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Spectrum of Unicyclic Graph

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ABSTRACT
Let G be a simple graph with n vertices and let A(G) be the (0, 1)-adjacency matrix of G. The characteristic polynomial of the graph G with respect to the adjacency matrix A(G), denoted by $\chi(G,\lambda)$ is a determinant of $\big(\lambda I - A(G)\big)$, where I is the identity matrix. Suppose that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the adjacency eigenvalues of G. The spectrum of the graph G, denoted by Spec(G), is the multiset of its adjacency eigenvalues. Unicyclic graph is connected graph containing exactly one cycle. In this paper we determine the spectrum of unicyclic graph containing cycle with length 6.

***Keywords**: Characteristic polynomial, Spectrum of the graph, Unicyclic graph.*

1. INTRODUCTION

Let G be any graph with the vertex set V(G) and the edge set E(G). The order of G, written as $|V(G)| = n$ denotes the number of vertices of G. Let v ∈ V(G), the degree of a vertex v, denoted by $d(v)$, is the number of edges incident with v. The largest degree among the vertices of G is called the maximum degree of G and is denoted by Δ(G). The minimum degree of G is denoted by δ(G).

For an integer $n \geq 3$, the cycle C_n is a graph of order n and size n whose vertices can be labeled by $v_1, v_2, v_3, \ldots, v_n$ and whose edges are $v_1 v_n$ and $v_i v_{i+1}$ for $i = 1, 2, 3, \ldots, n-1$. A unicyclic graph is a connected graph containing exactly one cycle.

Let A(G) be the adjacency matrix of G. The characteristic polynomial of G is the characteristic polynomial of A(G), denoted by $\chi(G,\lambda) = \det\big(\lambda I - A(G)\big)$, where I is the identity matrix. The eigenvalues of G, denoted by $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$, are the eigenvalues of A(G), arranged in a non-increasing order, where n is the order of G. The spectrum of a graph G, denoted by $Spec(G)$ is the set of eigenvalues of A(G) together with their multiplicities.

A graph spectral theory was first introduced by L. Collatz and U. Sinogowitz in 1957. Many researches have been done in the field of spectral graph theory. The results obtained are still limited to several classes of graphs such as complete graphs, path graphs, cycle graphs, bipartite graphs, and some other simple graphs, see [1 – 4]. In particular for unicyclic graph, G.H. Xu [5] studied the spectral characterization of unicyclic graphs. They obtained that the unicyclic graph G satisfies $\lambda_2 = 1$ if and only if G is cycle C_6 or the unicyclic graphs in Figure 1 and the unicyclic graph that satisfies $\lambda_2 < 1$ are C_4, C_5, or one of the graphs in Figure 2.

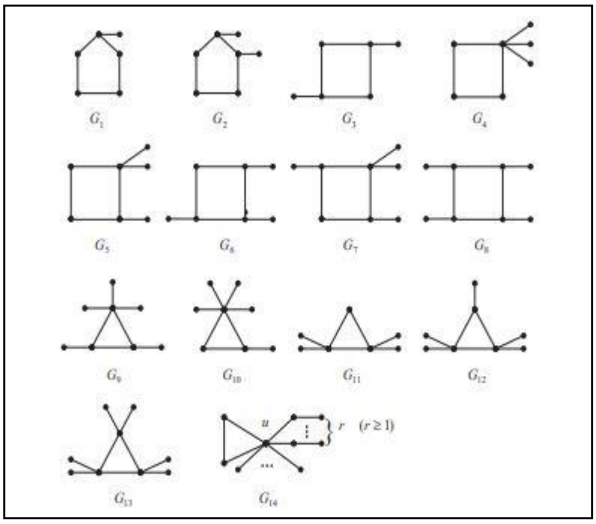

Figure 1 Unicyclic graph with $\lambda_2 = 1$.

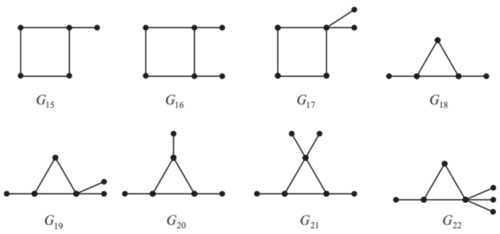

Figure 2 Unicyclic graph with $\lambda_2 < 1$.

Then these results are continued by Xiaoling, Qiongxiang, Fenjin [6]. They showed that the unicyclic graphs in [5] are not co-spectral. In 2020, Song Haizhou and Tian Lulu study the properties and structure of the maximal-adjacency-spectrum unicyclic graphs with given maximum degree. In this paper, we focus on finding a spectrum of unicyclic graph containing cycles of length 6.

2. MAIN RESULTS

In this section, we will discuss the characteristic polynomial and the spectrum of a unicyclic graph containing cycles of length 6. A unicyclic graph containing cycles of length 6 denoted by C_6^n, $n > 0$, where *n* represent the number of pendant vertices in the unicyclic graph. In this study, the unicyclic graphs containing cycles of length 6 are shown in Figure 3.

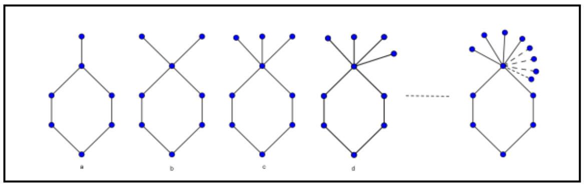

Figure 3 Unicyclic graphs $C_6^1, C_6^2, C_6^3, C_6^4$, and C_6^n.

Theorem 2.1 The characteristic polynomial of a unicyclic graph C_6^n is

$$\chi(C_6^n, \lambda) = \lambda^n(\lambda-1)(\lambda+1)(\lambda^4-(n+5)\lambda^2+(2\Delta+n))$$

Proof. The adjacency matrix of a unicyclic graph C_6^n as follows

$$A(C_6^n) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

First, we determine $(A(C_6^n) - \lambda I)$. We get:

$$A(C_6^n) - \lambda I = \begin{pmatrix} -\lambda & 1 & 0 & 0 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & -\lambda & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -\lambda & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -\lambda & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -\lambda & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 1 & -\lambda & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\lambda \end{pmatrix}$$

By eliminating the matrix using the Gauss Elimination method on software of maple 18, we get an upper triangular matrix as follows.

$$\begin{pmatrix} -\lambda & 1 & 0 & 0 & 0 & 1 & 1 & \cdots & \cdots & \cdots & 1 \\ 0 & -\frac{\lambda^2-1}{\lambda} & 1 & 0 & 0 & \frac{1}{\lambda} & \frac{1}{\lambda} & \cdots & \cdots & \cdots & \frac{1}{\lambda} \\ 0 & 0 & -\frac{\lambda(\lambda^2-2)}{\lambda^2-1} & 1 & 0 & \frac{1}{\lambda^2-1} & \frac{1}{\lambda^2-1} & \cdots & \cdots & \cdots & \frac{1}{\lambda^2-1} \\ 0 & 0 & 0 & -\frac{(\lambda^4-3\lambda^2+1)}{\lambda(\lambda^2-2)} & 1 & \frac{1}{\lambda(\lambda^2-2)} & \frac{1}{\lambda(\lambda^2-2)} & \cdots & \cdots & \cdots & \frac{1}{\lambda(\lambda^2-2)} \\ 0 & 0 & 0 & 0 & -\frac{\lambda(\lambda^4-4\lambda^2+3)}{(\lambda^4-3\lambda^2+1)} & \frac{(\lambda^4-3\lambda^2+2)}{(\lambda^4-3\lambda^2+1)} & \frac{1}{(\lambda^4-3\lambda^2+1)} & \cdots & \cdots & \cdots & \frac{1}{(\lambda^4-3\lambda^2+1)} \\ 0 & 0 & 0 & 0 & 0 & -\frac{(\lambda^4-5\lambda^2+4)}{\lambda(\lambda^2-3)} & \frac{(\lambda^2-2)}{\lambda(\lambda^2-3)} & \cdots & \cdots & \cdots & \frac{(\lambda^2-2)}{\lambda(\lambda^2-3)} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\lambda(\lambda^4-6\lambda^2+7)}{(\lambda^4-5\lambda^2+4)} & & & \cdots & \frac{\lambda(\lambda^2-3)}{(\lambda^4-5\lambda^2+4)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & & \cdots & \frac{\lambda(\lambda^2-3)}{(\lambda^4-6\lambda^2+7)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & \cdots & \frac{\lambda(\lambda^2-3)}{(\lambda^4-7\lambda^2+10)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\lambda(\lambda^4-(n+4)\lambda^2+(2\Delta+n-3))}{(\lambda^4-(n+3)\lambda^2+(2\Delta+n-6))} & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{\lambda(\lambda^4-(n+5)\lambda^2+(2\Delta+n))}{(\lambda^4-(n+4)\lambda^2+(2\Delta+n-3))} \end{pmatrix}$$

Furthermore, we calculate the determinant by multiplying all the elements on the main diagonal.

$$\det(A(C_6^n) - \lambda\} = (-\lambda)\left(-\frac{\lambda^2-1}{\lambda}\right)\left(-\frac{\lambda(\lambda^2-2)}{\lambda^2-1}\right)\left(-\frac{(\lambda^4-3\lambda^2+1)}{\lambda(\lambda^2-2)}\right)\left(-\frac{\lambda(\lambda^4-4\lambda^2+3)}{(\lambda^4-3\lambda^2+1)}\right)\left(-\frac{(\lambda^4-5\lambda^2+4)}{\lambda(\lambda^2-3)}\right)\left(-\frac{\lambda(\lambda^4-6\lambda^2+7)}{(\lambda^4-5\lambda^2+4)}\right)\cdots\left(-\frac{\lambda(\lambda^4-(n+4)\lambda^2+(2\Delta+n-3))}{(\lambda^4-(n+3)\lambda^2+(2\Delta+n-6))}\right)\left(-\frac{\lambda(\lambda^4-(n+5)\lambda^2+(2\Delta+n))}{(\lambda^4-(n+4)\lambda^2+(2\Delta+n-3))}\right)$$

$$= \frac{\left(\lambda^4-(n+5)\lambda^2+(2\Delta+n)\right)\lambda^n(\lambda^4-4\lambda^2+3)}{(\lambda^2-3)}$$

$$= \frac{(\lambda^4-(n+5)\lambda^2+(2\Delta+n))\lambda^n(\lambda^2-3)(\lambda^2-1)}{(\lambda^2-3)}$$

$$= \lambda^n(\lambda-1)(\lambda+1)(\lambda^4-(n+5)\lambda^2+(2\Delta+n).$$

Thus, the characteristic polynomial of a unicyclic graph C_6^n is $\lambda^n(\lambda-1)(\lambda+1)(\lambda^4-(n+5)\lambda^2+(2\Delta+n))$.

Theorem 2.2 The spectrum of a unicyclic graph C_6^n is

$$Spec\ C_6^n = \begin{pmatrix} \sqrt{\frac{(n+5)+\sqrt{n^2-2n+9}}{2}} & \sqrt{\frac{(n+5)-\sqrt{n^2-2n+9}}{2}} & 1 & 0 & -1 & -\sqrt{\frac{(n+5)-\sqrt{n^2-2n+9}}{2}} & -\sqrt{\frac{(n+5)+\sqrt{n^2-2n+9}}{2}} \\ 1 & 1 & 1 & n & 1 & 1 & 1 \end{pmatrix}$$

Proof. From Theorem 2.1, the characteristic polynomial of a unicyclic graph C_6^n is:

$$\chi(C_6^n, \lambda) = \lambda^n(\lambda-1)(\lambda+1)(\lambda^4-(n+5)\lambda^2+(2\Delta+n))$$

Observe that the maximum degree of a unicyclic graph is $n+2$, so we can rewrite the characteristic polynomial of a unicyclic graph C_6^n as

$$\chi(C_6^n,\lambda)=\lambda^n(\lambda-1)(\lambda+1)(\lambda^4-(n+5)\lambda^2+(3n+4)).$$

So that, we obtained the eigenvalues as follows.

$$\lambda_1=\sqrt{\frac{(n+5)+\sqrt{n^2-2n+9}}{2}},$$

$$\lambda_2=\sqrt{\frac{(n+5)-\sqrt{n^2-2n+9}}{2}},\ \lambda_3=1,\lambda_4=0,$$

$$\lambda_5=-1,$$

$$\lambda_6=-\sqrt{\frac{(n+5)-\sqrt{n^2-2n+9}}{2}}$$

$$\lambda_7=-\sqrt{\frac{(n+5)+\sqrt{n^2-2n+9}}{2}}$$

Next, we will determine the multiplicity of each the eigenvalue. The multiplicity of λ is the dimension of its eigenspace. The dimension of the eigenspace corresponding to $\lambda_i, i=1,2,3,4,5,6,7$ are equal to the number of zero rows in the matrix $(A(C_6^n)-\lambda_i I)$ after being reduced to a row echelon reduced matrix.

For $\lambda_1=\sqrt{\frac{(n+5)+\sqrt{n^2-2n+9}}{2}}$.

By software of maple 18, we obtained that the number of zero rows the matrix $(A(C_6^n)-\lambda_1 I)$ after being reduced to row echelon reduced matrix is 1.

Hence, the multiplicity of $\lambda_1=\sqrt{\frac{(n+5)+\sqrt{n^2-2n+9}}{2}}$ is 1.

Using the same method, we get , the multiplicity of $\lambda_2=\sqrt{\frac{(n+5)-\sqrt{n^2-2n+9}}{2}}$ is 1, the multiplicity of $\lambda_3=1$ is 1, the multiplicity of $\lambda_4=0$ is n, the multiplicity of $\lambda_5=-1$ is 1, the multiplicity of $\lambda_6=-\sqrt{\frac{(n+5)-\sqrt{n^2-2n+9}}{2}}$ is 1, and the multiplicity of $\lambda_7=-\sqrt{\frac{(n+5)+\sqrt{n^2-2n+9}}{2}}$ is 1.

So, we can show that the spectrum of a unicyclic graph C_6^n is

$$Spec\ C_6^n=\begin{pmatrix}\sqrt{\frac{(n+5)+\sqrt{n^2-2n+9}}{2}} & \sqrt{\frac{(n+5)-\sqrt{n^2-2n+9}}{2}} & 1 & 0 & -1 & -\sqrt{\frac{(n+5)-\sqrt{n^2-2n+9}}{2}} & -\sqrt{\frac{(n+5)+\sqrt{n^2-2n+9}}{2}}\\ 1 & 1 & 1 & n & 1 & 1 & 1\end{pmatrix}$$

Example (The spectrum of the unicyclic graph C_6^2).
The adjacency matrix of a unicyclic graph C_6^2 is

$$A(C_6^2)=\begin{pmatrix}0&1&0&0&0&1&1&1\\1&0&1&0&0&0&0&0\\0&1&0&1&0&0&0&0\\0&0&1&0&1&0&0&0\\0&0&0&1&0&1&0&0\\1&0&0&0&1&0&0&0\\1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\end{pmatrix}$$

The characteristic polynomial of a unicyclic graph C_6^2 as follows.

$$\begin{aligned}\chi(C_6^2,\lambda)&=\lambda^8-8\lambda^6+17\lambda^4-10\lambda^2\\&=\lambda^2(\lambda-1)(\lambda+1)(\lambda^4-7\lambda^2+10)\\&=\lambda^2(\lambda-1)(\lambda+1)(\lambda^2-2)(\lambda^2-5)\end{aligned}$$

The eigenvalues is

$\lambda_1=\sqrt{5},\ \lambda_2=\sqrt{2},\ \lambda_3=1,\lambda_4=0,\ \lambda_5=-1$

$\lambda_6=-\sqrt{2}\ \ \lambda_7=-\sqrt{5}$

Next, we will determine the multiplicity of each the eigenvalue.

For $\lambda_1=\sqrt{5}$.
By software of maple 18, we get the row echelon reduced matrix $(A(C_6^2)-\lambda_1 I)$ as follows.

$$\begin{pmatrix}-\sqrt{5}&1&0&0&0&1&1&1\\0&-\frac{4}{5}\sqrt{5}&1&0&0&\frac{1}{5}\sqrt{5}&\frac{1}{5}\sqrt{5}&\frac{1}{5}\sqrt{5}\\0&0&-\frac{3}{4}\sqrt{5}&1&0&\frac{1}{4}&\frac{1}{4}&\frac{1}{4}\\0&0&0&-\frac{11}{15}\sqrt{5}&1&\frac{1}{15}\sqrt{5}&\frac{1}{15}\sqrt{5}&\frac{1}{15}\sqrt{5}\\0&0&0&0&-\frac{8}{11}\sqrt{5}&\frac{12}{11}&\frac{1}{11}&\frac{1}{11}\\1&0&0&0&0&-\frac{2}{5}\sqrt{5}&\frac{3}{10}\sqrt{5}&\frac{3}{10}\sqrt{5}\\1&0&0&0&0&0&-\frac{1}{2}\sqrt{5}&\frac{1}{2}\sqrt{5}\\1&0&0&0&0&0&0&0\end{pmatrix}$$

Observe that the number of the zero rows of the above matrix is 1. So, the multiplicity of $\lambda_1=\sqrt{5}$ is 1.

For $\lambda_2=\sqrt{2}$.
By software of maple 18, we get the row echelon reduced matrix $(A(C_6^2)-\lambda_2 I)$ as follows.

$$\begin{pmatrix}-\sqrt{2}&1&0&0&0&1&1&1\\0&-\frac{1}{2}\sqrt{2}&1&0&0&\frac{1}{2}\sqrt{2}&\frac{1}{2}\sqrt{2}&\frac{1}{2}\sqrt{2}\\0&0&1&-\sqrt{2}&1&0&0&0\\0&0&0&1&0&1&1&1\\0&0&0&0&-\sqrt{2}&0&-1&-1\\1&0&0&0&0&-\sqrt{2}&0&0\\1&0&0&0&0&0&-\frac{1}{2}\sqrt{2}&\frac{1}{2}\sqrt{2}\\1&0&0&0&0&0&0&0\end{pmatrix}$$

Observe that the number of the zero rows of the above matrix is 1. So, the multiplicity of $\lambda_2 = \sqrt{2}$ is 1.

For $\lambda_3 = 1$.
By software of maple 18, we get the row echelon reduced matrix $(A(C_6^2) - \lambda_3 I)$ as follows.

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Observe that the number of the zero rows of the above matrix is 1. So, the multiplicity of $\lambda_3 = 1$ is 1.

For $\lambda_4 = 0$.
By software of maple 18, we get the row echelon reduced matrix $(A(C_6^2) - \lambda_4 I)$ as follows.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Observe that the number of the zero rows of the above matrix is 2. So, the multiplicity of $\lambda_4 = 0$ is 2.

For $\lambda_5 = -1$.
By software of maple 18, we get the row echelon reduced matrix $(A(C_6^2) - \lambda_5 I)$ as follows.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Observe that the number of the zero rows of the above matrix is 1. So, the multiplicity of $\lambda_5 = -1$ is 1.

For $\lambda_6 = -\sqrt{2}$.
By software of maple 18, we get the row echelon reduced matrix $(A(C_6^2) - \lambda_6 I)$ as follows.

$$\begin{pmatrix} \sqrt{2} & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & \frac{1}{2}\sqrt{2} & 1 & 0 & 0 & -\frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ 0 & 0 & 1 & \sqrt{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Observe that the number of the zero rows of the above matrix is 1. So, the multiplicity of $\lambda_6 = -\sqrt{2}$ is 1.

For $\lambda_7 = -\sqrt{5}$.
By software of maple 18, we get the row echelon reduced matrix $(A(C_6^2) - \lambda_7 I)$ as follows.

$$\begin{pmatrix} \sqrt{5} & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & \frac{4}{5}\sqrt{5} & 1 & 0 & 0 & -\frac{1}{5}\sqrt{5} & -\frac{1}{5}\sqrt{5} & -\frac{1}{5}\sqrt{5} \\ 0 & 0 & \frac{3}{4}\sqrt{5} & 1 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{11}{15}\sqrt{5} & 1 & -\frac{1}{15}\sqrt{5} & -\frac{1}{15}\sqrt{5} & -\frac{1}{15}\sqrt{5} \\ 0 & 0 & 0 & 0 & \frac{8}{11}\sqrt{5} & \frac{12}{11} & \frac{1}{11} & \frac{1}{11} \\ 1 & 0 & 0 & 0 & 0 & \frac{2}{5}\sqrt{5} & -\frac{3}{10}\sqrt{5} & -\frac{3}{10}\sqrt{5} \\ 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{5} & -\frac{1}{2}\sqrt{5} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Observe that the number of the zero rows of the above matrix is 1. So, the multiplicity of $\lambda_7 = -\sqrt{5}$ is 1. Hence, the spectrum of Unicyclic Graph C_6^2 is

$$Spec\ C_6^2 = \begin{pmatrix} \sqrt{5} & \sqrt{2} & 1 & 0 & -1 & -\sqrt{2} & -\sqrt{5} \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

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