Simulating Prices of the Barrier Option Based on Black and Scholes Model

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ABSTRACT

In the financial markets nowadays, the option is one of the popular financial products that has been increasingly traded. Exotic options, including barrier options, were invented to cater to the special needs of the investors. However, the pricing model of these financial instruments is complicated, and few studies have assessed the pricing model of barrier options. In this article, the present value of the knock-in barrier call option is evaluated through the traditional Black and Scholes model. To make the barrier option representative, we chose the underlying to be the S&P 500 index. Through a stochastic process in Excel, we managed to simulate the barrier option of S&P 500 with a maturity of 1 month. The simulation generates a graph that shows the relationship between payoff and the S&P 500 price, facilitating the categorization of barrier options. A series of sensitivity analyses are conducted to examine the soundness of the simulation, while several important factors that have a bearing on the value of the barrier option are presented and discussed, such as spot price, strike price, and barrier price. The simulation can serve as a helpful tool for investors to evaluate barrier options. Yet, it is constrained in a short period, so the pricing of the long-term barrier option may not behave in the same way and should be studied in the future.

Keywords: *Black and Scholes Model, Barrier Option, Valuation*

1. INTRODUCTION

Options trading is important nowadays in the financial market. On January 21, 2021, the Futures Industry Association (FIA) released data on the trading of derivatives markets on more than 80 exchanges worldwide in 2020. Table 1 shows that global options volume and positions hit record highs in 2020. Specifically, options volume increased by 39.3% to 21.22 billion.

Exotic Option is a derivation of a standard European and American Option which is traded over the counter. Barrier options are a very important type of option in today's exotic options trading market. Despite being an exotic option, barrier options have very similar financial attributes to both European and American options, and barrier options are also heavily used in the areas of asset optimization and risk hedging. A barrier option has a lower price than a standard option, which means that the owner of the option can hedge their risk and make money at a lower price if they feel that the barrier price will be reached at some point before the option expires. In this passage, we especially focus on the analysis of the barrier option of the S&P 500 index, to find out how the barrier options of S&P 500 should be priced.

Table 1. FLA Annual reports on the Options Market Trading Volume in 2020

Type	Jan-Dec 2020 Vol	Jan-Dec 2019 Vol	Vol % Change	2020 December OI	2019 December OI	OI % Change
Options	1,221,739,033	15,234,299,392	39.3%	741,167,934	652,153,487	13.6%

In our article, we cited a total of 11 articles related to the topics of our article to increase credibility. For the pricing of options and corporate liabilities, it talks about the Black and Scholes model of option pricing as a widely used mathematical method to the valuation of options. Under the assumption that the market cannot be predicted, we can use the model to simulate the price of the barrier options [1]. In The trouble with stock options, the article states that The number of options kept increasing and option grants in S&P 500 firms reached $119 billion in 2000, but critics of options often claim that traditional options are economically insufficient due to their costs. However, the emergence of exotic options may resolve some of the problems. Barrier options, for example, can be issued at a relatively lower price [2]. In Mispricing in the Black-Scholes model: an exploratory analysis, it shows that despite the Black and Scholes model of pricing lacking accuracy in many scenarios, some studies have shown that it can perform a prediction of at-the-money options well. Therefore, for the precision of the simulation of the barrier option in this article, it was set to be an at-the-money option, whereas other strike prices are discussed in the sensitivity analysis [3]. In Comparison of Black Scholes and Heston Models for pricing index options, it indicates that another flaw of the Black and Scholes model is the decreasing accuracy when the time to maturity becomes shorter. This problem should be considered as a limitation of the simulation in this article, since simulating a longer time to maturity will consume a significant amount of computing power in Excel for a path-dependent option [4]. In Pricing and hedging barrier options, “European options are a significant financial product. Barrier options, in turn, are European options with a barrier constraint. The investor may pay less buying the barrier option obtaining the same result as that of the European option whenever the barrier is not breached. Otherwise, the option's payoff cancels.” Therefore, we can use the B-S model for the valuation of the barrier options [5]. The article Conditioning on One-Step Survival for Barrier Option Simulations uses the special structure of barrier options to develop a variance reduction technique, which is suitable for general simulation problems with similar structures. We use measured changes at each step of the simulation to reduce the variance due to the possibility of crossing obstacles on each monitoring date. This method helps us reduce the error in the experiment [6]. In Pricing external barrier options under a stochastic volatility model, “The main reasons why barrier options have become so popular are that they have resilience and low premiums compared with vanilla options. Barrier options are extinguished (knock-out options) or activated (knock-in options) when the value of the barrier variable hits a certain level either from above (down-options) or below (up-options).” This supports the result of the graph we made [7]. The 2020 annual trends in Futures and Options Trading provides the data for the options trading volume [8]. In Path-dependency and path-creation perspectives on migration trajectories: The economic experiences of Vietnamese migrants in Slovakia, demonstrates the definition of path dependence is that the value of an option depends not only on the price of the underlying asset but also on the path that asset took during all or part of the life of the option. In other words, "Path dependence exists when a feature of the economy (institution, technical standard, the pattern of economic development, etc.) is not based on current conditions, but rather has been formed by a sequence of past actions each leading to a distinct outcome" [9]. In Standard & Poor's midcap 400 guide, it compared to another index, Dow Jones Index, S&P 500 included more companies from a wider variety of industries, so that the risk of S&P 500 is more diversified, and the change of index can show how the market is fluctuating more clearly. It is a daily snapshot of the financial health and activity of five hundred of the largest publicly traded companies [10]. The article indicates that the investor can manage the volatility risk using options since the price of the options "depends on the volatility of a given financial asset (a stock, a commodity, an interest rate, etc.)" [11].

While few studies applied the Black and Scholes model to analyze the present value of barrier options in Excel, we succeeded in running a day-to-day simulation of the price of S&P 500 to calculate the expected payoff of barrier options in different market paths. We found an approach that is not only able to obtain an estimated present value of a specific barrier option but also to be evaluated under certain parameters. These parameters are analyzed in the sensitivity test, where we classified the effect of the strike price, spot price, and barrier price on the price of the barrier option. We also explore the intuitive ways to explain the advantages of the barrier option over the standard European options in terms of the simulation.

Regarding the structure of this article, 4 sections are the introduction of the definition of the barrier option, the method of simulation, the discussion of the simulation and the sensitivity analysis, and the conclusion. First, the introduction explains how the barrier option is defined, especially the one of simulation. Second, the details of our simulation and the data used are presented in the method. Third, the discussion section covers our deduction and analysis of the simulation. Lastly, the conclusion extracts the important insights of the discussion and summarizes the limitations of this article.

2. OPTIONS OF S&P 500 INDEX

2.1. The basic idea of Standard & Poor's 500 index

Standard & Poor’s are normally shorted as S&P500. It is an index developed and maintained by a company named S&P Dow Jones Indices LLC. It is an average record of the U.S. stock market from 1957 onwards, looking at 500 public companies in the United States. Compared to another index, Dow Jones Index, S&P 500 included more companies from a wider variety of industries, so that the risk of the S&P 500 is more diversified, and the change of index can show how the market is fluctuating more clearly. It is a daily snapshot of the financial health and activity of five hundred of the largest publicly traded companies, including health care, information technology, materials, real estate, financials, energy, etc. Well-known companies included in this index are Apple, 3M, Boeing, IBM, Intel, etc.

2.2. The basic idea of Options

An option is a derivative contract in which the buyer pays a sum of money to the seller or the writer and receives the right to either buy or sell an underlying asset at a fixed price either on a specific expiration date, which is a European option or at any time before the expiration date, which is the American option. There are two types of basic options: call option and put option. The holder of the call option will have the right to buy an asset at the exercise price at the expiration date, and the holder of the put option will have the right to sell an asset at a specific exercise price at the expiration date. Options can be traded either on the floor or over the counter.

Compared to the future, we should focus more here that the options are giving the holder to have the right to execute the power, which means the holder can also disregard such right. Although it is not needed to pay extra money to hold a futures asset, while in the options, the holder needs to pay some extra money to own such right.

It is necessary to find out the appropriate amount of money that the person needed to pay to hold a call or put option. The key to pricing the derivatives of the financial market is fairness between the seller and buyer of the contracts under a specific underlying asset. The price needs to ensure both sides have the same rights and interests under a certain risk. The way to price the option will be introduced in the next section of this essay.

Since the price is influenced by the risk of the assets, so the market can view the option not only as a derivate to earn money through increasing or decreasing the price of the underlying asset. The investor can use the options price as the measurement of the market risk. Investors are often exposed to two types of market risk: directional risk and volatility risk. Directional risk can be hedged by buying and selling futures contracts but buying and selling futures do not enable hedging of volatility risk. But in option pricing, the impact of market volatility on the price of the underlying asset is considered, not just the time factor and the market interest rate factor. So, we can help manage the market risk, directional risk, and volatility risk, faced by investors more effectively through the price of options. For example, the investor can manage the volatility risk using options since the price of the options "depends on the volatility of a given financial asset (a stock, a commodity, an interest rate, etc.)"

The price of the option is low compared to the price of the futures of the underlying assets. That's why investors often think of options as insurance policies. For example, an investor who buys a pool of stocks and then worries that the price of those stocks will drop sharply in the future can buy a put option, which is like an insurance policy that costs a small amount of money and allows for a promptly stop loss if the stock drops.

In addition to the price advantage, the flexibility of the options structure gives options a unique advantage. Through the combination of different expiration dates, different exercise prices, and buy-sell options, different strategies can be created to meet the different needs of investors. Notable combinations include butterfly spreads, long straddle options, etc.

Given such flexibility, investors find out that they take more advantage of the options by adding some conditions, and then they become options what is known as an exotic options.

2.3. Basic Idea of Exotic Options

Exotic Options are more complicated than the normal options which people normally call plain vanilla products. This type of product arises for a variety of reasons, sometimes to meet the specific hedging needs of certain investors. Sometimes it is for tax, legal, and funding reasons. There are also times when they are designed to predict the direction of financial markets for certain market variables.

2.4. Types of Exotic Option

Some options will be introduced shortly here.

2.4.1. Lookback Option

The payoff of a lookback option is related to the maximum and minimum values reached by the price during the term of the option. The return on a floating lookback call option is equal to the difference between the final price of the underlying asset and the minimum price of the underlying asset during the term of the option. The return on a floating put option is equal to the difference between the highest price of the underlying asset over the list price of the underlying asset during the term of the option.

2.4.2. Asian Option

The return on an Asian option is related to the arithmetic average of the price of the underlying asset over the life of the option. The return on an average price call option is $max(0, S_{ave} - X)$, and the return on an average price put option is $max(0, X - S_{ave})$, where S_{ave} is the average of the underlying asset prices.

2.4.3. Barrier Option

A barrier option depends on whether the price of the underlying asset reaches a specific level within a specific A barrier option depends on whether the price of the underlying asset reaches a specific level within a specific time interval.

A variety of different barrier options are often traded in the over-the-counter market. Barrier options can be divided into two categories: knock-out options and knock-in options. Knock-out options cease to exist when the price of the underlying asset reaches a certain level; knock-in options begin to exist when the price of the underlying asset reaches a certain level.

2.5. Reasons to choose the Barrier Option

Barrier Option has many advantages over other types of options. The first advantage is the weak path dependence. The definition of path dependence is that the value of an option depends not only on the price of the underlying asset but also on the path that asset took during all or part of the life of the option [12]. In other words, "Path dependence exists when a feature of the economy (institution, technical standard, a pattern of economic development, etc.) is not based on current conditions, but rather has been formed by a sequence of past actions each leading to a distinct outcome."

Asian Option is a strong path dependence under such definition. The reason weak path-dependent is more advantageous compared to the strong path-dependent option is that the investors care about whether the barrier is reached, while they do not need any other information about the path. It is different from the Asian Option, which is a strong path-dependent option. In the Asian option, we need to calculate the average price. In this case, the information of the path will become the new independent variable. While in this Barrier option, we don't need the price of every time point and we just need to know whether the barrier price is reached. If we use the computer to do this simulation, the velocity can be faster, and more energy can be saved.

Another advantage is the price advantage. According to the simulation, which is shown in Table 2, and Fig.1 here shows the price of the barrier option under the same strike price and expiration time of a certain underlying asset can be much cheaper. So, this option is really attractive to a person who believes that the barrier price will be achieved!

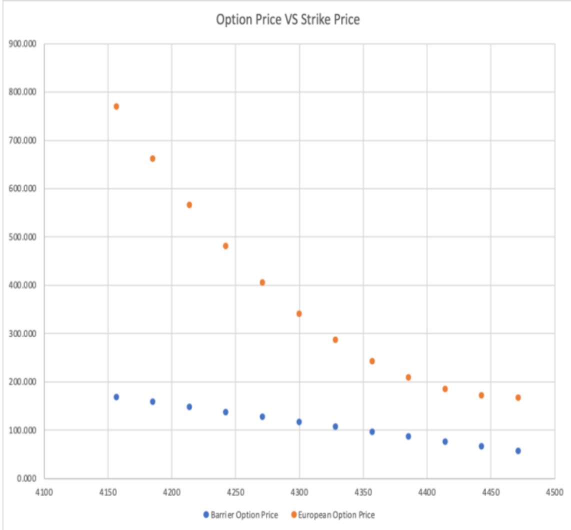

Figure 1 Comparison of Price of Barrier knock-in Price and European Call Option

Table 2. Comparison of Price of Barrier knock-in Price and European Call Option

Strike Price, $	Barrier knock in Option Price, $	European Call option Price, $
4156.44	169.004	776.444079
4,185.07	158.641	668.745424
4213.7	148.278	571.841448
4,242.33	137.915	485.621174
4270.96	127.580	409.975628
4,299.59	117.276	344.797788
4328.22	107.071	289.982535
4,356.85	96.910	245.426608
4385.48	86.784	211.028552
4,414.11	76.750	186.688683
4442.74	67.071	172.309038
4,471.37	57.760	167.793336

What's more, the option is a kind of financial derivative caring about volitivity. Normally, people will have exact opinions toward the market. If one believes that the price will not reach a certain point (Barrier Price) and he wants to get the return as a normal European call but doesn't want to pay for all the possibilities, then he will probably buy the knock-out barrier option since the price is much lower. The closer the barrier price with the spot price, the more probably that the option will be knocked out. The cheaper will be the option. (Good in the non-volatile market)

Same idea on the knock-in price: The higher the barrier price, the cheaper the option. (Good in a volatile market)

2.6. Source of Data

The following sections of the article will talk about how the option will be analyzed quantitively. The paper uses the data from Yahoo Finance. The data from Yahoo Finance include Spot Price. The volatility and interest rate of return are achieved by regression of the futures of the S&P 500 index. Table 3 shows such progression.

Table 3. Data used for the Progression of achieving δ.

Time to Maturity/ yr	Price of S&P 500 Futures,$
1/6	4390.00
2/5	4382.25
2/3	4372
1	4365.25
7/6	4356

We use those data to simulate the δ, the dividend rate by linear regression using excel. Fig.2 shows the graph of such simulation.

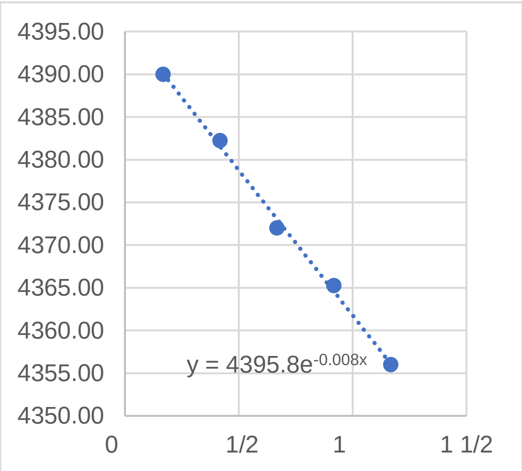

Figure 2 Linear Regression of S&P 500 Futures

From the graph linear regression, according to the formula of pricing of the financial futures. $F(0,T)$ here is the price of the futures with the parameter, T, which is the time to maturity of the futures. r here is the interest rate of free risk. δ here is the dividend rate of the S&P stock market.

$$F(0,T) = S_0 e^{(r-\delta)T} \tag{1}$$

As we get $-0.008 = r - \delta$ in the formula, we choose treasury bill rate as r, the day we get the number of the free risk value is 0.001.

Then we can get $\delta = 0.90\%$ by the following step.

$$\begin{aligned} -0.008 &= 0.001 - \delta \\ \delta &= 0.009 = 0.9\% \end{aligned} \tag{2}$$

Then we get σ, the standard deviation of the return by the following data. Table 4 shows the data that we used to find σ.

Table 4. Data used to find out the one-year standard deviation of return

Date	S&P ETF	Return
2020/11/1	357.03064	10.88%
2020/12/1	368.686493	3.26%
2021/1/1	366.484985	-0.60%
2021/2/1	376.675262	2.78%
2021/3/1	392.49057	4.20%
2021/4/1	414.610931	5.64%
2021/5/1	417.333313	0.66%
2021/6/1	425.301605	1.91%
2021/7/1	437.10965	2.78%
2021/8/1	450.117981	2.98%
2021/9/1	427.769592	-4.97%
2021/10/1	438.660004	2.55%

With those data, with the help of excel, it can get the standard deviation of return of the S&P 500 market is 13%

In conclusion, Table 4 shows the data that we used for simulation. r here is the risk-free interest, using Federal Reserve Treasury Bills Rate, δ here is the dividend rate of S&P stock market. T here is time to maturity of the assets. σ is the standard deviation of the return.

This article sets the strike price and the barrier price randomly to do the simulation.

Table 5. Data used for the simulation in the latter part.

Spot Price	Strike Price	Barrier Price	r	δ	T	σ
4471.37	4471.37	4600	0.12%	0.90%	0.08333	13%

In this article, the research is about how the option should be priced and this essay is about the pricing of the barrier price, mainly focused on how the option price of the barrier price is influenced by different prices like spot price, strike price, and barrier price.

3. METHOD AND RESULT

This part of the article will introduce the steps and results of our simulation process. First, take a good look at the basic data of our simulation. First of all, in our simulation, the risk-free rate is 0.12%. According to the SP 500 data that we found on the Internet, the spot price and strike price here are both set to 4471.37. The delta is 0.9%. Because this simulation is simulating data for one month, the time to maturity is 1/12 and because there are 22 trading days in a month, this simulation will simulate 22 days of data. The volatility is 13%. We set the barrier price to 4600. We simulated a total of 500 times.

The first step is to copy the simulated random number so that we won't re-simulate the situation without entering new data. If each input will regenerate a new set of data, then this will cause errors in our simulation. This will make our subsequent simulation more convenient and more accurate. The second step is to use the Black-Scholes equation (1) to simulate 500 times of data. What we need to pay attention to is that the spot price on the first day of simulation is our preset price 4471.37. However, our simulation did not use this preset price for the whole simulation, but for the rest of the days, the stimulation used the data of the previous day like the spot price. This is because every time we simulate the data of a new day, if we only use the initial spot price, this will cause a lot of errors in our subsequent 21-day simulation, so in order to avoid errors, we need the previous day's data as the spot price.

$$S_T = S_0 e^{(-122)T + ZT} \tag{3}$$

The third and final step is graphing. We need to find the value of S_T. The logic of how we get the value of S_T is that if there is a value greater than the barrier price we set during the 22 days of our simulation, then we consider the option to be activated, in other words, it is knocked in. After the knock-in occurs, subtract the strike price from the price of the last day to get the payoff of this option. If there is no knock-in, then the payoff is 0. In this way, the simulation can get the graph of payoff and stock price. Graph 1 meets our expectations for the barrier option. We can see that there is a period that exceeds the strike price, but the payoff is still 0. This is because although the price exceeds the strike price, the historical price does not exceed the barrier price, the payoff is still 0. In the end, the gain of this barrier option we simulated was 57.7603. We believe that the value of the Barrier, strike price, and spot price are all factors that will affect our final simulation, so next, we conducted sensitivity tests on these three factors.

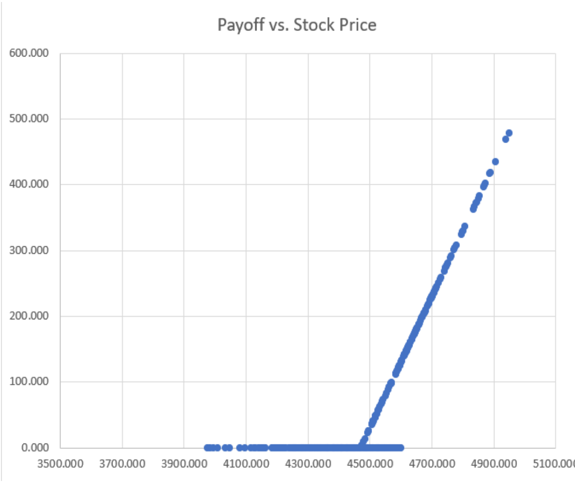

Figure 3. Payoff vs. Stock Price

4. SENSITIVITY ANALYSIS

To understand the factors that will influence the price of the barrier options, we carried out sensitivity analysis based on the strike price, the spot price of S&P 500, and the barrier price. The tests are all done in Excel with the data table feature in What-if Analysis. We also evaluated the bearing of these factors theoretically so that we can check whether the approach of simulation is reasonable.

4.1. Strike Price

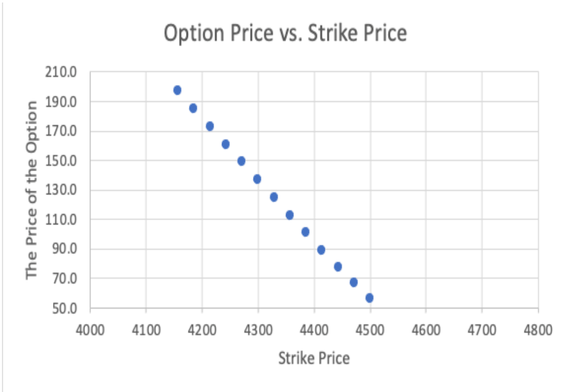

Figure 4. Option price with a different strike price

The sensitivity test between the barrier option price and the strike price displays a negative linear correlation in Figure 4. Since we simulated a knock-in barrier call option, we know that an increase in the strike price will result in a decrease in the expected payoff and, in turn, the barrier option price. Thus, the result of the sensitivity is consistent with our theory of the barrier option. We notice that the linear correlation is perfect due to the fixed z-score numbers. The slope of the best-fit line is approximately -0.30, which means that for every dollar increase in the strike price, the barrier option price will drop by 0.30 dollars.

4.2. Spot Price

Figure 5. Option price with the different spot price

Another factor we tested was the spot price of S&P 500. According to Figure 5, the test on spot price demonstrates an exponential positive correlation. The option price starts from zero around the S&P price at $4,150. The exponentiality indicates that the option price is more volatile than the spot price. As the spot price increases to more than $4,600, the correlation seems to be linear. This change is similar to the relation between the European option price and the spot price in Black and Scholes' study. Therefore, we find that the sensitivity test on the spot price is within our expectations.

4.3. Barrier Price

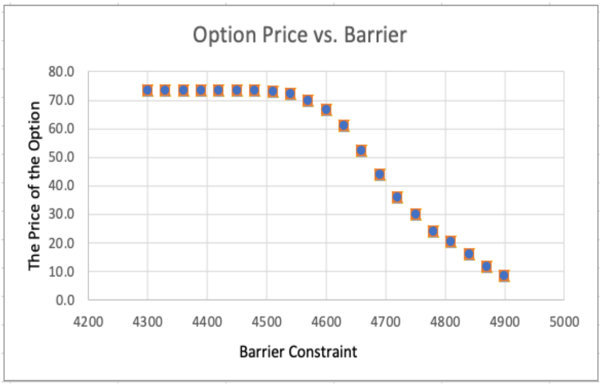

Figure 6. Option price with different Barrier price

The last sensitivity test we have done was on the barrier price. It is important to understand the effect of the barrier price because it is customizable and marks the ingenuity of the derivative option. In Figure 6, we can see that there is a point that separates the two segments of different relations. In the first half, the graph shows a constant option price of around $68. The reason for the constant price is that, when the barrier price is lower than the spot price, the contract of the barrier option will be immediately executable. Therefore, if the barrier does not exceed the current price of the S&P 500 index, the barrier call option will always be identical to a standard European option. On the other hand, when the barrier price is above the spot price, there is an imperfect negative linear correlation. With the increase of the barrier, the option price drops gradually. In general, elevating the barrier will decrease the probability of the execution of the option, reducing the expected payoff. Therefore, the trend matches our simulations. However, the impishness of the linear relationship needs further study.

5. CONCLUSION

In this article, the research is about how the option should be priced and this essay is about the pricing of the barrier price, mainly focused on how the option price of the barrier price is influenced by different prices like spot price, strike price, and barrier price. The data of the research is mainly extracted from Yahoo Finance, including spot prices. Volatility and returns were obtained by regressing the S&P 500 futures. In this paper, the strike price and the barrier price are set randomly for simulation.

In our study, we adopt Black and Scholes' model on option pricing to simulate the price of the knock-in barrier call option of the S&P 500 index. We used Excel to generate a series of random z-score numbers to simulate the market price of the S & P 500 index within a month of trading. Comparing the price at each trading day with the barrier price, we determined the final payoff in each trial. Based on the lognormal equations, we manage to derive the prices of the barrier option, providing insights for all the other types of barrier option derivatives. Moreover, we categorize the expected payoff of the barrier option in graphs and conduct sensitivity analysis to examine critical factors that quantitatively affect the price of the option. The results not only present parameters that are useful for evaluating the barrier option but also prove the robustness of our simulation. Equipped with the knowledge of our simulation, investors can grasp a more thorough understanding of the barrier option and its advantage in lowering cost. Our simulation on S&P 500 index barrier option epitomizes path-dependent options based on other financial instruments. Similar approaches can be applied to the pricing of these options.

We recognize that our simulation is still inadequate. For example, the time span of our simulation is not long enough. We only simulated one month's data. The financial market is changing rapidly, and every month's market will be very different. Our one-month data may be quite different from the actual situation, so it is not convincing enough. In the future, we will try a longer simulation period, such as a one-year or three-year simulation. Longer simulation time also means that our simulation will have higher accuracy. And, in future research, we will also consider simulations of other markets, such as Nasdaq, or some foreign markets. We will try to find out the correlations and differences between different markets through our simulations.

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