L(2,1) Labeling of Lollipop and Pendulum Graphs

Kusbudiono*, Irham Af'idatul Umam, Ikhsanul Halikin, Mohamat Fatekurohman

Graph, Combinatorics, and Algebra Research Group, Department of Mathematics, FMIPA, Universitas Jember
Corresponding author. Email: kusbudiono@unej.ac.id

ABSTRACT
One of the topics in graph labeling is $L(2,1)$ labeling which is an extension of graph labeling. Definition of $L(2,1)$ labeling is a function that maps the set of vertices in the graph to non-negative integers such that every two vertices u, v that have a distance one must have a label with a difference at least two. Furthermore, every two vertices u, v that have a distance two must have a label with a difference at least one. This study discusses the $L(2,1)$ labeling on a lollipop graph $L_{m,n}$ with $m \geq 3$ and n positive integers. The purpose of this study is to determine the minimum span value from the $L(2,1)$ labeling on the lollipop graph $L_{m,n}$ and we can symbolize $\lambda_{2,1}(L_{m,n})$ and to determine the minimum span value from the $L(2,1)$ labeling on the pendulum graph. In addition, it also builds a simulation program for $L(2,1)$ labeling lollipop graphs up to tremendous values of m and n. In this paper, we obtained that the minimum span of a lollipop graph is $\lambda_{2,1}(L_{m,n}) = 2m - 2$, and the minimum span of a pendulum graph, let P_n^k with $k \geq 4$ and $n \geq 5$, is $k + 1$.

Keywords: $L(2,1)$ *Labeling, Lollipop graph, Pendulum graph.*

1. INTRODUCTION

Graph theory is a branch of mathematics that has undergone many developments. There are many topics studied in graph theory, one of which is graph labeling. A graph G expressed in $(V(G), E(G))$ is a pair of two sets. $V(G) = \{v_1, v_2, \dots, v_n\}$ is a non-empty set of elements called vertices and $E(G) = \{e_1, e_2, \dots, e_n\}$ is a possibly empty set of an unordered pair $\{v_1, v_2\}$ of two vertices $v_1, v_2 \in V(G)$, called the edge set of G [1]. In general, graph labeling discusses labeling in integers at graph points, graph edges, or both. In its development, we added some rules to graph labeling related to distance. One graph labeling based on this is labeling $L(h,k)$ [2].

Then, we will focus on $L(2,1)$ labeling as a function f which maps the vertex set $V(G)$ to non-negative integers such that if $d(u,v) = 1$, then $| f(u) - f(v) | \geq 1$ and if $d(u,v) = 2$, then $| f(u) - f(v) | \geq 1$ where $d(u,w)$ is the distance between vertices u and v. A number k such that an $L(2,1)$ labelling exist is called as span of $L(2.1)$-labelling if there is no label greater than k. The span of a graph G can be more than one, the minimum value of the span of a graph G is notated by $\lambda_{(2,1)}(G)$.

Some researchers have applied $L(2,1)$ labeling to several graphs and have obtained the minimum span of the graph. Some of these graphs include star graph [2]; cycle, path, and complete graph [3]; $K_{1,n}$ -free graphs [4]; supercycle graph [5]; lotus, fan, wheel, and $K_1 \odot (P_n \cap F_n)$ graphs [6]; and Sierpinski graph [7]. In this study, we discuss the $L(2,1)$ labeling of the lollipop graph $L_{m,n}$ With $m \geq 3$, n positive integers and $L(2,1)$ labeling of the pendulum graph.

This study aims to determine the minimum span value of the $L(2,1)$ labeling on the lollipop graph. Determine the minimum span value of the $L(2,1)$ labeling on the pendulum graph. Build a simulation program that can handle $L(2,1)$ labeling on a lollipop graph up to tremendous values of m and n.

We will use three theorems as the basis for finding the minimum span value of the $L(2,1)$ labeling as follows.

Lemma 1.1 [4] If H is a subgraf of G, then $\lambda_{2,1}(H) \leq \lambda_{2,1}(G)$.

Lemma 1.2 [4] For any n natural numbers, $\lambda_{2,1}(K_n) = 2n - 2$.

Lemma 1.3 [4] For any n natural numbers, $\lambda_{2,1}(P_n) = 4$, with $n \geq 5$.

Lemma 1.4 [2] Let $S_{1,n}$ be a star, then $\lambda_{2,1}(S_{1,n}) = n + 1$.

2. RESULT AND DISCUSSION

We can obtain a lollipop graph by combining the complete graph K_m and the path graph P_n with a bridge (edge) so that the lollipop graph notation is $L_{m,n}$ for any natural number m and n [8]. The notation of the vertex set and the edge set of the lollipop graph is as follows, $V = \{u_i \mid 1 \le i \le m\} \cup \{v_j \mid 1 \le j \le n\}$ and $E = \{u_i u_k \mid 1 \le i \le m - 1 i + 1 \le k \le m\} \cup \{u_1 v_1\} \cup \{v_j v_{(j+1)} \mid 1 \le j \le n-1\}$. Figure 1 illustrates the notation of a lollipop graph.

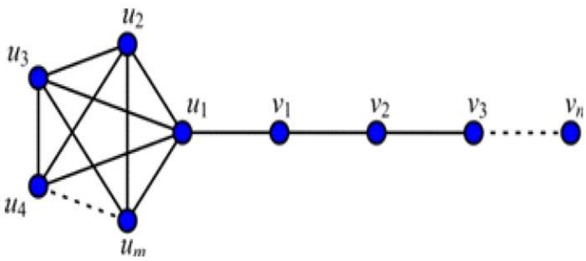

Figure 1 Lollipop graph notation $L_{m,n}$.

Theorem 2.1 For any m, n natural number, lollipop graph $L_{m,n}$ with $m \ge 3$ has the minimum span is on the $L(2,1)$ labeling is $2m - 2$.

Proof. To prove that the minimum span of a lollipop graph is $\lambda_{2,1}(L_{m,n}) = 2m - 2$, then we have to show that $\lambda_{2,1}(L_{m.n}) \ge 2m - 2$ and $\lambda_{2,1}(L_{m,n}) \le 2m - 2$.

Both a complete and path are subgraphs of a lollipop. Based on Lemma 1.2, a complete graph (K_m) has a minimum span value of $\lambda_{2,1}(K_m) = 2m - 2$, and according to Lemma 1.3, a path graph P_n has a minimum span of $\lambda_{2,1}(P_n) = 4$ for $n \ge 5$, otherwise $\lambda_{2,1}(P_n) < 4$. Thus $\lambda_{2,1}(K_m) \ge \lambda_{2,1}(P_n)$. Furthermore, based on the fact from Lemma 1.1 that $\lambda_{2,1}(L_{(m,n)}) \ge \lambda_{2,1}(K_m) = 2m - 2$ causes $\lambda_{2,1}(L_{(m,n)}) \ge 2m - 2$. To show that $\lambda_{2,1}(L_{(m,n)}) \le 2m - 2$, it is enough to do the labeling by,

$$f(u_i) = 2m - 2i; 1 \le i \le m$$

$$f(v_j) = \begin{cases} 1; j = 1 \\ 3; j = 2 \\ 2a; 3 \le j \le n; a \text{ is the remainder} \\ \qquad \text{of } j - 3 \text{ divided by } m \end{cases}$$

In the lollipop graph $L_{m,n}$, the vertices formed from the complete graph $(u_i; 1 \le i \le m))$ have a distance of one. Then the vertices u_1 and v_1 also have a distance of one, and the last one where the distance between the two vertices is one, namely the vertices v_j and v_{j+1} with $1 \le j \le n - 1$. Next, the two vertices in the lollipop graph that have a distance of two are vertex u_i, $i \ne 1$ with v_1, u_1 with v_2, and v_j with v_{j+2}, $1 \le j \le n - 2$. From the labeling function f above, the labels given have met the labeling requirements of $L(2,1)$. Furthermore, based on the mapping rule of the function, the largest labeling is $2m - 2$. Thus, we can prove that $\lambda_{2,1}(L_{(m,n)}) \le 2m - 2$. So based on the above, the author concludes that the minimum span value for $L(2,1)$ labeling of Lollipop graph $L_{m,n}$ is $2m - 2$. ■

In this study, the author makes a labeling simulation program $L(2,1)$ on a lollipop graph $L_{m,n}$. We can see the display of the program in Figure 2.

The following are some of the results of labeling carried out using a simulation program.

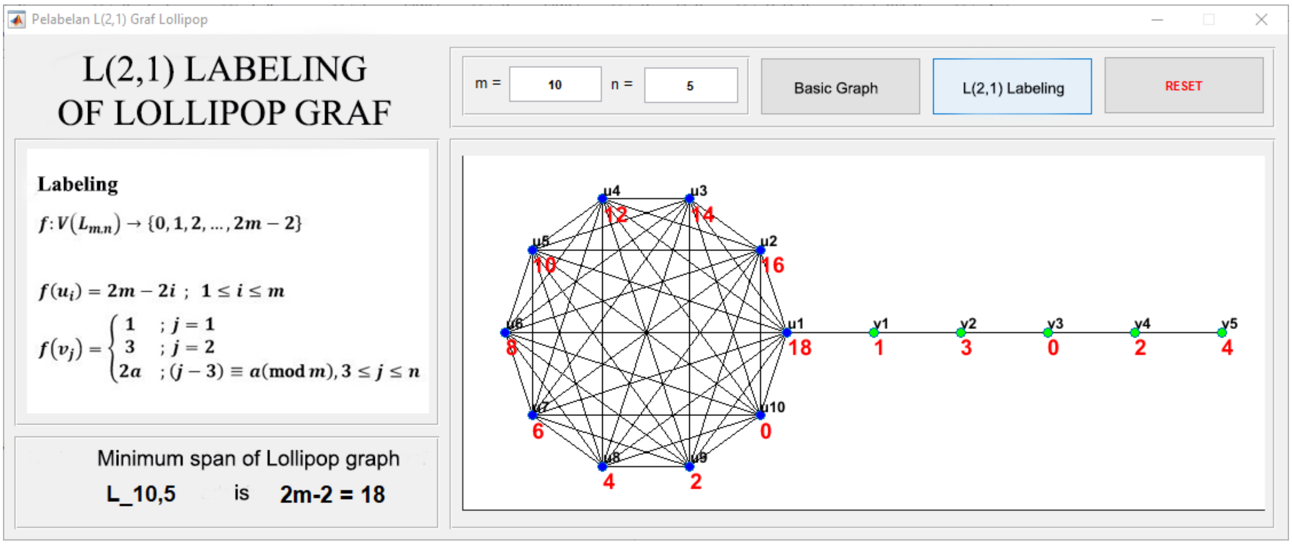

Figure 2 Simulation Program Display.

1. $L(2,1)$ Labeling of lollipop graph $L_{3,5}$

In Figure 3, the $L(2,1)$ labeling of the lollipop graph $L_{3,5}$ has the largest label value of four and the smallest label value of zero, so the labeling has a span of four. The span value follows Theorem 2.1, which is $2m - 2 = 2(3) - 2 = 4$.

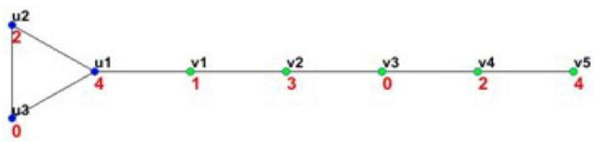

Figure 3 $L(2,1)$ labeling on the lollipop graph $L_{3,5}$.

2. $L(2,1)$ Labeling of lollipop graph $L_{4,8}$

In Figure 4, the $L(2,1)$ labeling of the lollipop graph $L_{4,8}$ has the largest label value of four and the smallest label value of zero, so the labeling has a span of four. The span value follows Theorem 2.1, which is $2m - 2 = 2(4) - 2 = 6$.

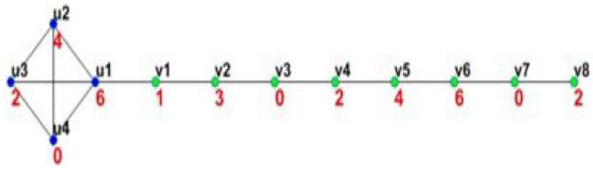

Figure 4 Labeling $L(2,1)$ of lollipop graph $L_{4,8}$.

3. $L(2,1)$ Labeling on the lollipop graph $L_{10,10}$

In Figure 5, the $L(2,1)$ labeling of the lollipop graph $L_{10,10}$ has the largest label value of four and the smallest label value of zero, so the labeling has a span of four. The span value follows Theorem 2.1, which is $2m - 2 = 2(10) - 2 = 18$.

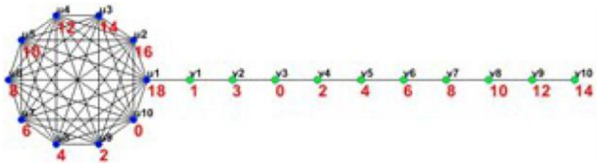

Figure 5 $L(2,1)$ Labeling on the lollipop graph $L_{10,10}$.

4. $L(2,1)$ Labeling on the lollipop graph $L_{20,10}$

In Figure 6, the $L(2,1)$ labeling of the lollipop graph $L_{20,10}$ has the largest label value of four and the smallest label value of zero, so the labeling has a span of four. The span value follows Theorem 2.1, which is $2m - 2 = 2(20) - 2 = 38$

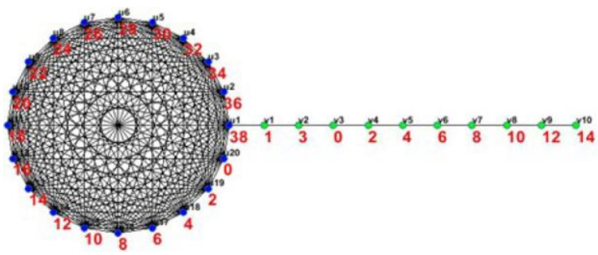

Figure 6 $L(2,1)$ Labeling on the lollipop graph $L_{20,10}$.

We defined pendulum graph P_n^k as a graph obtained by attaching a cycle graph C_n to every leaf of star graph S_n. Consider Figure 7 which illustrates graph P_n^k. Suppose that we notated vertices and edges of pendulum graph P_n^k as follows:

$$V(P_n^k) = \{u_0\} \cup \{v_j^i;\ i \in [1,k] \text{ and } j \in [1,n]\}$$

$$E(P_n^k) = \{v_0v_1^i, v_1^iv_n^i, v_j^iv_{j+1}^i;\ i \in [1,k] \text{ and } j \in [1,n]\}$$

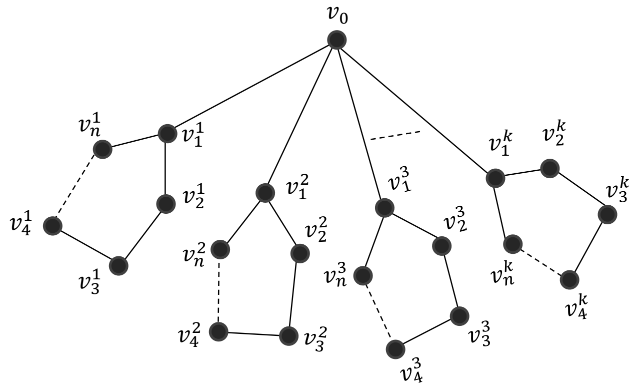

Figure 7 Pendulum graph P_n^k.

Theorem 2.2 Let P_n^k be a pendulum graph with $k \geq 4$ and $n \geq 5$, then $\lambda_{2,1}(P_n^k) = k + 1$.

Proof. Let P_n^k be a pendulum graph with $k \geq 4$ and $n \geq 5$. Since a star $S_{1,k}$ is a subgraph of pendulum graph P_n^k, then based on the Lemma 1.4, we get $\lambda_{2,1}(P_n^k) \geq \lambda_{2,1}(S_{1,k}) = k + 1$. Next, we will show that $\lambda_{2,1}(P_n^k) \leq k + 1$ by constructing the $L(2,1)$ labelling of pendulum graph P_n^k. For this labelling, we will consider three cases.

- Case 1: $n \equiv 0 \mod 3$

$$f(v_0) = 0$$

$$f(v_j^1) = \begin{cases} 2; j = 1 \\ 4; j = n \\ 1; j = n-1 \\ 5; j = 2 \mod 3,\ j \neq n-1 \\ 0; j \equiv 0 \mod 3,\ j \neq n \\ 3; j \equiv 1 \mod 3,\ j \neq 1 \end{cases}$$

$$f(v_j^2) = \begin{cases} 3; j \equiv 1 \mod 3 \\ 1; j \equiv 2 \mod 3 \\ 5; j \equiv 0 \mod 3 \end{cases}$$

$$f(v_j^3) = \begin{cases} 4; j = 1 \\ 1; j = n \\ 2; j = 2 \\ 5; j \equiv 0 \mod 3, \quad j \neq n \\ 0; j \equiv 1 \mod 3, \quad j \neq 1 \\ 3; j \equiv 2 \mod 3, \quad j \neq 2 \end{cases}$$

$$f(v_j^i) = \begin{cases} i+1; j = 1 \mod 3, i \geq 4 \\ 3; j \equiv 2 \mod 3 \\ 1; j \equiv 0 \mod 3 \end{cases}$$

- Case 2: $n \equiv 1 \mod 3$

$f(v_0) = 0$

$$(v_j^1) = \begin{cases} 2; j \equiv 1 \bmod 3,\ j \neq n \\ 4; j \equiv 2 \bmod 3 \\ 0; j \equiv 0 \bmod 3 \\ 5; j = n \end{cases}$$

$$(v_j^2) = \begin{cases} 3; j = 1 \\ 5; j = 2, n-1 \\ 1; j = n \\ 2; j \equiv 0 \bmod 3,\ j \neq n-1 \\ 4; j \equiv 1 \bmod 3, j \neq n, j \neq 1 \\ 0; j \equiv 2 \bmod 3, j \neq 2 \end{cases}$$

$$(v_j^3) = \begin{cases} 5; j \equiv 0 \bmod 3, j \neq n \\ 3; j \equiv 1 \bmod 3, j \neq 1 \\ 0; j \equiv 2 \bmod 3, j \neq 2 \\ 4; j = 1 \\ 2; j = 2 \\ 1; j = n \end{cases}$$

$$f(v_j^i) = \begin{cases} i+1; j \equiv 1 \bmod 3, j \neq n, 4 \leq i \leq k \\ 2; j \equiv 2 \bmod 3 \\ 0; j \equiv 0 \bmod 3 \\ 3; j = n \end{cases}$$

- Case 3: $n \equiv 2 \bmod 3$

$f(v_0) = 0$

$$(v_j^1) = \begin{cases} 2; j \equiv 1 \bmod 3, j \neq n-1 \\ 4; j \equiv 2 \bmod 3, j \neq n \\ 0; j \equiv 0 \bmod 3 \\ 3; j = n-1 \\ 5; j = n \end{cases}$$

$$(v_j^2) = \begin{cases} 3; j = 1 \\ 1; j = n \\ 5; j = 2 \\ 2; j \equiv 0 \bmod 3, j \neq n, j \neq 2 \\ 4; j \equiv 1 \bmod 3, j \neq 1 \\ 0; j \equiv 2 \bmod 3 \end{cases}$$

$$(v_j^3) = \begin{cases} 4; j \equiv 1 \bmod 3, j \neq n-1 \\ 2; j \equiv 2 \bmod 3, j \neq n \\ 0; j \equiv 0 \bmod 3 \\ 3; j = n-1 \\ 1; j = n \end{cases}$$

$$(v_j^i) = \begin{cases} i+1; j \equiv 1 \bmod 3, j \neq n-1, 4 \leq i \leq k \\ 2; j \equiv 2 \bmod 3, j \neq n \\ 0; j \equiv 0 \bmod 3 \\ 3; j = n-1 \\ 1; j = n \end{cases}$$

In the same way as Theorem 4, it is easy to prove that every two vertices with distance one receive labels that differ by at least two, and every two vertices at a distance two receive labels that differ by at least one.■

3. CONCLUSION

Based on the results and discussion described in section 2. We conclude that the minimum span value of the $L(2,1)$ labeling) of the lollipop graph $L_{m,n}$ is $2m-2$ with $m \geq 3$ and n positive integers, and the minimum span value of the $L(2,1)$ labeling of the pendulum graph P_n^k is $k+1$ with $k \geq 4$ and $n \geq 5$. The minimum span value for a lollipop graph is the same as the minimum span value for a complete graph.

Further research suggests $L(2,1)$ labeling vertices and edges in several graphs or other graph operations

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