A Monte Carlo Simulation Scheme for Basket Options with Barriers

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ABSTRACT

In this paper we introduce simulation pricing of a sample barrier call option on a basket of stocks under the multivariate Black-Scholes-Merton scheme. Ten S&P 500 stocks from different industries were chosen to compose the basket, with the weights assigned to be the optimizer of the basket's Sharpe ratio. Knock-in and knock-out barriers of the options were set to be monitored daily. For simulation the model assumed that the prices of underlying assets were lognormal and correlated with constant drift rates and volatilities. Historical estimations to volatilities and correlations were used, while EWMA model were employed to obtain more representative results. Cholesky decomposition was introduced to generate correlated random vector. Further, we conducted sensitivity analysis over barrier prices, strike prices, and volatilities through our Macro program. 3D diagrams were drawn to illustrate changes in price with respect to multi variables. We drew conclusion that the barrier calls proposed had varying sensitivity to strike price and barrier level under different volatilities. We also discovered that while most barrier options had positive Vegas, the Vega of a knock-out option might be negative when the volatility was relatively too high for its barrier price.

Keywords: *Basket option, Barrier option, Monte-Carlo simulation, Volatility*

1. INTRODUCTION

Basket option, as a popular finance instrument, has been widely traded over the counter (OTC) among institutional buyers. Used as a risk management tool for hedging purposes or the basis for structured products, its payoff is based on the average performance of a basket of underlying stocks, which allows for customized weights tailored for portfolio holders.

Besides, the upper and lower bounds on the prices are applicable whenever the joint characteristic function of the vector of log-returns is known [1].

Industry-standard models are used to price this basket product. The process involves depicting the movement of each individual basket component by applying a matrix of correlation to independent stochastic drivers for various models.

Barrier option, widely used for structured products, has been favored by the speculators who consider it a cheaper but risky alternative to vanilla types. Barrier option incurs more possibilities of a zero payoff, when the price of the underlying hit certain pre-specified barriers i.e., being knocked in or out, the option comes into existence or terminated. In other words, a knock-in option is worthless if the underlying price hasn't reached the barrier during the life of the option, while a knock-out option is worthless if the barrier has been reached.

Barrier options are considered by investors who presume the underlying value will not hit a certain level during the life of the option, or buyers who need to cover their exposure during periods of sharp market volatility. For sellers and structurers of the option, the choice of barrier level is worth a careful investigation.

Barrier options are one typical type of path-dependent option-- its payoff depends not only on the price of underlying assets at expiration but also on the path that the asset price took during the life of the option [2]. There are different barrier types defined by the frequency the underlying price has been monitored, namely daily, weekly, monthly, or even continuously monitored barriers. Its pricing model involves inquiry into prices at each monitored spot, which introduces more complexity towards its formulation and stochastic analysis.

There are also lots of literature about valuing the option. For derivatives whose closed-form solution is hardly defined, simulation is applied. Monte-Carlo simulation, named by Stanislaw Ulam and John Von Neumann [3] when they mentioned it to solve neutron diffusion problems in the mid-1940s, is a well-known method that uses random variables to model the probabilities that cannot be analytically approached. It is based on the law of large numbers (LLN) [4], a theorem behind random sampling which describes the convergence of the result by performing the same experiment a large number of times. The larger the number of trials performed, the closer the average of results obtained from those trials should be to the expected value. The method provides general approach to options' risk-neutral valuation under certain calibrated stochastic model.

Black-Scholes (BS) model is the industry standard for modeling prices' random behaviors. It was first introduced by Black and Scholes [5], later developed by Merton. This model formulates stochastic dynamics of stock prices as lognormally distributed with a constant drift rate and volatility, i.e., standard deviations. According to the risk-neutral assumptions the model proposes, the drift rate is set to be the risk-free interest rate of the market, conforming to the no-arbitrage principle under risk-neutral measures.

Assumed by the Black Scholes model, geometric Brownian motion, also called exponential Brownian motion [6] could be seen as a lognormal adjustment to Brownian motions in continuous time, where the logarithm of random variables follows Brownian motion. The motion is defined by the geometric Wiener process in stochastic differential equations. This method is used to simulate stock price in the Black-Scholes model.

Later, for multi-assets derivatives like basket options, to describe jointly the dependent dynamics of stock prices composing the basket, Bjork extended the Black and Scholes model to multivariate case, where the basket dynamics were depicted by correlated Geometric Brownian motions. A Gaussian copula was used for the correlation structure, where the marginal features of single variate were derived from implied, or historical estimators.

Historical Volatility (i.e., HV) [7] is an estimator to an asset's volatility by the past performance of stock prices. It is a statistical measure of the dispersion of returns for a given security or market index over a given period. In the stock market, HV reflects past volatility of underlying stock price, while it is sometimes to provide an estimation to or a reference for future volatilities.

Favored approaches to historical volatilities include GARCH, EWMA models, and so on. After Engle mentioned the ARCH model to analyze heteroscedasticity of time sequence in 1982, Bollerslev [8] mentioned the GARCH model in 1986. GARCH is a statistical model used in analyzing volatility of time-series data where the variance error is believed to be serially autocorrelated. It is an important method used to formulate HV. Exponentially Weighted Moving-Average (EWMA), introduced firstly by Roberts (1959) [9] and fully established by Hunter (1986), is a moving average of past variances where later data are given exponentially larger weights than earlier entries. Each EWMA point combines all the messages of all the subgroups and observe values before. We could use EWMA to check any skewing with different values, for EWMA could spy and show all the control processes in the graph.

In this paper, a multivariate Black-Scholes model, historical estimators of the volatility were used. Also, this paper performs a Monte Carlo simulation to price the basket and barrier option.

2. INVESTIGATED FIRMS

This paper chose 10 different stocks in 10 different companies to constitute a basket option. Through using variable modern financial methods, this paper simulated different combination results. We collect the data for the following firms.

2.1. WMT

Walmart Inc. (Formerly Wal-Mart Stores, Inc.) is an American retail corporation. The company was founded by Sam Walton in 1962. As of July 31, 2021 [10], Walmart has 10524 stores and clubs in 24 countries. Walmart is the world's largest company by revenue, with US$548.743 billion, according to the Fortune Global 500 list in 2020.

2.2. AMZN

Amazon.com, Inc, is an American multinational technology company that focuses on e-commerce, cloud computing, digital streaming, and artificial intelligence [11]. It is one of the Big Five companies in the U.S. information technology industry, along with Google, Apple, Microsoft, and Facebook. It was founded by Jeff Bezos in 1994. It is the world's largest online marketplace and its value once surpassed Walmart as the most valuable retailer in the United States in 2017.

2.3. AAPL

Apple Inc is an American multinational technology company that specializes in consumer electronics, computer software, and online services. Apple is the world's largest technology company by revenue, totally $274.5 billion in 2020, and, since January 2021, the world's most valuable company. It was founded by Steve Jobs, Steve Wozniak, and Ronald Wayne in 1976. It is also one of the Big Five companies as we mentioned above [12].

2.4. FB

Facebook Inc is an American multinational technology company that mainly runs the namesake social networking service called Facebook [13], which has 2.9 billion monthly users by 2021. It was founded by Mark Zuckerberg, Eduardo Saverin, Andrew McCollum, Dustin Moskovitz, and Chris Hughes in 2004. It is one of the world’s most valuable companies and is considered as one of the Big Five companies as we mentioned above.

2.5. MSFT

Microsoft Corporation is an American multinational technology company that runs businesses over computer software, consumer electronics, personal computers, and related services. Microsoft was founded by Bill Gates and Paul Allen in 1975 [14]. Microsoft ranked No.21 in the 2020 Fortune 500 rankings of the largest United States corporations by total revenue. It was the world’s largest software maker by revenue as of 2016. In 2019, Microsoft reached the trillion-dollar market cap, becoming the third U.S. public company to be valued at over $1 trillion after Apple and Amazon. As of 2020, Microsoft has the third-highest global brand valuation. It is considered as one of the Big Five companies as we mentioned above.

2.6. SPG

Simon Property Group, Inc, is an American real estate investment trust that invests in shopping malls, outlet centers, and community/ lifestyle centers. The company dates to 1960 [15].It is the largest owner of shopping malls in the United States.

2.7. XRX

Xerox Holdings Corporation is an American corporation that sells print, and digital document products and services in more than 160 countries [16][17]. It was founded in 1906. As a large developed company, it is consistently placed in the list of Fortune 500 companies.

2.8. CINF

Cincinnati Financial Company offers property and casualty insurance [18]. It is ranked as the 20th largest insurance company by market share in the U.S. It was founded by John Jack Schiff and Robert Cleveland Schiff in 1950.

2.9. DAL

Delta Air Lines, Inc, (i.e., DAL), typically referred to as Delta, is one of the major airlines of the United States and a legacy carrier. It was founded in 1925. It is ranked second among the world’s largest airlines by the number of scheduled passengers carried, revenue passage kilometers flown, and fleet size. It is ranked 69th on the Fortune 500 [19].

2.10. COTY

COTY Inc is an American multinational beauty company founded in 1904 [20]. It is one of the world’s largest beauty companies and the largest fragrance company, with over $9 billion in revenue for the fiscal year ending in 2018. Coty’s stock was, as of 2020, the smallest S&P 500 component by market capitalization.

3. METHODOLOGY

The experiment introduces an integrated resolution to the problem. The proposed Macros set is an open-source macro toolkit developed for pricing customized derivatives with pay-off based on performance of arbitrary basket of assets. It provides Microsoft Excel user an expedient way of evaluating one typical type of over-the-counter products. Comparing to Python based pricing the built-in macro reads and writes data within excel spreadsheet, with adjustable parameters configurated in the user-friendly interfaces, achieving an easy-to-handle and first-hand analysis for starters.

3.1. Monte-Carlo simulations and Law of Large Numbers

Monte Carlo simulations was used for evaluation of the products’ performance and has been used to value options [21]. As the sample size increases, the random sampled mean would con- verge asymptoticly to the expected return of the products as according to the law of large number. The standard error and confidence interval could be generated according to the specific problem and sample size to evaluate the pricing.

$$Y_N = \frac{1}{N}\sum_{i=1}^{N} Y_i \quad (1)$$

$$SE = \frac{\sigma_Y}{\sqrt{N}} \qquad (2)$$

Where N is the sample size, Y_i stands for the ith sample and Y_N is the sample mean. σ_Y is the standard deviation of the simulated problem and SE stands for the standard error of the sampled mean.

3.2 Multivariate Black-Scholes Dynamics of Basket Prices

Our Macro program simulated the dynamic of assets in the basket under the multivariate Black-Scholes-Merton scheme. The dynamic of each stock was formulated as geometric Brownian motion, where the ratio of prices between two consequent days were lognormally distributed, with mean equal to risk free rate and standard deviation equal to the volatility estimator. The process for each individual stock involves one standard normal random variable.

$$\frac{dS_i}{S_i} = \mu_i dt + \sigma_i \mathrm{d}b \qquad (3)$$

$$S_i(t) = S_i(0)e^{\left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i\sqrt{t}b} \qquad (4)$$

Where $S_i(t)$ stands for the price of the i th underlying asset at time $t, \sigma i$ is the estimated volatility assumed constant of the ith asset, and μ is taken to be the average risk-free rate of the period according to risk neutral measures of the Black Scholes scheme.

To simulate the Gaussian correlated structure of prices as designed by the multivariate model, Cholesky decomposition was used to generate the mutually dependent random normal vector based on the desired covariance matrix.

$$LL^T = Q \qquad (5)$$

$$V = LU \qquad (6)$$

Using historical correlations, Q denotes the desired covariance matrix of the random vector V for simulation, particularly it is equal to the correlation matrix of the historical returns. U is one mutually independent random normal vector.

Applying the above correlated geometric Brownian motions of the simulated underlying stock prices, by the rule of risk-neutral valuation, the theoretical price of the option is determined to be the present value of the expected payoff $E[R(\mathrm{T})]$ at expiration T.

$$P(0) = e^{-rT}E[R(T)] \qquad (7)$$

3.3 Volatility Estimators

The experiment performs under current data the comparison of performance of historical and implied volatilities. The historical volatility was introduced by EWMA and GARCH model. EWMA model assigns to the prices the weights that exponentially increase with date. More recent data weighs into the average with certain specified constant.

$$\sigma_{i,n}{}^2 = (1 - \alpha)r_{i,n-1}{}^2 + \alpha\sigma_{i,n-1}{}^2 \quad (8)$$

In the above expression, σn denotes the estimated volatility at day n, rn denotes the daily return at the same day. The α is a constant chosen between 0 and 1. GARCH model considers the weight for the long-time average variance of daily returns.

$$\sigma_{i,n}{}^2 = \gamma V^2 + \beta r_{i,n-1}{}^2 + \alpha\sigma_{i,n-1}{}^2 \quad (9)$$

$$\alpha + \beta + \gamma = 1 \qquad (10)$$

The implied volatility comes from applying the BSM model to the spot price of the currently traded vanilla call or put option. The up-to-date measures of implied volatility was quoted for each stock. The type of volatility estimators used could be predefined as an adjustable parameter in the interface.

3.4. Daily Monitored Basket Price Against Barriers

Path dependent features of the products such as daily monitored barriers would require our simulation to monitor prices of each spot. Our simulation can record the daily rolling return of the basket assets and determine upon this whether the barrier has been approached. An illustrative chart of the path of daily return of the basket $r_{basket}(t)$ is displayed by preference.

$$r_i(t) = \frac{S_i(t) - S_0(t)}{S_0(t)}, \qquad i = 1..N \quad (11)$$

$$r_{basket}(t) = \sum_{i=1}^{N} w_i r_i(t) \qquad (12)$$

4. RESULTS AND DISCUSSION

This section demonstrates and evaluates the pricing of one performance of the Macro process and its. We will examine under this situation the price of the call option with or without the barrier. The sector compares among volatility estimators generated by three methods, namely EWMA, GARCH, and implied volatilities.

The simulation tool probes into the detail of changes in prices with different preset parameters (e.g., strike prices, barriers, etc.) and presumed volatilities. The outcome is illustrative since it gives the researcher a basic recognition of how prices are influenced under possibly different backgrounds, and how traders using the same scheme might evaluate a product based on beliefs.

The analysis is divided into two schemes by either varying strike prices or barriers. In one scheme strike prices ranging from 100% to 120% are considered, which influences the product's payoff as a call. In another scheme, knock-in or knock-out barriers are discussed within reasonable intervals. The variables construct a possible decision space from a structure point of view. To account for the difference in volatility estimations, all the schemes incorporate an extra axis of the presumed volatility ranging from 0.6 to 1.8 of one 'preferred' estimation. Sensitivity along this axis is commonly referred to as Vega of the option, which is hardly obtainable by analytical methods for exotic products. Under each scheme price of the basket call with no barriers is obtained as the baseline (Figure 1).

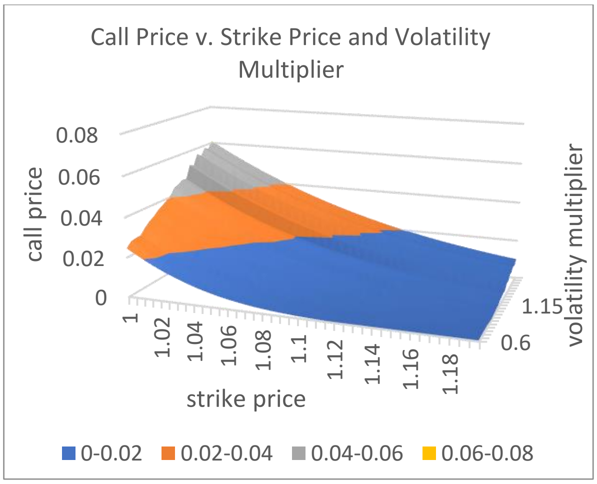

Figure 1 Call Price with Different Strike Prices and Volatilities

The prices of a basket call are analyzed. Analyzing the influence of strike price towards the call option when no barriers are involved, as is shown in (Figure 1), the price of the option decreases and gradually converge to zero as the strike price increases, with the absolute value of the slope decreasing until convergence. Notably at strike price close to 100%, the slope of change versus strike price for different volatility measures appears to be consistent (See Figure 2). Lower volatility isn't associated with significant higher rate at which price decreases before it converges to zero. Increments of presumed volatility unsurprisingly influence the price in a positive way, indicating a positive Vega, while this sensitivity is more eminent given lower strike prices, indicating that the prices are more volatile to volatility, marked by a Vega growing from 0 to approximately 3.8% per unit of multiplier at 100% strike price (See Figure 3). To sum up, the decrease in strike prices and the increase in volatility estimator will lead to increases in payoff.

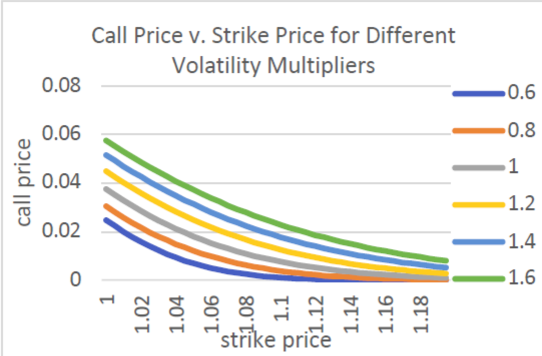

Figure 2 Call Price v. Strike Price with Different Volatilities

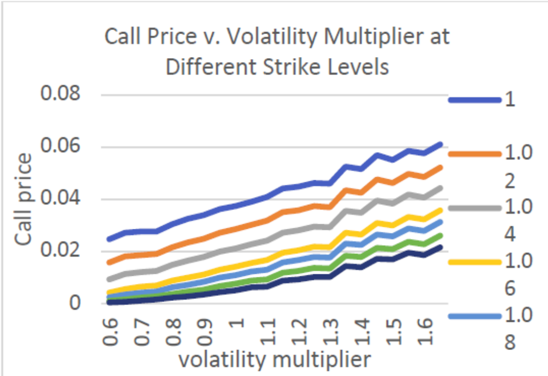

Figure 3 Call Price v. Volatility with Different Strike Levels

When up-and-in barrier of 110% has been applied, the prices of the products fall to different extents (Figure 4). The shrinkage is especially significant when volatility is lower, measured by approximately 49% at the point of (strike =100%, volatility multiplier=0.6). Under this situation, the slope along axis of strike price varies for different volatility multipliers. As is shown in figure 5. Lower volatility is associated with smaller slope, where prices are more insensitive to strike prices. The average Vega along volatility axis increases from 0 to approximately 3.8% per unit of multiplier at lower strike prices.

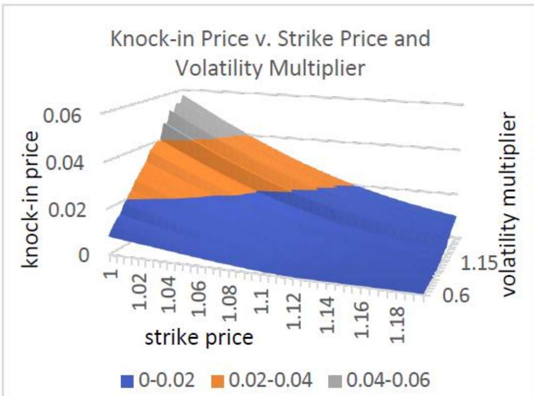

Figure 4 Knock-in Price with Different Strike Prices and Volatilities

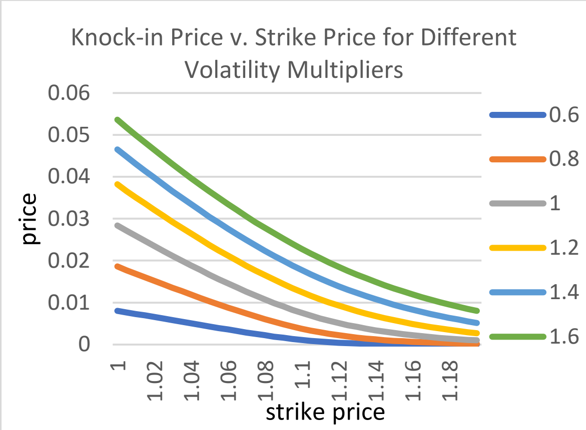

Figure 5 Knock-in Price v. Strike Price at Different Volatilities

When up-and-out barrier of 110% is applied, the option price is mutually complementary to those with the same level of knock-in barrier, with up-and-out value equal to call value subtracted by up-and-in value (Figure 6). It is remarkable that for this up-and-out option, higher volatility seems to imply lower sensitivity of the option's value to changes in strike prices (Figure 7).

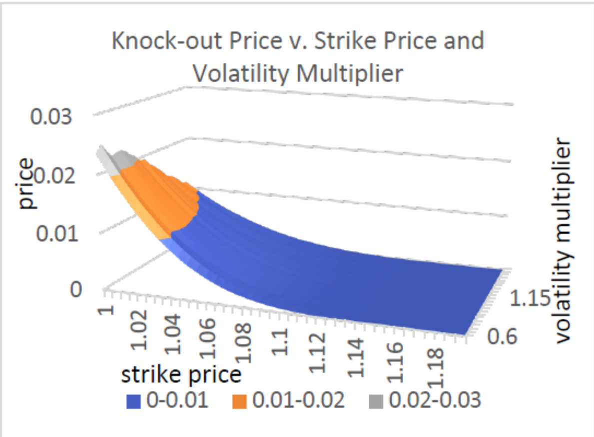

Figure 6 Knock-out Price with Different Strike Prices and Volatilities

Figure 7 Knock-out Price v. Strike Price at Different Volatilities

One critical decision for traders of a barrier option is to specify barriers for their exotic products. The process could be based on realistic considerations, limitations, and trusted market predictions. Our toolkit helps investigate Black Scholes sensitivity of prices, knock-in and knock-out probabilities towards barriers, giving an insight to how barrier settings are influencing moneyness of the derivative. In the following research, strike price of the call option is set at 100%.

The analysis first dug into the details of knock-in and knock-out probability. Generally, the probabilities are both lower for higher upper barriers (see Figure 8 illustrating knock-out probability as example), which means, less unlikely for the value of the basket to reach at any monitoring day before expiration. Moreover, these probabilities appear to bend towards lower volatility measures (Figure 9). Value of the barrier options under lower volatility would have higher sensitivity to barrier levels.

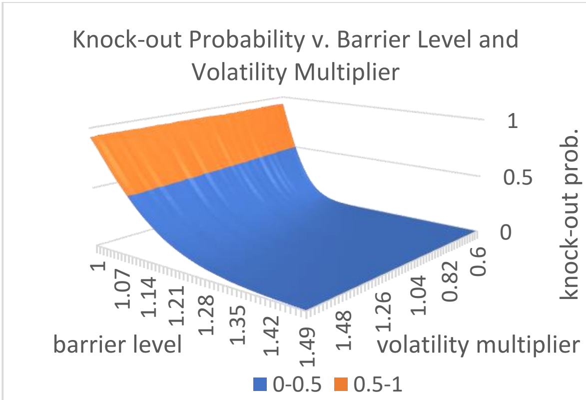

Figure 8 Knock-out Probability with Different Barrier Levels and Volatilities

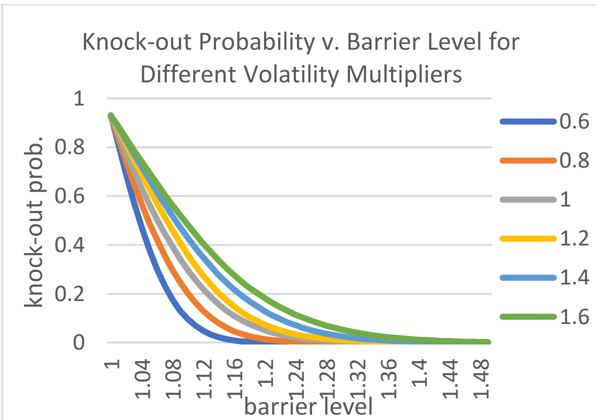

Figure 9 Knock-out Probability v. Different Barrier Levels at Different Volatilities

Figures 10 and 11 illustrate that, Price of an up-and-in call options falls from call value towards 0 with a rising barrier, while the price of an up-and-out call option grows from 0 to call value with a rising barrier. In each case the rate of change increases at first, and then decreases when the price converges (Figure 12 and 13). Noticeably, the average rate of decrease doesn't seem to differ comparing under different volatility multipliers. The only difference is for lower volatility, the call value tends to be lower, and the shape of the price-to-barrier curve assembles shrunken version of those with greater volatilities.

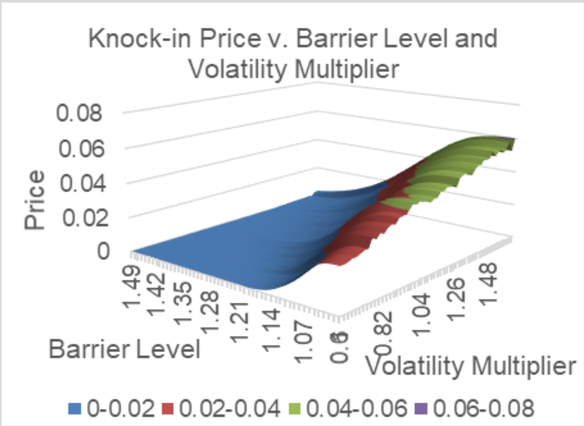

Figure 10 Knock-in Price v. Barrier Level and Volatility

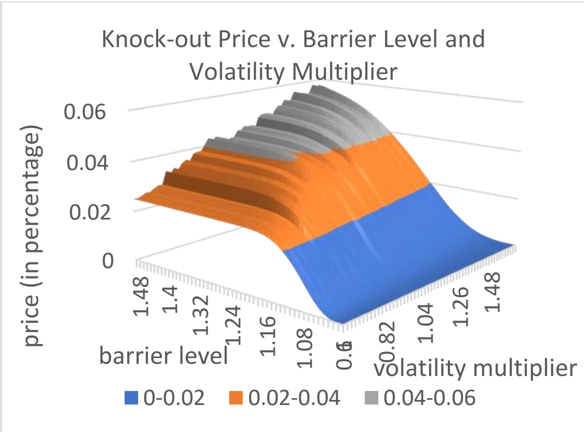

Figure 11 Knock-out Price v. Barrier Level and Volatility

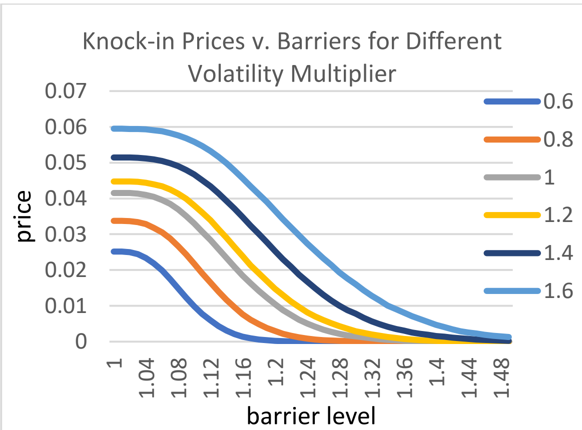

Figure 12 Knock-in Price v. Barrier Level at Different Volatility

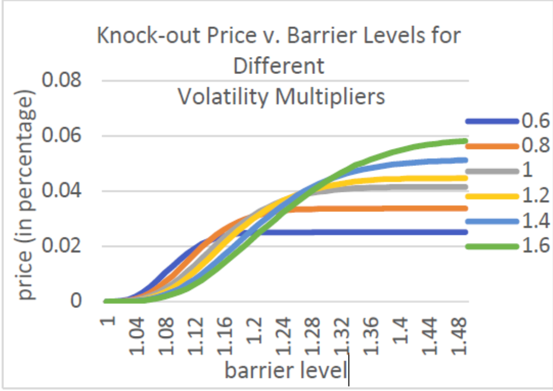

Figure 13 Knock-out Price v. Barrier Level at Different Volatility

Along the volatility axis, the price changes in rate Vega of the corresponding barrier option. It's revealed that all the up-and-in call options has positive Vegas (Figure 14), while the up-and-out call with barrier level smaller than 120% have generally negative Vegas (Figure 15), meaning their prices would possibly decrease when higher volatility is applied.

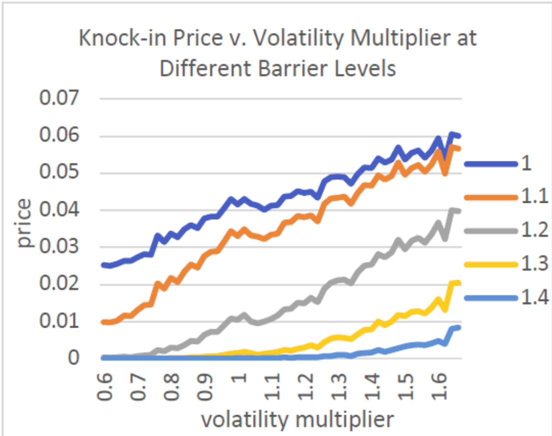

Figure 14 Knock-in Price v. Volatility with Different Barrier Levels

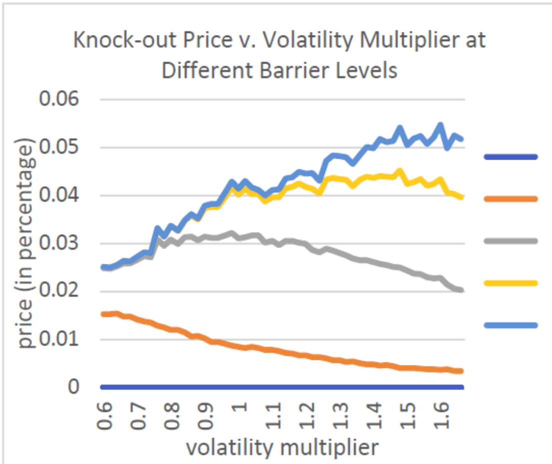

Figure 15 Knock-out Price v. Volatility with Different Barrier Levels

5. CONCLUSION

The research has revealed the following insights on changes of barrier levels, strike prices and volatilities could made on the Black-Scholes price of the sample basket call options. First, as a call option, regardless of barrier type, its value decreases as strike price increases. If no barriers are applied, the rate at which it decreases with strike price despite different volatilities. However, when up-and-in barriers are applied, the rate of decrease is greater for higher volatilities, while it is the opposite when up-and-out barriers are applied. This could be because that knock-in and knock-out barriers have curtailed the moneyness of the option especially under higher or lower volatilities.

Second, adding knock-in or knock-out barriers will cause a significant discount price of the option compared to call option with no barriers. The devaluation arises from restrictions imposed by barriers on the expected payoff. The value of the barrier option is also sensitive to changes in barrier levels. When the level of up-and-in barrier rises from 100%, the value of the option decreases from call value and converges to 0. When the level of up-and-out barrier rises from 100%, the value of the option increases from 0 and converges to call value. The average rate of the above changes before convergence doesn’t seem to differ under different volatilities. Moreover, the shapes of the graphs of option value vs. barrier levels at different volatilities resemble each other.

Third, volatility influences prices of the basket options in different ways. The investigated up-and-in calls with any barriers and up-and-out call with barrier levels higher than approximately 120% all seem to have a positive Vega, indicating that an increased estimated volatility would imply an increase in expected payoff, and so the option’s price. However, up-and-out calls with barrier levels lower than 120% is shown to have a generally negative Vega, i.e., the price of the option will most probably fall with an increasing expected volatility.

Although the Black-Scholes-Merton can successfully price basket option with barriers, it still has many limitations. First, the model simply assumes constant, instead of stochastic volatility. Advanced model run simulations based on carefully calibrated implied volatility surfaces, where the sensitivity analysis should require bumping of the whole volatility surface as a varying input. Second, the assumption of risk-neutral valuation upon which the model relies still shows imperfections. Moreover, implied instead of historical correlations should be used to further depict future correlated movement of the underlying assets.

Possible future research includes Implement of control variate into Macro set to improve the efficiency of the simulation. number of simulations to reach the same level of standard error, and the error given same number of trials is expected to be reduced. Other related topics such as continuous barrier options are to be visited. Though given analytical solutions, the simulation approach to exotic option with continuous barriers needs to be further investigated.

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