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Edge Magic Total Labeling of (n, t)-Kites

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ABSTRACT
An edge magic total (EMT) labeling of a graph $G = (V, E)$ is a bijection from the set of vertices and edges to a set of numbers defined by $\lambda: V \cup E \to \{1,2, \dots, |V| + |E|\}$ with the property that for every $xy \in E$, the weight of xy equals to a constant k, that is, $\lambda(x) + \lambda(y) + \lambda(xy) = k$ for some integer k. This paper gives the construction of EMT labeling for certain classes and some variations of (n, t)-kites.

Keywords: *Magic Labeling, Edge Magic Total Labeling, Kites.*

1. INTRODUCTION

Given a graph $G = (V, E)$, an *edge magic total (EMT) labeling* of a graph $G = (V, E)$ is a bijection from the set of vertices and edges to a set of numbers defined by $\lambda: V \cup E \to \{1,2, \dots, |V| + |E|\}$ with the property that for every $xy \in E$, the weight $w(xy)$ of xy equals to a constant k, that is $w(xy) = \lambda(x) + \lambda(y) + \lambda(xy) = k$, for some integer k. If $\lambda(V) = \{1,2, \dots, |V|\}$ and $\lambda(E) = \{|V|, |V| + 1, \dots, |V| + |E|\}$ then the bijection λ is called *super edge magic total (SEMT) labeling* [1]. The integer k is called the *magic constant* of the labeling λ.

In this paper we consider labeling for (n, t)-kite, graph obtained by connecting a vertex of a cycle C_n and a path P_t using an additional edge. Other names for (n, t)-kites are (n, t)-tadpoles or dragon graphs. Finding an EMT labeling for (n, t)-kites for odd n and $t \neq 1$ is posted as an open problem in [2] and there are not many known results for case even n. In Section 2 we list the known results on (n, t)-kites. In Section 3 we state the constructions of the new results on (n, t)-kites, some of which were first given in [3] but there are mistakes unproved constructions. In this paper proofs are given, extensions added, and necessary corrections are made. In Section 4 we expand the results on some variations of kites that are first introduced in [3] and [4].

2. KNOWN RESULTS

Denote the vertex set of the cycle part of an (n, t)-kite by ordered vertices $v_1, v_2, \dots, v_n$ where v_1 is the only vertex of degree three, and the vertex set of the path part by ordered vertices $u_1, u_2, \dots, u_t$ where u_1 be the only vertex of degree one, as shown in Figure 1.

To date, the results on EMT/SEMT labeling of (n, t)-kites fix the length of the kite's tail with $t \in \{1,2\}$ and there are no known results yet for longer tail [5].

Theorem 2.1. [6] An $(n, 2)$-kite has an SEMT labeling if and only if n is even.

Theorem 2.2. [7,8] An $(n, 1)$-kite has an SEMT labeling when n is odd.

Theorem 2.3. [9] An $(n, 1)$-kite has an EMT labeling with $k = \frac{1}{2}(5n + 9)$ when n is odd and an EMT labeling with $k = \frac{1}{2}(5n + 10)$ when n is even.

Wallis, Baskoro, Miller, Slamin in [9] also stated the duality property of EMT labeling: Let $M = |V| + |E| + 1$. Given a labeling λ with magic constant k, its dual labeling λ' with magic constant $k' = 3M - k$ is defined by $\lambda'(\star) = M - \lambda(\star)$ for any $\star \in V \cup E$. This duality property is important to check that any obtained new labeling is not the dual of an already known labeling.

3. RESULTS ON (n, t)-KITES

Unlike the known results, we first consider fixing the value of n. For $n = 3$ and $n = 4$, EMT labeling exists for (n, t)-kite with any positive integer t. These results are given in Theorem 3.1 and Theorem 3.2, respectively.

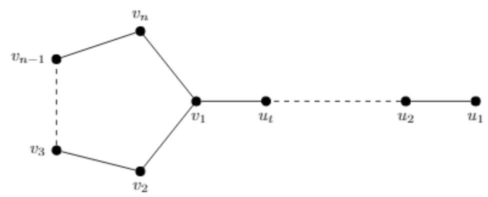

Figure 1 $(n. t)$-kite.

Theorem 3.1. A $(3,t)$-kite has an EMT labeling with $k=\frac{1}{2}(5t+20)$ when t is even and with $k=\frac{1}{2}(5t+21)$ when t is odd.

Proof.

Case 1. For even t, define the labeling as follows:

$$\lambda(v_1)=t+3$$
$$\lambda(v_2)=\frac{1}{2}(t+2)$$
$$\lambda(v_3)=t+5$$
$$\lambda(u_i)=\begin{cases}\frac{1}{2}(5+t+i) & ,i \text{ odd}\\ \frac{1}{2}i & ,i \text{ even}\end{cases}$$
$$\lambda(v_1v_2)=t+6$$
$$\lambda(v_2v_3)=t+4$$
$$\lambda(v_3v_1)=\frac{1}{2}(t+4)$$
$$\lambda(v_1u_t)=t+7$$
$$\lambda(u_iu_{i+1})=2t+7-i \qquad ,1\le i<t$$

Verifying the weights is straightforward:

For i odd we have:

$$w(u_iu_{i+1})=\frac{5+t+i}{2}+\frac{(i+1)}{2}+(2t+7-i)$$

for i even we have:

$$w(u_iu_{i+1})=\frac{5+t+(i+1)}{2}+\frac{i}{2}+(2t+7-i)$$

and for the remaining edges we have:

$$w(v_1v_2)=(t+3)+\frac{1}{2}(t+2)+(t+6)$$
$$w(v_2v_3)=\frac{1}{2}(t+2)+(t+5)+(t+4)$$
$$w(v_3v_1)=(t+5)+(t+3)+\frac{1}{2}(t+4)$$
$$w(v_1u_t)=(t+3)+\frac{1}{2}t+(t+7)$$

It follows that all weights w equals $\frac{1}{2}(5t+20)$.

Case 2. For odd t, define the labeling as follows:

$$\lambda(v_1)=t+3$$
$$\lambda(v_2)=\frac{1}{2}(t+3)$$
$$\lambda(v_3)=t+5$$
$$\lambda(u_i)=\begin{cases}\frac{1}{2}(i+1) & ,i \text{ odd}\\ \frac{1}{2}(5+t+i) & ,i \text{ even}\end{cases}$$
$$\lambda(v_1v_2)=t+6$$
$$\lambda(v_2v_3)=t+4$$
$$\lambda(v_3v_1)=\frac{1}{2}(t+5)$$
$$\lambda(v_1u_t)=t+7$$
$$\lambda(u_iu_{i+1})=2t+7-i \qquad ,1\le i<t$$

Verifying the weights is straightforward:

For i odd we have:

$$w(u_iu_{i+1})=\frac{i+1}{2}+\frac{5+t+(i+1)}{2}+(2t+7-i)$$

for i even we have:

$$w(u_iu_{i+1})=\frac{5+t+i}{2}+\frac{(i+1)+1}{2}+(2t+7-i)$$

and for the remaining edges we have:

$$w(v_1v_2)=(t+3)+\frac{1}{2}(t+3)+(t+6)$$
$$w(v_2v_3)=\frac{1}{2}(t+3)+(t+5)+(t+4)$$
$$w(v_3v_1)=(t+5)+(t+3)+\frac{1}{2}(t+5)$$
$$w(v_1u_t)=(t+3)+\frac{1}{2}(t+1)+(t+7)$$

It follows that all weights w equals $\frac{1}{2}(5t+21)$. ∎

Theorem 3.2. A $(4,t)$-kite has an EMT labeling with $k=\frac{1}{2}(5t+24)$ when t is even and with $k=\frac{1}{2}(5t+23)$ when t is odd.

Proof.

Case 1. For even t, define the labeling as follows:

$$\lambda(v_1)=\frac{1}{2}(t+2)$$
$$\lambda(v_2)=t+3$$
$$\lambda(v_3)=\frac{1}{2}(t+4)$$
$$\lambda(v_4)=t+6$$
$$\lambda(u_i)=\begin{cases}\frac{1}{2}(i+1) & ,i \text{ odd}\\ \frac{1}{2}(t+4+i) & ,i \text{ even}\end{cases}$$
$$\lambda(v_1v_2)=t+8$$
$$\lambda(v_2v_3)=t+7$$
$$\lambda(v_3v_1)=t+4$$
$$\lambda(v_3v_1)=t+5$$
$$\lambda(v_1u_t)=t+9$$
$$\lambda(u_iu_{i+1})=2t+9-i \qquad ,1\le i<t$$

It is easy to verify that all weights equal $\frac{1}{2}(5t+24)$.

Case 2. For odd t, define the labeling as follows:

$$\lambda(v_1)=\frac{1}{2}(t+1)$$
$$\lambda(v_2)=t+3$$
$$\lambda(v_3)=\frac{1}{2}(t+3)$$
$$\lambda(v_4)=t+6$$
$$\lambda(u_i)=\begin{cases}\frac{1}{2}(t+4+i) & ,i \text{ odd}\\ \frac{1}{2}i & ,i \text{ even}\end{cases}$$
$$\lambda(v_1v_2)=t+8$$
$$\lambda(v_2v_3)=t+7$$
$$\lambda(v_3v_1)=t+4$$
$$\lambda(v_3v_1)=t+5$$
$$\lambda(v_1u_t)=t+9$$
$$\lambda(u_iu_{i+1})=2t+9-i \qquad ,1\le i<t$$

It is easy to verify that all weights equal $\frac{1}{2}(5t+23)$. ∎

Illustration for Theorem 3.1 and Theorem 3.2 are given in Figure 2 and Figure 3, respectively.

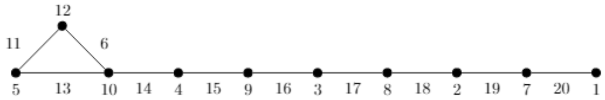

Figure 2 EMT labeling for (3,7)-kite with $k = 28$.

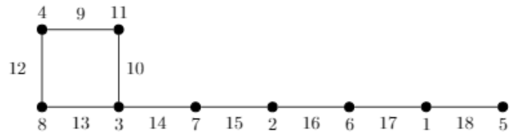

Figure 3 EMT labeling for (4,5)-kite with $k = 24$.

For $n = 5$ and $n = 6$, while (5,3)-kite and (6,3)-kite have several EMT labelings as shown in Figure 4, 5, 6, 7, the generalization to larger values of t is still under further investigation.

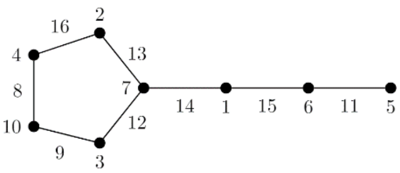

Figure 4 EMT labeling for (5,3)-kite with $k = 22$.

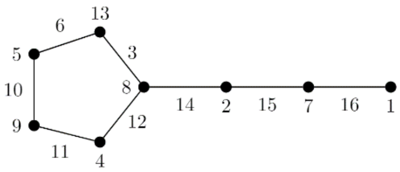

Figure 5 EMT labeling for (5,3)-kite with $k = 24$.

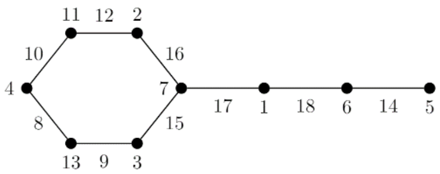

Figure 6 EMT labeling for (6,3)-kite with $k = 25$.

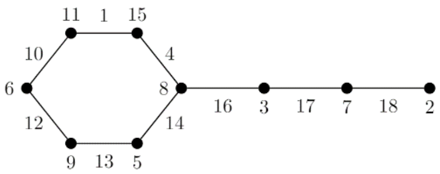

Figure 7 EMT labeling for (6,3)-kite with $k = 27$.

We consider two possible approaches for generalizing the results on (5,3)-kite and (6,3)-kite: extending the tail or the body of the kite.

Open Problem 1. Find EMT labeling for (n, t)-kites when $n \in \{5,6\}$ and $t \geq 4$.

Open Problem 2. Find EMT labeling for (n, t)-kites when $n \geq 7$ and $t = 3$.

The next result is related to the known results stated in Section 2. Theorem 2.1 guarantee that for $(n, 2)$-kites, SEMT labeling only exists when n is even. Theorem 3.3 consider the case when n is odd and show that $(n, 2)$-kites have EMT labeling for any odd n.

Theorem 3.3. A $(n, 2)$-kite has an EMT labeling with $k = \frac{1}{2}(5n + 13)$ when n is odd.

Proof. For convenience let $v_{n+1} = v_1$.

Define the labeling as follows:

$$\lambda(v_i) = \begin{cases} n & i = 1 \\ \frac{1}{2}(n-1) & i = 2 \\ n+3 & i = 3 \\ \frac{1}{2}(n+i-1) & i = 4, 6, \dots, n-1 \\ \frac{1}{2}(i-3) & i = 5, 7, \dots, n \end{cases}$$

$$\lambda(u_i) = \begin{cases} n+1 & i = 1 \\ \frac{1}{2}(n+1) & i = 2 \end{cases}$$

$$\lambda(u_1u_2) = n + 5$$
$$\lambda(v_1u_2) = n + 6$$
$$\lambda(v_1v_2) = n + 7$$
$$\lambda(v_2v_3) = n + 4$$
$$\lambda(v_3v_4) = n + 2$$
$$\lambda(v_iv_{i+1}) = 2n + 8 - i \qquad i \geq 4$$

Verify the weights:

For odd $i \geq 4$:

$$w(v_iv_{i+1}) = \frac{n+i-1}{2} + \frac{(i+1)-3}{2} + (2n + 8 - i)$$

for even $i \geq 4$:

$$w(v_iv_{i+1}) = \frac{i-3}{2} + \frac{n+(i+1)-1}{2} + (2n + 8 - i)$$

and for the remaining edges we have:

$$w(u_1u_2) = (n + 1) + \frac{1}{2}(n + 1) + (n + 5)$$
$$w(v_1u_2) = n + \frac{1}{2}(n + 1) + (n + 6)$$
$$w(v_1v_2) = n + \frac{1}{2}(n - 1) + (n + 7)$$
$$w(v_2v_3) = \frac{1}{2}(n - 1) + (n + 3) + (n + 4)$$
$$w(v_3v_4) = (n + 3) + \frac{1}{2}(n + 4 - 1) + (n + 2)$$

It follows that all weights equal $\frac{1}{2}(5n + 13)$. ∎

Illustration for Theorem 3.3 is given on Figure 8.

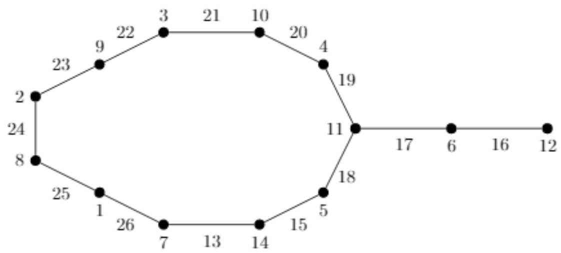

Figure 8 EMT labeling for (11,2)-kite with $k = 34$.

Now consider the latter part of the result for $(n,1)$-kites for even n on Theorem 2.3. An alternative construction for the same class of graphs is given in Theorem 3.4. The construction in Theorem 3.4 is not the dual of the construction in Theorem 2.3 since for $(n,1)$-kites we have $M = 2n+3$ and so the dual on Theorem 2.3 has $k' = \frac{1}{2}(7n+8) \neq \frac{1}{2}(7n+2)$.

Theorem 3.4. A $(n,1)$-kite has an EMT labeling with $k = \frac{1}{2}(7n+2)$ when n is even.

Proof. For convenience let $v_0 = v_1$.

Case 1. For $n \equiv 0 \bmod 4$, define the labeling as follows:

$$\lambda(v_i) = \begin{cases} n & i = 0 \\ \frac{n}{2} - 1 & i = 1 \\ n+1 & i = 2 \\ \frac{1}{2}(4n+1-i) & i = 3,5,\dots,\frac{n}{2}+1 \\ \frac{1}{2}(3n+6-i) & i = 4,6,\dots,\frac{n}{2} \\ \frac{3}{4}n - 1 + i & i = \frac{n}{2}+2,\ \frac{n}{2}+3 \\ \frac{5}{4}n & i = \frac{n}{2}+4 \\ \frac{1}{2}(3n+4-i) & i = \frac{n}{2}+6,\ \frac{n}{2}+8,\dots,n-2 \\ \frac{1}{2}(4n+3-i) & i = \frac{n}{2}+5,\ \frac{n}{2}+7,\dots,n-1 \\ n+2 & i = n \end{cases}$$

$$\lambda(v_0v_1) = 2n+2$$
$$\lambda(v_1v_2) = 2n+1$$
$$\lambda(v_2v_3) = \frac{1}{2}(n+2)$$

$$\lambda(v_iv_{i+1}) = \begin{cases} i-2 & 3 \le i \le \frac{n}{2} \text{ and } 4 \le i \le n-1 \\ \frac{n}{2} & i = \frac{n}{2}+1 \\ n-2 & i = \frac{n}{2}+2 \\ n-1 & i = \frac{n}{2}+3 \end{cases}$$

$$\lambda(v_nv_1) = 2n$$

Verify the weights:

$$w(v_0v_1) = n + \left(\frac{n}{2} - 1\right) + (2n+2)$$
$$w(v_1v_2) = \left(\frac{n}{2} - 1\right) + (n+1) + (2n+1)$$
$$w(v_2v_3) = (n+1) + \frac{1}{2}(4n+1-3) + \frac{1}{2}(n+2)$$
$$w(v_{n-1}v_n) = \frac{1}{2}\big(4n+3-(n-1)\big) + (n+1) + \big((n-1)-2\big)$$

for $3 \le i \le \frac{n}{2}$ and i is odd:

$$w(v_iv_{i+1}) = \frac{4n+1-i}{2} + \frac{3n+6-(i+1)}{2} + (i-2)$$

for $3 \le i \le \frac{n}{2}$ and i is even:

$$w(v_iv_{i+1}) = \frac{3n+6-i}{2} + \frac{4n+1-(i+1)}{2} + (i-2)$$

for $i = \frac{n}{2}+1$ we have $w(v_iv_{i+1}) =$

$$\frac{1}{2}\left(4n+1-\left(\frac{n}{2}+1\right)\right) + \left(\frac{3}{4}n - 1 + \frac{n}{2} + 2\right) + \frac{n}{2}$$

for $i = \frac{n}{2}+2$ we have $w(v_iv_{i+1}) =$

$$\left(\frac{3}{4}n - 1 + \frac{n}{2} + 2\right) + \left(\frac{3}{4}n - 1 + \frac{n}{2} + 3\right) + (n-2)$$

for $i = \frac{n}{2}+3$ we have $w(v_iv_{i+1}) =$

$$\left(\frac{3}{4}n - 1 + \frac{n}{2} + 3\right) + \frac{5}{4}n + (n-1)$$

for $\frac{n}{2}+4 \le i \le n-1$ and i is odd:

$$w(v_iv_{i+1}) = \frac{3n+4-i}{2} + \frac{4n+3-(i+1)}{2} + (i-2)$$

for $\frac{n}{2}+4 \le i \le n-1$ and i is even:

$$w(v_iv_{i+1}) = \frac{4n+3-i}{2} + \frac{3n+4-(i+1)}{2} + (i-2)$$

It follows that all weights equal $\frac{1}{2}(7n+2)$.

Case 2. For $n \equiv 2 \bmod 4$, define the labeling as follows:

$$\lambda(v_i) = \begin{cases} n & i = 0 \\ \frac{n}{2} - 1 & i = 1 \\ n+1 & i = 2 \\ \frac{1}{2}(4n+1-i) & i = 3,5,\dots,\frac{n}{2} \\ \frac{1}{2}(3n+6-i) & i = 4,6,\dots,\frac{n}{2}+3 \\ \frac{1}{4}(5n+2) & i = \frac{n}{2}+2 \\ \frac{1}{2}(4n+3-i) & i = \frac{n}{2}+4,\ \frac{n}{2}+6,\dots,n-1 \\ \frac{1}{2}(3n+4-i) & i = \frac{n}{2}+5,\ \frac{n}{2}+7,\dots,n \end{cases}$$

$$\lambda(v_0v_1) = 2n+2$$
$$\lambda(v_1v_2) = 2n+1$$
$$\lambda(v_2v_3) = \frac{1}{2}(n+2)$$

$$\lambda(v_iv_{i+1}) = \begin{cases} i-2 & 3 \le i \le \frac{n}{2} \text{ and } 4 \le i \le n-1 \\ n-2 & i = \frac{n}{2}+1 \\ n-1 & i = \frac{n}{2}+2 \\ \frac{n}{2} & i = \frac{n}{2}+3 \end{cases}$$

$$\lambda(v_nv_1) = 2n$$

Similar to Case 1, it is straightforward to verity that all weights equal $\frac{1}{2}(7n+2)$. ■

Illustration for Theorem 3.4 is given on Figure 9.

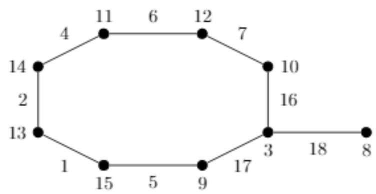

Figure 9 EMT labeling for (8,1)-kite with $k = 29$.

For $n \ge 7$, the problem of finding EMT of (n,t)-kite is still open.

Open Problem 3. Find EMT labeling for (n,t)-kites when $n \ge 7$, and $t \ge 3$.

4. SOME VARIATIONS OF (n, t)-KITES

For $n \geq 3$ and $p \geq 1$ the graph $C_n + A_p$ is a graph with $V(C_n + A_p) = \{v_i: 1 \leq i \leq n\} \cup \{u_j: 1 \leq j \leq p\}$ and $E(C_n + A_p) = \{v_i v_{i+1}: 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{v_i u_j: 1 \leq j \leq p\}$, as introduced in [1]. In other words, $C_n + A_p$ is a kite with p tails of length one incident to the same vertex. In [4] it has been shown that $C_n + A_p$ has SEMT labeling for $p \geq 1$ and odd $n \geq 3$.

Cichacz, Froncek, and Singgih in [3,8] introduced an algorithm that can be used to expand an EMT labeling of a graph into EMT labeling of 'larger' graphs using Kotzig Array. Applying this method to family of graphs constructed from cycles and paths, what the algorithm does is extending the length of the cycles and copying the number of paths with the same factor. For example, an EMT labeling for $C_n \cup P_t$ will be expanded into an EMT labeling for $C_{rn} \cup rP_t$ where r can be any odd integers. Applying this expansion algorithm to (n, t)-kites and to $C_n + A_p$ gives other EMT/SEMT labeling for variations of (n, t)-kites.

Define a (m, n, t)-kites as a kite having m tails each of length t that are equidistant to each other across n vertices of the kite's body. This family of graphs are first introduced as mutated (m, n, t)-tadpole in [3]. Applying the expansion in algorithm in [8] to Theorem 2.1 and Theorem 2.3 gives Corollary 4.1. Applying the expansion to Theorem 2.2 and Theorem 3.3 gives Corollary 4.2. Applying it to Theorem 3.1 and Theorem 3.2 gives Corollary 4.3. Figure 10 gives an SEMT labeling for a (3,12,2)-kites obtained by applying the expansion to the SEMT labeling of a (4,2)-kites given by Theorem 2.1.

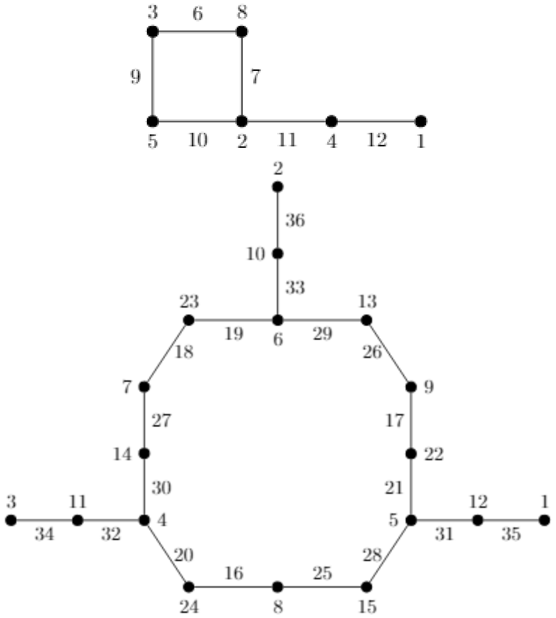

Figure 10 EMT labeling for $(4,2)$-kite with $k = 17$ (top) and the corresponding expanded $(3,12,2)$- kite with $k = 48$ (bottom).

Corollary 4.1. For $n, r \geq 0$, $\big((2r+1), (2r+1)n, 1\big)$-kites has an SEMT labeling when n is odd and an EMT labeling when n is even.

Corollary 4.2. For $n, r \geq 0$, $\big((2r+1), (2r+1)n, 2\big)$-kites has an SEMT labeling when n is even and an EMT labeling when n is odd.

Corollary 4.3. For $t, r \geq 0$, $\big((2r+1), 3(2r+1), t\big)$-and $\big((2r+1), 4(2r+1), t\big)$-kites has an EMT labeling.

Applying the expansion algorithm to $C_n + A_p$ gives SEMT labeling for $C_{rn} + rA_p$, which is a kite with body of length rn having clump of p tails of length one incident to r vertices that having distance n from each other. Corollary 4.4 states the result and Figure 11 illustrate an SEMT labeling of $C_9 + 3A_2$ obtained by applying the expansion to the SEMT labeling of $C_3 + A_2$ given on [4].

Corollary 4.4. For any odd $r \geq 1$, odd $n \geq 3$, and any $p \geq 1$, the graph $C_{rn} + rA_p$ has an SEMT labeling.

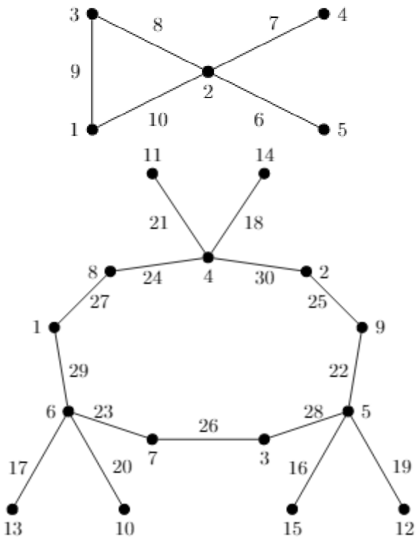

Figure 11 SEMT labeling for $C_3 + A_2$ with $k = 13$ (top) and the corresponding $C_9 + 3A_2$ with $k = 36$ (bottom).

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