i lives in $1 \le i \le n-2$ and even i lives in $2 \le i \le n-1$, then label the edges u_iu_{i+1} is $f(u_iu_{i+1}) = 6n + 2i$.

The weight of each vertex is the sum of the label of edges incident to that vertex, as follow:

1. $w(v_1^i) = w(v_5^i) = 10n, \ \ 1 \le i \le n,$
2. $w(v_2^i) = w(v_4^i) = 10n + 1, \ \ 1 \le i \le n,$
3. $w(v_3^i) = 11n + 2, \ \ 1 \le i \le n,$
4. $w(u_1) = 45n - 2,$
5. $w(u_i) = \begin{cases} 51n - 2, & 3 \le i \le n-1, i \text{ odd}, \\ 47n + 2, & 2 \le i \le n-2, i \text{ even}. \end{cases}$
6. $w(u_n) = 39n + 2.$ ■

An example of the local antimagic coloring of graph $P_5 \circ P_5$ is shown in Figure 8.

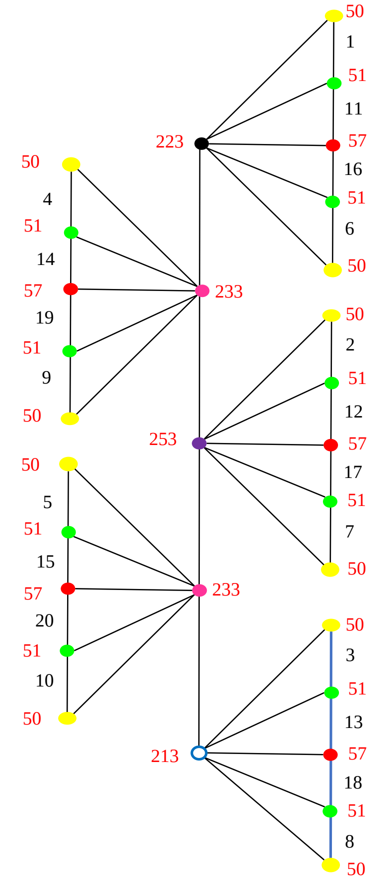

Figure 8 Graph $P_5 \circ P_5$.

3. CONCLUSION

The chromatic number of local anti-magic vertex coloring $P_n \circ P_k, for\ k = 1,3,5$ are $\mathcal{X}_{la}(P_n \circ P_1) = n + 2,\ for\ n \ge 4,\ \mathcal{X}_{la}(P_n \circ P_3) = 6\ for\ n \ge 4$ and $\mathcal{X}_{la}(P_n \circ P_5) \le 7$, for $n \ge 5$

For further research, we can look at the case of k odd and $k > 5$ and k even.

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