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Extensions for Macaulay Duration, Modified Duration, and Further Study on Immunization Strategy

Yuhang Dai[1, *, †], Kejun Guo[2, †], Zikai Jin[3, †] and Yuxin Wu[4, †]

[1]*Zhongshan No.1 Middle School, No.68, Xingwen Road, Zhongshan, Guangdong Province, People's Republic of China*

[2]*Ulink in Shanghai, No.559, Laiting South Road, Songjiang District, Shanghai, People's Republic of China*

[3]*No.1 Middle School of Jiangyin, No.88 Changjiang Road, Jiangyin, Wuxi, Jiangsu Province, People's Republic of China*

[4]*Dulwich International High School Suzhou, No. 360, Gangtian Road, Suzhou Industrial Park, Suzhou, Jiangsu Province, 215000, People's Republic of China*

*Corresponding author. *E-mail addresses: guanghua.ren@gecacademy.cn*

[†]*These authors contributed equally.*

ABSTRACT

This article introduces these contents in detail about Macaulay duration, Modified duration, and immunization strategy. The article is based on Macaulay Duration to study the relevant information of the financial market. This study takes two different information of the bond. Immunization needs to be determined according to the actual situation. For example, it is necessary to immunize bonds according to the real condition rather than rely entirely on the theory to operate by studying the limitation of immunization. This work would help the investors to better establish the portfolio.

***Keywords**: Bond, Macaulay duration, Modified duration, Immunization*

1. INTRODUCTION

Duration is the metric that clearly determines the "effectiveness" of a financial instrument [1,2]. Frederick Robertson Macaulay first derived the measure for the average maturity of the bond's guaranteed cash flows in 1938 [3]. This is called Macaulay Duration named after its inventor. When managers try to analyze the maturity of a bond or its interest rate sensitivity, the most used tool is the "Macaulay Duration" in terms of its simplicity. It reveals the average time to each payment [4-6]. Also, it is broadly used by investors and portfolio managers because it is related to the time a bond is exposed to the interest rate volatility when there is a change in the yield to maturity, which is identified as the modified duration.

As an indicator, Macaulay duration is also widely utilized by portfolio managers who use duration matching as an immunization strategy. Immunization strategies are what portfolio managers frequently used to alleviate the risk of being exposed to interest rate change causing a capital loss through constructing an immunized portfolio. Strategies are usually cash-flow matching, duration matching, convexity matching, and options on bonds [7]. This essay will only focus on the duration matching strategy.

As the duration was invented, it is a feasible tool to help to understand and alleviate the interest rate risk.

This article begins by elaborating the basic knowledge and computations of Macaulay Duration, afterward, its limitations. It then identifies the extension about the modified duration and its limitations. At last, it extends the applications of duration as a duration matching strategy and its limitation.

In general, it is found that the Macaulay duration is a useful tool to apply in some cases, but it does come with some non-negligible limitations, which are the focus of what further research needs to address.

In this article, Section 2 elaborating an introduction to Macaulay duration and its extension. The modified duration would be shown in Section 3. Section 4 has a very detailed explanation of the immunization strategy. Lastly, it summarizes the conclusion in Section 5.

2. MACAULAY DURATION

To start with, this paper would introduce Macaulay duration which is the weighted average term to maturity of the cash flows from a bond. According to the formula, Equation (1), Macaulay duration is calculated by summing up all the multiples of the present values of cash flows and at each period respectively, and then dividing the sum by the market bond price. In these figures, PV(CFt) represents the present value of the cash flow at period t; 'n' means the total number of periods to maturity. And there is another form of formula which is more specific that is used to calculate Macaulay duration. In this case, 'c' means the periodical coupon rate and 'y' means the yield in each period.

$$Macaulay\ Duration = \frac{\sum_{t=3}^{n} PV(CF_t)*t}{Market\ Bond\ Price} = \frac{\sum_{t=1}^{n} \frac{t*C}{(1+Y)^t} + \frac{n*M}{(1+Y)^t}}{Market\ Bond\ Price} \quad (1)$$

As a result, this research can derive some factors influencing Macaulay duration by deducing the more specific formula. These factors have relationships with the Macaulay duration. It is suggested that coupon rate and yield are both indirectly proportional to the Macaulay duration. As the coupon rate or yield increases, the Macaulay duration will decrease. However, the maturity of the bond is directly proportional to the Macaulay duration so the Macaulay duration will increase followed by the increase in maturity.

Additionally, there are also some characteristics of Macaulay duration [8,9]. Firstly, the duration of a bond is always equal to or less than its maturity. So in what situation will the duration be equal to the maturity? There is an extreme situation: when a bond has a zero-coupon rate, the duration is equal to its maturity. For example, there is a bond with a face value of $100. Its current bond price is $85.48. In this bond, there is no coupon payment and the maturity is 4 years. It is necessary to calculate the Macaulay duration of the bond. You might be struggling with the bond price of $85.48 because it looks very tricky. However, this example implies the characteristic of duration. This means that the duration of this bond should be zero due to the coupon payment of zero. Another characteristic of Macaulay duration is that as maturity increases, the duration tends to be closed to the perpetuity duration. And the perpetuity level of the duration is equal to one divided by the sum of one and the yield. One important feature is that Macaulay duration measures the interest rate elasticity of the bond price, which means it measures the sensitivity of the price change of a bond given a change in the interest rate. In this case, the higher the duration, the higher the price change given a change in the interest rate.

There is another example shown in table 1. There is a bond with a face value of $100 as well. But the current bond price is $100. The annual coupon is different which is 4. The maturity is 2 years. There will be a risk at a level of 4%. So for the first year, the bond's cash flow should be equal to the annual payment of 4, so it can calculate the present value of the first-year coupon pay, 4/(1+4%) to get $3.846. The tPV is equal to 3.846*1 to get 3.846. In the second year, the cash flow will be $104. The present value would be $96.154. So the tPV equals 192.3. The sum of the present value of the bond in 2 years should be $196.2. So the duration should be 0.962.

Table 1. Example for Macaulay duration

Year	CF	PV	tPV
1	4	3.846	3.846
2	104	96.154	192.3
		100	196.2

**Unit in US dollars*

However, there are also limitations of Macaulay duration. In the theory, duration assumes a linear and indirect relationship between interest rate and bond price. However, the relationship between them is likely to be curvilinear. The more convex the relationship between interest rate and bond price, the more inaccurate the rate is for measuring the sensitivity of the interest rate. Therefore, Macaulay duration will predict a lower price than the actual price and large interest rate changes. It will just give an approximate estimate of the sensitivity of the interest rate. The modified duration will improve this problem in the next part.

3. MODIFIED DURATION

After introducing the basic concept Macaulay duration, there is a necessity to add one more concept, so this paper will describe the modified duration as an extension of the Macaulay duration. As it is known that the Macaulay duration is being built based on the change of the bond's price under the condition of the small change in yield to maturity. But one thing to be notified is that the Macaulay duration only forms a similar proportional relationship with these two key terms (the yield to maturity and the bond's price), moreover, this formula only success when the yield to maturity of the bond is very small. While in a financial market, everything has to be as precise as possible. So the modified duration is being invented. This article could describe these two are in the relationship that the modified duration helps to improve the flaw of the Macaulay duration.

This part will start with the deduction of the equation of the modified duration. First of all, it is necessary to be clear about the bond's present value, which is the total present value of the future cash flow in each maturity.

$$P = \sum_{t=1}^{T} \text{PV}(C_t) = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \quad (2)$$

As the modified duration is an extension of the Macaulay duration, it could first place the formula of the Macaulay duration

$$D_{mac} = \frac{\sum_{t=1}^{T}\left[t \cdot \frac{C_t}{(1+y)^t}\right]}{P} \quad (3)$$

Differentiate the formula of the bond's price as the equation between P and 1+y

$$\frac{dP}{d(1+y)} = -\sum_{t=1}^{T} \frac{t \cdot C_t}{(1+y)^{t+1}} = -\frac{1}{1+y} \cdot \left[t \cdot \frac{C_t}{(1+y)^t}\right] \quad (4)$$

Divided both sides by P

$$\frac{dP}{dy}\frac{1}{P} = -\frac{1}{1+y} \cdot D_{mac} = -D_{mod} \quad (5)$$

Far from now, this study could deduce that the small changes in p and y could be written as

$$\frac{\Delta P}{P} = -D_{mod} \cdot \Delta y \quad (6)$$

Therefore, it could be seen that the percentage of the change in bond's price is equal to the multiple of modified duration and the change in the yield to maturity, as in Equation (6). What could be seen behind Equation (6) is that it also proves the usefulness of modified duration when testing the risk that could happen of the bond when there are changes in the interest rate. So far, this research could define the modified duration. Which is an index when measuring the sensitivity of the bond's price to the changes in the interest rate. When there are changes in the interest rate in the market, the bond with a bigger modified duration, the percentage of the price change will be bigger. There is a much simple equation of the modified duration:

$$D_{mod} = \frac{D_{mac}}{1+y} \quad (7)$$

**Where, the D_{mac} means the Macaulay duration, the D_{mod} means the Modified duration.*

From this simple formula, Equation (7), when there are small changes in the yield to maturity, the change in the bond's price and the modified duration is proportional, so it could be said that the modified duration is an improvement while considering the yield to maturity y. As a result, the measurement will be more precise.

So how could use modified duration to judge the risk of the bond. According to the magnitude of the modified duration, when it is big, the ability to resist the risk when there is a rising interest rate is weak while the ability to resist the risk when there is a decreasing interest rate is strong. Oppositely, when the magnitude of the modified duration is small. The ability to resist the risk when there is a rising interest rate is strong while the ability to resist the risk when there is a decreasing interest rate is weak.

While, there is a problem with modified duration is that it only works when the yield to maturity is very small. This rule can be found by recalculating the bond's price by using different percentage changes of yield to maturity.

For example, this research calculates the modified duration of each bond. By discounting the price of the bond under two different conditions. One is that there is a 0.1 percentage increase of the yield to maturity. the other one is that there is a 10 percent increase of the yield to maturity. It can be found out that in the case of a 0.1 percent increase, the price is very similar to what has been deduced in the first place but the case of 10 percent increase. There is a very large gap between the price that has been calculated and the original price. This is because it's just a linear approximation and the bond price as a function of the yield is convex. Taking this convexity into account would lead to a better approximation. The limitation of both Macaulay duration and modified duration is that this two duration is a meaningful measurement of interest rate sensitivity when appraising very small yield changes.

How to use modified duration in the reality? For the investors. When assuming there is a potential rising in the interest rate, so the investors could focus on investing those with short maturity and shorten the duration of the bond. When assuming there is a potential decrease in the interest rate, it could lengthen the duration of the bond and enlarge the investment on the bond with a long-term duration as this is helpful for the investors to get higher profit in the rising of the bond market.

4. IMMUNIZATION

Having learned that the Macaulay Duration is the weighted average of the times to each coupon or principal remittance [10]. Duration is an essential concept in portfolio management strategies, because duration matching strategy is one of the portfolio management strategies. It can be used to tackle problems in terms of security portfolio.

In the fickle financial market, there are always financial institutions, such as insurance companies and retirement funds, seek to control the risk of their bond portfolio from being put under exposure to the interest rate movements owing to several unaware factors on the market like cuts to interest rates and required reserve ratios. The institutions usually use a passive portfolio management strategy to achieve the status of immunization, which means that their institution portfolio is well-prevented from the interest rate volatility [11].

4.1. Calculation for Immunization strategy

Consider, for instance, the government that issues Treasury note, or T-note for brief, for $1,000. (T-note is a marketable U.S. Government debt security with a fixed interest rate and a maturity which is usually within 2 to 10 years). Assuming that there is a T-note with a 5-year maturity and an interest rate of 5%, the government guarantees to pay $1,000 × (1.05)^5 = $1,276.28 in 5 years as coupon payment.

So in order to sustain the ability to fund its liability of $1,276.28, they purchase a bond for $1,000 of 5% annual coupon rate, and selling them out at face value. Hence, as long as the interest rate remains at 5%, they will have full ability to fund the liability of $1,276.28, as the bond's value matches accurately with the present value of the liability of $1,276.28. But what if interest rates vary, the fund will be influenced and the bond may not be promised to increase to the anticipated number of $1,276.28. If interest rates increase, the par value of the bond will show a loss, the bond's ability will be exploited to satiate the liability. Hence, the bond's value will be less in 5 years than the expected value. However, at a higher interest rate, if you reinvested your previous coupon payments, the reinvented gain will increase faster, offsetting the capital loss in face value. As the interest rates roar up, the capital gain decreases correspondingly, but meanwhile, reinvested income grows at a faster rate. If the bond portfolio duration is constructed properly, these two effects will offset each other exactly, as a result, the portfolio is well-protected against the volatility in interest rate.

So, how to construct an immunized portfolio by duration matching? As the name implies, duration matching is a risk-mitigating strategy that aligns the matches of the duration of the portfolio with the duration of liabilities and isolates the value of the portfolio from interest rate movements. For most financial institutions, usually hold naturally imperfect matching between the duration of the obligation and the duration of their portfolios, so they want their assets as much as possible to perfectly match their liability, which immunizes the portfolio from interest rate movement.

Suppose there is a one-off tuition payment of $17,000 for a bachelor's degree in 3 years, so this tuition pay is your liability, and it should be paid after 3 years, so it's the liability duration is 3. There are 2 bonds on the market, Bond A and Bond B.

Bond A: Face value=$100, current price = $85.48, no coupon payments, remaining life = 4 years

Bond B: Face value=$100, current price=$100, annual coupon of 4, remaining life = 2 years

Assume all debt considered in this exercise has the same level of default risk. The benchmark yield curve for fixed-income security investments of the same riskiness is flat at 4% per annual year.

Firstly, it works out the duration of two different bonds; for Bond A the duration is 4 years, owing to that for zero-coupon bonds, the duration is equal to its yields to maturity (YTM), for there is only one-off principal payment, so the time average equals to its bond maturity, 4 years.

For Bond B:

$$\sum_{t=1}^{3} t\frac{PV(CF_t)}{P} = \frac{3.85}{100} \times 1yr + \frac{3.7}{100} \times 2yrs + \frac{3.56}{100} \times 3yrs = 2.89\ yrs。 \quad (8)$$

This study uses the formula for the duration to solve the exact duration for Bond B is 2.89 years.

Then it starts to match the liability with the portfolio, as in Equation (9), which by just simply putting them in each side of the equation and solve the asset portfolio that equates the duration of assets and the liability duration. The total percentage is 1, suppose the ratio of Bond B is b, so for Bond A is (1-b), it needs to solve 'b' in the expression:

$$3 = b \times 2.89 + (1 - b) \times 4. \quad (9)$$

b (proportion for Bond B) = 0.9(90%)

So the manager should invest 90% of the portfolio in Bond B, and the rest 10% for Bond A. The present value of the liability is $17,000/(1+4%)^4=$15,113.

$$\text{Bond B}=15{,}113\times 90\% = \$13{,}602. \quad (10)$$

$$\text{Bond A}=15{,}113\times 10\% = \$1{,}511. \quad (11)$$

So you should construct a bonds portfolio that invests $13,602 in Bond B and $1,511 in Bond A as shown in Equations (10) and (11). Therefore the portfolio has been immunized and you have already fully funded the liability.

The fact that, for now, the current portfolio is immunized, however, the fund manager still cannot leave, due to that the fore-mentioned assumption that the interest rate is flat and unchanged, so this research neglects the need for rebalancing in terms of volatility in interest rates. What's more, even if rates do not change, as time goes, the duration of both liability and assets change, thereby requires rebalancing. Next, the limitations for immunization strategies will be seen.

4.2. Limitations of Immunization

As implementing duration to assist to be insulated from an interest rate change, this factor, duration, gives us an interpretation of the interest rate sensitivity of the bonds. Therefore, duration matching is by and large used to construct an immunized portfolio. However, nothing is perfect, when using duration, it supposes a flat yield curve, which signifies that there is little difference in short and long term rate for bonds, and a little parallel shift of the yield curve, at which interest rates across all maturities change by the same number of basis points. In other words, the portfolio is immunized only under specific and limited types of interest rate volatility, which is inconceivable in the real financial markets. The immunized portfolio is restricted by these assumptions which can hardly be achieved in the real world. Not yet, there are more restrictions.

For investors work in banks, they are familiar with the concept "duration", and they have different financial calculation tools to assist, so they can get the accurate number of the up-to-date duration at any time, so they can administer the immunization proficiently and accurately, but for general investors, they cannot have access to those, so they need something easier to conduct.

Besides, immunization can only help with the mismatch problem; when the duration of your assets does not match the duration of your liability, you can use the immunization strategy to match them correctly equal to isolate from IR change, but not includes other financial risks such as the default risk or credit risk.

5. CONCLUSION

The objective of this article is to explore the main tool that usually use when analyzing the financial market which is the Macaulay duration. as it is known when investing something in the financial market, the investors must be aware of the risk of each investment, thus some tool must be invented to predict the risk of each investment. Macaulay duration is a very effective tool to use. The research method used is analyzing the data in the reality, and then it could use the data to test the effectiveness of the duration and find out its limitation

After the research, it has also been explored that the practicality of the extended application of duration and the limitation of the duration with the limitation of the immunization. It found out that the duration is a very useful tool for analyzing the risk of bonds in the financial market under different circumstances. But also with quite a lot limitations the related to the specific required magnitude of the price. But still, there is no doubt about the usefulness of the duration.

There are also some inevitable limitations in this paper. It assumes a linear and indirect relationship between interest rate and bond price for the duration, so the outcome may be lower than the actual value. Also for immunization strategy, it is supposed that yield curve that is flat, and only a little parallel shift of the yield curve. However, how to avoid or reduce these assumptions is not involved in this paper. The more effective and accurate methods should further be studied like interest-rate sensitivity and cash-flow matching strategy.

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