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An Empirical Study on Markowitz and Single Index Model

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ABSTRACT
In this paper, the Markowitz Model and the Single-Index Model were used to study U.S. stocks. Four industries of U.S. stocks were selected to empirically analyze the data from 2002 to 2021, and the combination selection results given by the two models under five restrictions were compared. The results turned out that with all five constraints, the Sharpe ratio of the optimal portfolio provided by the Index Model, 1.432, 1.523, 1.597, 1.279, 1.333, respectively, was higher than that by the Markowitz Model,1.416, 1.503, 1.540, 1.256, 1.329, respectively, which showed that although both models can give investors investment suggestions for portfolio selection, the single-index model performs better.

Keywords: *Markowitz Model, Index Model, Optimal Portfolio.*

1. INTRODUCTION

As we all know, investment is a science, and there have been a lot of scholars figuring out how to make up portfolios. In 1952, Harry Markowitz used probability theory and quadratic programming method to solve the problem of risk portfolio selection in his paper 'Portfolio Selection, which is the symbol of the birth of modern portfolio management theory [1]. His paper gave the main principles and methods of how to calculate the return and risk of portfolios and the Mean-variance Model was established, which solved the problem of optimal capital allocation among risky assets. In 1963, William Sharpe proposed an attractive candidate for the Markowitz Model, the Index Model, which solved the drawbacks of a large amount of calculation in the Markowitz Model [2]. Also, Lintner(1965) and Mossin(1966) gave their contribution and the CAPM was then built up [3][4]. After then, the multi-factor model was invented, such as Fama-French's Three-Factor Model, Ross's APT Model [5][6]. These models provided more options for portfolio selection.

In terms of the practical application effect of the model, many scholars have made relevant empirical analyses. Some scholars thought the Index Model was not suitable in the real market. Miller and Scholes (1972) used the data of 631 stocks listed on the NYSE for 10 years (1954-1963), and the results show that the coefficient does not explain the return on assets well [7]. However, some scholars Sharpe and Cooper (1972) also used the data of NYSE (1931-1967) and found that the coefficient has a high explanatory ability [8]. So, it seems that the Index Model can be applied to practice to a certain extent.

Up to now, the model has more expanded forms, for example, there are models that include consumption into analysis factors, CCAPM. However, Breeden, Gibbons, and Litzenberger (1989) studied the empirical meaning of CCAPM, and compared its performance with the results based on traditional CAPM and the results show that the performance is roughly the same [9].

Also, through empirical research and analysis, Kalman J. Cohen and Jerry A. Pogue (1967) concluded that for the strict field of common stock, the performance of the multi-factor model is not better than that of the single-index model [10].

In conclusion, the Single-Index Model still has its application value and the aim of this paper is to use the latest data to empirically analyze the utility of the Index Model, also comparing it with the Markowitz Model.

The structure of the paper is as follows. The first part is the data used in this paper. The second part is the description of the two models and the five constraints. The third part is the results, and lastly the conclusion.

2. DATA

The data which been used in the paper is a recent 20 years of historical daily total return data for ten stocks, which belong in groups to four different sectors according to Yahoo! finance. To reduce the non-Gaussian effects, daily data of these stocks is aggregated to the monthly observations. The ten stocks are AMZN, AAPL, CTXS, JPM, BRK/A, PGR, UPS, FDX, JBHT. LSTR, and they belong in consumer cyclical, technology, financial services, and industrials. Also, the data includes one equity index, S&P 500, and a proxy for the risk-free rate, which is a 1-month Fed Funds rate. The introduction of the stocks is as follows.

AMZN is Amazon.com, Inc. It belongs to the consumer cyclical. It is an online retailer that offers a wide range of products, which include books, music computers, electronics, and numerous other products.

Figure 1 demonstrates the monthly stock price of Amazon. We can see that the stock price was low in 2002, about 14.68 dollars per share, while within two decades, it soured up almost 200 times, to 3467.42 dollars per share.

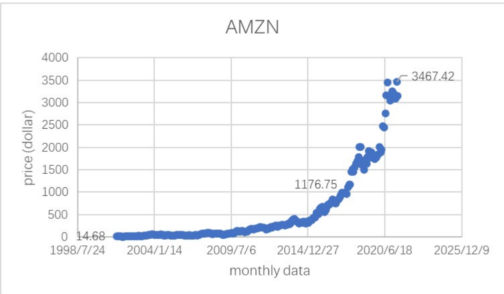

Figure 1. AMZN stock price chart

AAPL refers to Apple Inc. It is in technology groups. The company designs manufacture and markets personal computers and related personal computing and mobile communication devices along with a variety of related software, services, peripherals, and networking solutions. The second technology company, CTXS, is Citrix Systems, Inc. It designs, develops, and markets technology solutions that allow applications to be delivered, supported, and shared on-demand. The company develops and markets comprehensive solutions across all dimensions of application, server, and desktop virtualization, as well as application and network optimization.

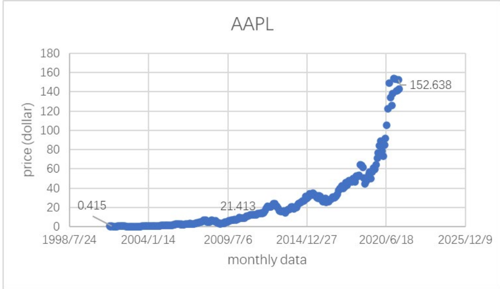

Figure 2. AAPL stock price chart

Figure 2 demonstrates the monthly stock price of Apple. The price was 0.416 dollars per share in 2002, while it went up to 152.638 dollars per share in 2022.

The next three companies JPM, BRK/A, and PGR belong to the financial services sector. The full name of JPM is JPMorgan Chase &Co. It offers global financial services and retail banking. Its services include investment banking, treasury and securities services, asset management, private banking, card member services, commercial banking, and home finance. BRK/A refers to Berkshire Hathaway Inc. and it is a holding company owning subsidiaries in a variety of business sectors. Its principal operations are insurance business. The third is PGR, Progressive Corporation is also an insurance holding company.

The last four companies belong to the industrial sector. United Parcel Service, Inc, which is UPS, not only offers packages and documents delivery services but also provides global supply chain services. FDX means FedEx Corp, which also provides packages and freight delivery services. JBHT means J. B. Hunt Transport Service, Inc. It offers logistics services. The last one is LSTR, referring to Landstar System, Inc, which is a North American truckload carrier

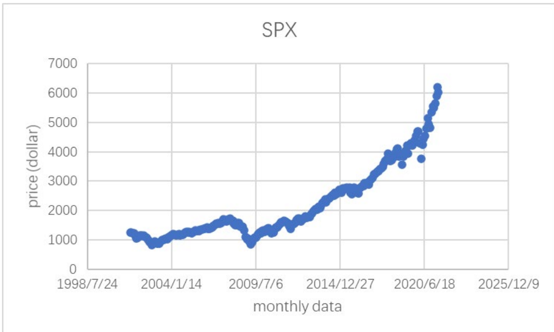

Figure 3. SPX price chart

Figure 3 shows the monthly price of an index, S&P500. We can see that S&P has increased almost five times in the past two decades.

3. METHOD

The method used in this paper to find the optimal portfolio is two models, the Markowitz Model, and the

Index Model, and to make the optimal portfolio can be practical in different situations, five constraints are used.

3.1. Markowitz Model

According to the assumptions of the Markowitz Model, the investors are risk-averse and would like to find a portfolio with minimum risk for a given level of return. Under the model, the return of the portfolio is the proportion-weighted combination of the constituent assets' returns. The volatility shows the portfolio's risk, so the proxy for the risk of the portfolio is the variance or the standard deviation.

The formulas used in the Markowitz Model are as follows.

$$E(r_p) = \sum w_i E(r_i) \quad (1)$$

where $E(r_p)$ means the expected return on the portfolio, w_i means the weight of component asset$i$, $E(r_i)$ is the expected return on the asset i.

$$\sigma_p = \sqrt{\sum_{i,j} w_i w_j cov(i,j)}$$
(2)

where $\sigma_p$means the standard deviation of the portfolio, so the variance should be σ_p^2, $cov(i,j)$ means the covariance of the periodic returns on the two assets, and also can be denoted as $\rho_{i,j}\sigma_i\sigma_j$, where $\rho_{i,j}$ means the correlation coefficient between the returns on asset i and asset j.

3.2. Index Model

The Index Model assumes that only one macro factor will affect the risk and return of stocks, and this macro factor can be replaced with the market index, such as S&P500. So the return of any stock can be deposed into three parts, which are the firm-specific expected value, unanticipated developments in the macroeconomy, and a firm-specific unexpected component.

The formulas used in the Index Model are as follows.

$$r_i = \alpha_i + \beta_i m + e_i \quad (3)$$

where r_i is the return of the asset i, α_i is the firm-specific expected value, $\beta_i m$ is the market factor surprise, and e_i is the firm-specific surprise, also called residuals.

So, the expected return of the portfolio can be denoted as follows.

$$E(r_i) = r_f + \alpha_i + \beta_i\big(E(r_m) - r_f\big)$$
(4)

where r_f is the risk-free return, $E(r_m)$ is the expected return of the market portfolio, so $E(r_m) - R_f$ means the market premium, the excess return on the market portfolio, β_i is the responsiveness of asset i to the market return. The expected return of e_i is zero.

Also, the formula can be shown to be

$$E(R_i) = \alpha_i + \beta_i E(R_m)$$
(5)

where $E(R_i) = E(r_i) - r_f$, the excess return of the asset i, $E(R_m) = E(r_m) - r_f$, the excess return of the market portfolio.

The standard deviation of asset i in the Index Model:

$$\sigma_i = \sqrt{\beta_i^2\sigma_m^2 + \sigma^2(e_i)}$$
(6)

The formula above means the total risk of asset i can be deposed into systematic risk and firm-specific risk. In formula 6, the former part is the systematic risk and the latter part is the firm-specific risk.

$$E(R_p) = \sum w_i\alpha_i + w_i\beta_i R_m$$
(7)

where $E(R_p)$ is the expected excess return of the portfolio.

$$\sigma_p = \sqrt{\sum w_i^2\beta_i^2\sigma_m^2 + w_i^2\sigma^2(e_i)}$$
(8)

3.3. Five Constraints

The first constraint is the sum of absolute values of all asset weights must be less than or equal to 2. This constraint is designed to simulate Regulation T by FINRA, which is authorized by Congress to protect America's investors by making sure the broker-dealer industry operates fairly and honestly. This constraint allows broker-dealers to allow their customers to have positions, 50% or more of which are funded by the customer's account equity.

$$\sum |w_i| \le 2$$
(9)

w_i means the weight of each asset in the portfolio.

The second constraint is the absolute value of each asset weight should be less than or equal to 1. This constraint is designed to simulate some arbitrary "box" constraints on weights, which may be provided by the client.

$$|w_i| \le 1 \quad (10)$$

The third constraint is no constraint and to see how the area of permissible portfolios and efficient frontier, in general, look like.

The fourth constraint is every weight of asset should be more than or equal to zero. This constraint is designed to simulate the typical limitations existing in the U.S. mutual fund industry: a U.S. open-ended mutual fund is not allowed to have any short positions.

$w_i \geq 0$
(11)

The last constraint is the weight of the index should be equal to zero to see whether the inclusion of the index in the portfolio has a positive effect.

$w_{spx} = 0$ (12)

4. RESULT ANALYSIS

This paper analyzed the results under different constraints from the perspective of the two models, and finally compares the performance of the two models

4.1. Results of the Markowitz Model

To find the optimal risky portfolio, the Sharpe ratio is introduced. Sharpe ratio is the difference between the return of the risky asset and the risk-free return, divided by the standard deviation of the risky asset. The optimal portfolio should have the maximum Sharpe ratio among all possible portfolios.

$Sharpe\ ratio = E(R_p)/\sigma_p$
(11)

Table 1 shows the excess return, the risk, and the Sharpe ratio of the optimal portfolio in MM(Markowitz Model) under five constraints.

We can conclude from the results in table 1 that when under constraint 1, the optimal risky portfolio's return is 26.44%, while the standard deviation is 18.67%. When under constraint 2, the return of the optimal risky portfolio is 33.2% and the risk is 22.09%. When under no constraint, the optimal return is 49.323%, while the risk is 32.023%. When no short positions are allowed, the optimal return is 22.106%, and the risk is 17.588%. Finally, when no index is included, the return of the optimal portfolio is 23.987%, and the risk is 17,978%.

Table 1. The indicators of each optimal risky portfolio of the MM under five constraints

Constraint	Return $E(R_P)$	Risk σ_p	$Sharpe\ ratio$
1	26.44%	18.67%	1.416514313
2	33.20%	22.09%	1.503242045
3	49.323%	32.023%	1.54025711
4	22.106%	17.588%	1.256907131
5	23.897%	17.978%	1.329272704

Table 3 shows the weights of each stock in the maximum Sharpe ratio portfolio in the MM(Markowitz Model) under five constraints.

4.2. Results of the Index Model

Table 2 shows the excess return, the risk, and the Sharpe ratio of the optimal portfolio in the IM(Index Model) under five constraints.

Table 4 shows the weights of each stock in the maximum Sharpe ratio portfolio in the IM(Index Model) under five constraints.

Table 2. The indicators of each optimal risky portfolio of the IM under five constraints.

Constraint	Return $E(R_P)$	Risk σ_p	$Sharpe\ ratio$
1	28.552%	19.927%	1.432825886
2	34.075%	22.365%	1.523585939
3	60.474%	37.856%	1.597478287
4	24.452%	19.109%	1.27961619
5	26.965%	20.215%	1.333920466

As Table 2 demonstrates, under constraint 1, the excess return of the portfolio is 28.552%, while the corresponding risk is 19.927%, and the Sharpe ratio is 1.4329. Under constraint 2, the portfolio's excess return is 28.552%, while the risk is 22.365%, and the Sharpe ratio is 1.5236. Under constraint 3, the excess return of the portfolio is 60.474%, while the corresponding risk is 37.856%, and the Sharpe ratio is 1.5975. Under constraint 4, the excess return of the portfolio is 24.452%, while the risk is 19.109%, and the Sharpe ratio is 1.2796. Under constraint 5, the excess return of the portfolio is 26.965%, while the risk is 20.215%, and the Sharpe ratio is 1.3340.

5. CONCLUSION

This paper selects ten stocks from four industries, adds a broad market index to the portfolio, and uses the Markowitz model and index model to calculate the optimal risky portfolio under different constraints. By comparing both results from the Markowitz Model and the Index Model, we can conclude the following result.

Under constraint 1, which is the Regulation T , we can see that the Index Model offered a higher Sharpe ratio than the Markowitz Model, which means the portfolio it provided is better. So, when we have to obey Regulation T, we can use the Index Model to figure out the optimal risky portfolio. Under constraint 2, the Index Model provided a higher Sharpe ratio, so it is better to choose the Index Model. Under constraint 3, which means no constraint, the Index Model still provided a higher Sharpe ratio, so Index Model offered a better portfolio. Under constraint 4, which means no short position is allowed, the Sharpe ratio of the Index Model was higher. So, it is better to choose Index Model. Under constraint 5, both models showed that the exclusion of an index in the portfolio had a negative effect, which means when we made up a portfolio, we should add an index to optimize the portfolio. Also, the Index Model gave a result with a higher Sharpe ratio, so we should choose the Index Model to help us find a better portfolio. In conclusion, under all five constraints, the Index Model performed better than the Markowitz Model.

The result shows that compared with the Markowitz Model, the Index Model does have its advantages such as less information needed, better performance, and the Index Model does provide the useful suggestion of the portfolio selection for different conditions, which is useful for the investors.

However, the model used in this paper is a little simple and does not take into account other factors affecting the stock price in reality. Therefore, in future research, more complex models will be used for analysis.

Table 3. The weights of each asset in the optimal risky portfolio of the Markowitz Model under five constraints.

	Weight										
	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR
1	-48.21%	16.37%	29.95%	-0.10%	-0.09%	41.42%	32.96%	-0.02%	-1.50%	12.48%	16.73%
2	-100.00%	22.30%	39.70%	-1.18%	-0.47%	62.53%	45.98%	-3.04%	-10.54%	20.83%	23.89%
3	-235.02%	36.74%	64.84%	0.41%	17.01%	91.14%	67.75%	1.29%	-8.69%	30.71%	33.81%
4	0.00%	12.92%	25.11%	0.00%	0.00%	19.43%	22.73%	0.00%	0.00%	8.76%	11.05%
5	0.00%	14.60%	26.63%	-3.42%	-15.50%	36.43%	31.94%	-12.06%	-13.18%	17.75%	16.79%

Table 4. The weights of each asset in the optimal risky portfolio of the Index Model under five constraints.

	Weight										
	SPX	AMZN	AAPL	CTXS	JPM	BRK/A	PGR	UPS	FDX	JBHT	LSTR
1	-47.26%	17.83%	30.33%	0.71%	-2.70%	22.66%	31.34%	1.37%	0.06%	19.09%	26.57%
2	-100.00%	21.79%	36.53%	3.09%	-8.73%	33.42%	39.96%	9.59%	5.65%	24.92%	33.78%
3	-326.25%	42.60%	69.30%	10.90%	8.64%	58.25%	71.82%	27.74%	24.69%	49.88%	62.42%
4	0.00%	15.39%	27.00%	0.00%	0.00%	3.51%	21.38%	0.00%	0.00%	13.93%	18.80%
5	0.00%	18.54%	31.59%	-0.33%	-21.27%	11.85%	27.87%	-5.34%	-5.51%	18.44%	24.15%

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