ATLANTIS PRESS
Advances in Computer Science Research, volume 96
Proceedings of the International Conference on Mathematics, Geometry, Statistics, and Computation (IC-MaGeStiC 2021)

Generalization of Chaos Game on Polygon

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ABSTRACT
The original chaos game has been applied to the triangular attractor points. With the rules for selecting attractor points randomly, the points generated in large iterations will form like a Sierpinski triangle. Several studies have developed it on the attractor points of quadrilaterals, pentagons, and hexagons which are convex in shape. The fractals formed vary depending on the shape of the attractor points. This paper will study the development of chaos game at attractor points in the form of arbitrary convex and non-convex polygons. The results obtained are consistent with previous results. The resulting fractal is in the form of a convex polygon built from the outermost points of its attractor.

Keywords: *Fractals, Chaos game, Attractor points, Convex polygon.*

1. INTRODUCTION

Chaos game is a game of drawing a collection of points in an equilateral triangle with certain rules that are repeated iteratively. The points drawn are the midpoints of the distance from the starting point or the previous point to one of the corner points of the triangle with random selection. In small iterations, the set of points looks chaotic and shapeless. However, if it is done on thousands of iterations, then the set of midpoints will approach the Sierpinski triangle shape.

The formation of the Sierpinski triangle can basically be explained through the affine transformation, namely the dilation. The dilation factor in this case is half and the centre of the dilation is at one of the vertices of the equilateral triangle. This dilation process with a dilation factor of half indicates a process similar to a chaos game which generates the midpoint of the distance between the previous point and one of the corner points randomly. In this case, the starting point used as an attractor point to determine the midpoints does not have to be inside the equilateral triangle, but it is also possible to be outside it. The midpoint that is outside an equilateral triangle in a certain iteration must be inside the equilateral triangle. And once inside the triangle, the point is impossible to get out of the triangle [1]. In this case it should be noted that the form of self-similarity that appears is the formation of an equilateral triangle at each vertex of the triangle whose size is half of the triangle in the previous iteration.

Chaos games with the more common three attractor points and with the initial object being an arbitrary polygon have also been discussed. The shape of the resulting fractal depends on the shape of the attractor point, which is a triangle. An arbitrary polygon as the initial object in a large iteration will look like a collection of points. Therefore, it is concluded that the shape of the fractal is determined by the shape of the attractor points [2].

Chaos game on square attractor points has been developed with rules for selecting corner points randomly and non-randomly [3]. With random rules, chaos game generates a fractal in the form of a square with a size of half a square in the previous iteration at each attractor point which is a corner point of the square. Meanwhile, with non-random or patterned rules, the resulting fractal properties do not appear. The longer the non-random rule pattern, the less fractal properties appear.

Chaos game on the pentagon-shaped attractor points also produces a collection of midpoints that form fractals [4]. Around the corner points of the pentagon appear points in the form of a pentagon with a size half of the previous pentagon. Likewise, at every vertex of a small pentagon, a pentagon with the size of half is also formed. Similar results were also obtained in the chaos game for hexagons [5].

Some of the research results related to the chaos game above are applied to convex regular polygons. The fractals formed at each corner point that function as attractor points are similar to the initial convex polygon shape. In this article, we will discuss chaos games on irregular polygons and non-convex polygons. The goal is to get fractal patterns generated in the chaos game with attractor points forming arbitrary polygons.

2. RESULTS AND DISCUSSION

Chaos game with three attractor points in general can be described as in Figure 1. If three attractor points are taken that form an arbitrary triangle, after the chaos game rules are carried out in thousands of iterations, then at each attractor point there will be a collection of points that form a triangle whose size is half of the previous triangle. The smaller triangle also contains a triangle that is half the size of the smaller triangle.

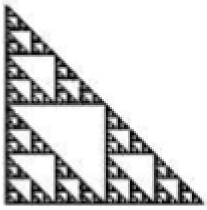


Figure 1 Chaos game result with three attractor points.

The results of the chaos game with four convex attractor points have a fractal shape according to the shape of the convex quadrilateral (Figure 2). At each attractor point there is a convex quadrilateral that is half the size of the previous one. Likewise, each small quadrilateral attractor point contains a similar quadrilateral that is half the size of the previous quadrilateral.

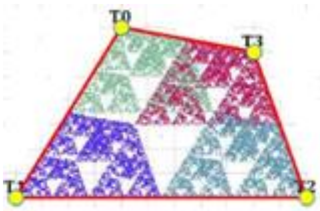


Figure 2 Chaos game result with four convex attractor points.

Chaos game with four non-convex attractor points has three attractor points that form a triangle plus one attractor point inside the triangle. The fractal generated by the chaos game is triangular (Figure 3) around its four attractor points. Likewise, at the four attractor points in the small triangle there is a similar triangle that is half the size of the previous triangle.

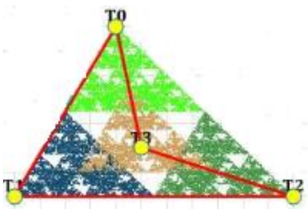


Figure 3 Chaos game result with four non-convex attractor points.

Chaos game results with five convex attractor points are similar to four convex attractor points. The results of the chaos game with five convex attractor points have a fractal shape according to the shape of the convex pentagon (Figure 4). At each attractor point there is a convex pentagon that is half the size of the previous one. Again, each attractor point of small pentagon contains a similar pentagon that is half the size of the previous pentagon.

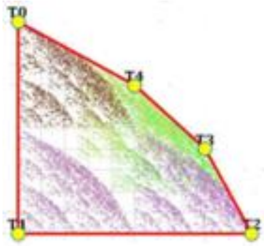
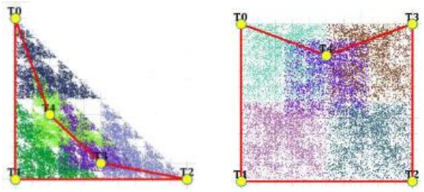


Figure 4 Chaos game result with five convex attractor points.

Chaos game with five non-convex attractor points can have three attractor points that form a triangle plus two attractor points inside the triangle. Otherwise, it can be four attractor points that form a convex quadrilateral plus one attractor point inside the quadrilateral. The fractal generated by the chaos game in first case is triangle around its five attractor points (Figure 5(a)). In second case, the fractal generated by the chaos game is quadrilateral around its five attractor points (Figure 5(b)).



(a) (b)

Figure 5 Chaos game results with five non-convex attractor points.

Chaos game results with six convex attractor points are similar to four and five convex attractor points. The results of the chaos game with six convex attractor points have a fractal shape according to the shape of the convex hexagon (Figure 6). At each attractor point there is a convex hexagon that is half the size of the previous one. Again, each attractor point of small hexagon contains a similar hexagon that is half the size of the previous hexagon.

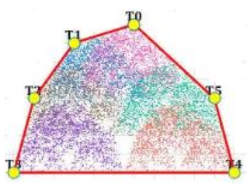
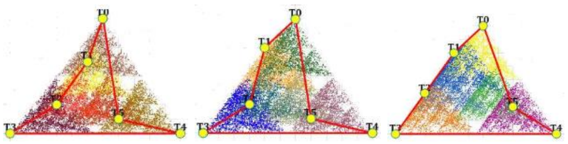


Figure 6 Chaos game result with six convex attractor points.

The six non-convex attractor points can have three, four, or five attractor points that respectively form a triangle, quadrilateral, and pentagon. There are respectively three, two, or one attractor points inside them. The fractal generated by the chaos game in first case is triangle around its six attractor points (Figure 7(a)). In second case, the fractal generated by the chaos game is quadrilateral around its six attractor points (Figure 7(b)). And in the third case the fractal is pentagon around its six attractor points (Figure 7(c)).



(a) (b) (c)

Figure 7 Chaos game results with six non-convex attractor points.

We will now discuss the above results. Numerically, in chaos game all generated points (x^i, y^i) at *i*-th iteration can be calculated by the formula in Equation (1).

$$\begin{bmatrix} x^{i+1} \\ y^{i+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x^i - a \\ y^i - b \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} \quad (1)$$

In this case point (a, b) is the coordinates of a randomly selected attractor point and serves as the centre of dilation. An attractor point will attract all the points generated in the chaos game following a dilation transformation with the centre at each attractor point and the dilation magnitude half. The dilation process that attracts all existing points will be carried out by all attractor points. Therefore, with a dilation with a centre at each attractor point and with a dilation of half the magnitude, then at each attractor point will be formed like the previous points but with half the size.

Each iteration in the chaos game will generate a midpoint between the previous point and one of the randomly selected attractor points. Assuming the selection of attractor points is done randomly, then each attractor point will have the same probability of being selected. Thus, if the chaos game on a polygon with *n* attractor points is carried out in *K* iterations, then the number of points generated in each attractor point is about *K/n*.

The fractal that appears is in the form of a convex polygon built from the outermost points of its attractor. This is due to the dilation process with each attractor point as the centre of the dilation. Assuming the probability distribution of each attractor point to be selected is the same, then the shape of the convex polygon around the attractor point is also relatively similar. This also applies to the case of attractor points that form convex and non-convex polygons. Likewise, this also applies to attractor points that form regular polygons and irregular polygons. Chaos games with more than six attractor points produce fractals similar to three to six attractor points.

3. CONCLUSION

Chaos games with attractor points forming convex and non-convex polygons will produce fractals as well as the outermost convex polygons constructed by attractor points. Each attractor point will attract the middle points in the chaos game, so that around each attractor point will appear the fractal of the convex polygon. This also applies to attractor points that are inside the convex polygon.

ACKNOWLEDGMENTS

The author would like to thank the students and lecturers who have provided input for the contents of this paper. Thank you also for the support from the Department of Mathematics and the Faculty of Mathematics and Natural Sciences, University of Jember.

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