

# Completing Market with One Step Trinomial Model: A European Call Approach

Yang Kuang<sup>1\*</sup>, Chengqian Lin<sup>2</sup>, Nan Yang<sup>3</sup>

<sup>1</sup>*John Muir College, University of California, La Jolla, CA 92093, United States*

<sup>2</sup>*Bayes Business School (formerly Cass) City, University of London, London EC1Y 8TZ, United Kingdom*

<sup>3</sup>*Queen Mary University of London, London, E1 4NS, United Kingdom*

\*Corresponding author. Email: [yakuang@ucsd.edu](mailto:yakuang@ucsd.edu)

## ABSTRACT

The fundamental defect of the market constructed under a trinomial tree model lies in its incompleteness with initial risk-free and risky assets only. This paper analyzes the first-step probability behaviors of the trinomial tree model and summarizes how could introduce derivatives being a general solution to such a market. The European call option is used for demonstration. This paper further discusses the first-step pricing method for this call option under arbitrary market parameters. Results show that any derivatives being linearly independent of initial primary assets could complete such a market. When pricing such derivatives, their portfolio should be bounded by three sub-binomial tree structures that recombined from three possible branches of the initial model.

**Keywords:** *Trinomial tree model, European call option pricing, incomplete market.*

## 1. INTRODUCTION

The trinomial tree model is a lattice-based model which can represent the discrete-time stochastic behavior of a risky asset and therefore useful in option pricing [1]. Derived from the binomial tree model, the trinomial one is known and well-studied for its more realistic and complex structure. One significant concept introduced by this model is the incomplete market, where economists cannot use the portfolio method to fully replicate the investment [2].

This paper is motivated by the nice probability expression  $p_u = \frac{R-d}{u-d}$  [2] in a no-arbitrage market under the binomial tree model, where  $\{u, d\}$  are the one-step up and down factors for the stock price,  $R = B_0(1+r)^t$  be the compound interest, and  $p_u$  be the probability for the stock price to go up. The concise equation is developed from the portfolio method and is fundamental in the pricing strategy, however only valid in the binomial tree model, which is not widely practiced in real-life markets. Therefore, this work lays emphasis on the trinomial tree model, discusses its similarities and differences from the binomial ones.

This paper is carried out as follows: Section 2 discusses the probability aspect of the trinomial model; Section 3 introduces certain limits in finance and why

economists cannot simply calculate the price in a market under the trinomial tree model. We provide the general idea to solve the incompleteness and apply the European call option as an example. In Section 4, we discuss how to price a derivative in such a market, based on the portfolio strategy but also apply mathematical approaches.

### 1.1. Notation

The purpose of this paper is to give a preliminary insight into financial pricing in a trinomial tree model, therefore we define several background concepts as follows:

The market initially includes primary assets only: (1) risk-free bond  $B$  in the money market, which has a compound interest rate  $R = B_0(1+r)^t$ , where  $B_0$  is the money value at time 0; and (2) risky stock  $S$  in the stock market with a three-jump behavior: up, middle, and down (denoted  $u, m, d$  respectively), defined by

$$S_t = \begin{cases} S_{t-1}u & \text{with probability } p_u \\ S_{t-1}m & \text{with probability } p_m \\ S_{t-1}d & \text{with probability } p_d \end{cases} \quad (1)$$

for some constant  $u > m > d > 0$ ,  $t \in N^+$  and  $p \in (0, 1)$ , where  $t$  represents discrete time periods of the market. We will further assume zero transaction cost and

perfect information in the market.

A portfolio is a combination of accessible assets committed to representing the payoff space. The market is defined to be complete if such a portfolio can be built [2].

This paper only discusses the first step of the market, i.e.,  $t = 1$ . However, arbitrary values for  $u, m, d$  are accepted as long as  $u > m > d > 0$ .

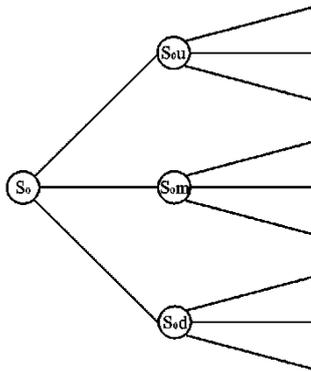


Figure 1 Trinomial Tree Model (First Step)

## 2. PROBABILITY IN A TRINOMIAL TREE MODEL

The no-arbitrage pricing exists in the binomial model, meaning that all possible combinations of assets in such a market, namely a portfolio, receive the same payoff if their initial investments are at the same amount. This harmonious property stipulates certain relationships between  $\{u, d\}$ , the up and down payoff of the stock price, and their corresponding probabilities  $\{p_u, p_d\}$ , which provide a convenient way for assets pricing at any given time. This session develops some concepts in the trinomial tree model. However, one may not assume they can be directly applied to a market under such model since no-arbitrage principle is not yet fulfilled.

### 2.1. Theorem

Assuming that the expected returns from money and stock market are the same, i.e.  $S_0u \cdot p_u + S_0m \cdot p_m + S_0d \cdot p_d = S_0R$ , then following relations between  $p_u, p_m, p_d$  are true.

$$\begin{cases} p_u = \frac{R + (d - m)p_m - d}{u - d} \\ 0 < p_m < \min\left(\frac{R - d}{m - d}, \frac{u - R}{u - m}\right) \\ p_d = \frac{R + (u - m)p_m - u}{d - u} \end{cases} \quad (2)$$

**Proof** Use the fact that  $p_u + p_m + p_d = 1$ , express  $p_u, p_d$  with  $p_m$ .

### 2.2. Remark

If  $R < m$ , then  $0 < p_m < \frac{R-d}{m-d}$ ; if  $R > m$ , then  $0 < p_m < \frac{u-R}{u-m}$ .

### 2.3. Corollary

When  $p_m = \left\{\frac{R-d}{m-d}, 0, \frac{u-R}{u-m}\right\}$ , the model is actually binomial since  $\{p_u, p_m, p_d\}$  will be 0 correspondingly.

One may notice that the probabilities are shown in ranges and dependent relations. This difference from the binomial tree model leads to its indeterminacy. This can be proved by linear equations.

## 3. INCOMPLETE MARKET UNDER THE TRINOMIAL TREE MODEL

### 3.1. Theorem

A market under the trinomial tree model with primary assets only is incomplete.

**Proof** Let  $F$  denote payoff function of the stock,  $N^B$  denote a unit of the risk-free asset and  $N^S$  denote a unit of the risky asset,  $\{N^B, N^S\}$  are linearly independent, then

$$\begin{cases} N^B R + N^S S_0 u = F(S_0 u) \\ N^B R + N^S S_0 m = F(S_0 m) \\ N^B R + N^S S_0 d = F(S_0 d) \end{cases} \Rightarrow \begin{bmatrix} R & S_0 u \\ R & S_0 m \\ R & S_0 d \end{bmatrix} \begin{bmatrix} N^B \\ N^S \end{bmatrix} = \begin{bmatrix} F(S_0 u) \\ F(S_0 m) \\ F(S_0 d) \end{bmatrix} \quad (3)$$

is an inconsistent system. This means market does not guarantee a general solution, with one exception in the following Corollary.

### 3.2. Corollary

A market under the trinomial tree model can be replicated if and only if

$$(u - d)F(S_0 m) = (m - d)F(S_0 u) + (u - m)F(S_0 d) \quad (4)$$

**Proof** Solving the first and third equation from Theorem 3.1, we have

$$\begin{bmatrix} R & S_0 u \\ R & S_0 d \end{bmatrix} \begin{bmatrix} N^B \\ N^S \end{bmatrix} = \begin{bmatrix} F(S_0 u) \\ F(S_0 d) \end{bmatrix} \quad (5)$$

and if the  $N^B R + N^S S_0 m = F(S_0 m)$  also satisfy the coefficient vector above, Corollary 3.2 must be true.

### 3.3. Remark

Payoff functions  $F(S) = S$  and  $F(S) = c$  satisfy Corollary 3.2. This indicates that a special market's payoff space will allow replication even under a trinomial tree model.

### 3.4. Completing the market

One may see that whether the market is complete depends on the consistency of the payoff equation set as in *Theorem 3.1*. Therefore, the easiest way to make the system consistent is by adding another derivative as a variable, i.e., the third asset. This paper will use the European call option as an example.

**Example** Let  $N^C$  denotes the unit of calls in our portfolio and  $C_{u,m,d}$  being payoffs of the call corresponding to the stock price at  $\{u, m, d\}$  respectively. If  $\{N^B, N^S, N^C\}$  are linearly independent, we

have

$$\begin{cases} N^B R + N^S S_0 u + N^C C_u = F(S_0 u) \\ N^B R + N^S S_0 m + N^C C_m = F(S_0 m) \\ N^B R + N^S S_0 d + N^C C_d = F(S_0 d) \end{cases} \Rightarrow \begin{bmatrix} R & S_0 u & C_u \\ R & S_0 m & C_m \\ R & S_0 d & C_d \end{bmatrix} \begin{bmatrix} N^B \\ N^S \\ N^C \end{bmatrix} = \begin{bmatrix} F(S_0 u) \\ F(S_0 m) \\ F(S_0 d) \end{bmatrix} \quad (6)$$

being a consistent system thus guaranteed solution to our payoff space because of full row and column rank. Consequentially,

$$\begin{bmatrix} N^B \\ N^S \\ N^C \end{bmatrix} = \begin{bmatrix} R & S_0 u & C_u \\ R & S_0 m & C_m \\ R & S_0 d & C_d \end{bmatrix}^{-1} \begin{bmatrix} F(S_0 u) \\ F(S_0 m) \\ F(S_0 d) \end{bmatrix} \quad (7)$$

where  $\begin{bmatrix} N^B \\ N^S \\ N^C \end{bmatrix}$  is the portfolio's coefficient vector.

Recall that the payoff function of any European call option is  $C(S_T) = \max(S_T - K, 0)$ , where  $T$  is the maturity time of the call, and  $K$  is the pre-determined strike price. In the following section, we further denote  $K = S_0 k$ , where  $k$  represent a growth factor relative to  $S_0$ , the initial investment into the stock market.

## 4. PRICING THE CALL OPTION

In order to conclude the European model into a market, reasonable pricing for this option is necessary. One may notice that the European call is also under the trinomial tree model and its replication is like which in *Theorem 3.1* hence overdetermined. However, the price of a call is rather "flexible" if it captures most payoffs [2,3]. By giving the degree of freedom to variables, we can calculate the second-best scenario and know the boundaries for pricing. Consider the critical cases that the number of constraining equations equals to the number of variables, in this case is 2.

### 4.1. Theorem

In our market, the payoff function of a European call option is replicable hence can be priced directly if  $k \leq d$  or  $k \geq u$ , and not necessarily replicable if  $d < k < u$ .

**Proof** See *Remark 3.3*. Notice that if  $k \leq d$ , the payoff is the stock price itself, and if  $k \geq d$ , the payoff is always 0.

### 4.2. Corollary

Assuming that  $d < k < u$ . If  $k \geq m$ , the payoff function of such a call option is

$$C = \max(S_T - K) = \begin{cases} S_0 u - K = S_0(u - k), & \text{if } S_1 = S_0 u \\ 0, & \text{if } S_1 = S_0 m \text{ (8)} \\ 0, & \text{if } S_1 = S_0 d \end{cases}$$

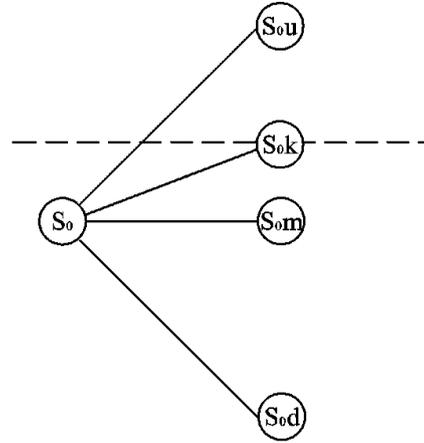


Figure 2  $k \geq m$

If  $k < m$ , the payoff function of such call option is

$$C = \max(S_T - K) = \begin{cases} S_0 u - K = S_0(u - k), & \text{if } S_1 = S_0 u \\ S_0 m - K = S_0(m - k), & \text{if } S_1 = S_0 m \text{ (9)} \\ 0, & \text{if } S_1 = S_0 d \end{cases}$$

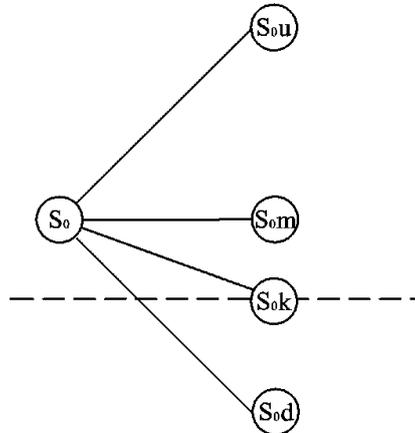


Figure 3  $k < m$

In each step, outcome above the dash line will exercise the call. The difference is whether the payoff of state  $S_0 m$  is 0. We will see later how value of  $k$  affects pricing.

### 4.3. Theorem

If  $d < k < u$ , the price of a European call option under the trinomial tree model is upper bounded by

$$\overline{N^B R + N^S S_0} \quad (10)$$

where

$$\overline{N^B} = R^{-1} \frac{uC(S_0d) - dC(S_0u)}{u-d}, \quad \overline{N^S} = \frac{C(S_0u) - C(S_0d)}{S_0(u-d)}$$

**Proof** One may notice that  $\overline{N^B}, \overline{N^S}$  are portfolio coefficients under a binomial tree model with branches  $u, d$ , and corresponding payoff functions  $C(S_0u), C(S_0d)$ . Let  $\overline{C}$  denote the payoff function for such a portfolio. If  $k \geq m$ ,

$$\overline{N^B R} + \overline{N^S S_0 u} = S_0(u - k) = C(S_0u)$$

$$\overline{C} = \frac{\overline{N^B R} + \overline{N^S S_0 m} = \frac{(m-d)S_0(u-k)}{u-d} > 0 = C(S_0m)}{u-d} \quad (11)$$

$$\left\{ \begin{array}{l} \overline{N^B R} + \overline{N^S S_0 d} = 0 = C(S_0d) \end{array} \right.$$

If  $k < m$ ,

$$\overline{N^B R} + \overline{N^S S_0 u} = S_0(u - k) = C(S_0u)$$

$$\overline{C} = \frac{\overline{N^B R} + \overline{N^S S_0 m} = \frac{(m-d)S_0(u-k)}{u-d} > S_0(m - k) = C(S_0m)}{u-d} \quad (12)$$

$$\left\{ \begin{array}{l} \overline{N^B R} + \overline{N^S S_0 m} = 0 = C(S_0d) \end{array} \right.$$

In both cases,  $\overline{C}$  has a better payoff in when  $S_1 = S_0m$ , and matches the payoff when  $S_1 = S_0u$  and  $S_1 = S_0d$ , therefore we say that  $\overline{C}$  super replicates  $C$ . Under the ideal market environment, the price of a derivative should reflect its payoff, therefore the price of  $C$  could at most be the initial investment of  $\overline{C}$  [4].

#### 4.4. Theorem

If  $d < k < u$ , the price of a European call option under the trinomial tree model is lower bounded by

$$\max(\overline{N^B R} + \overline{N^S S_0}, \overline{N^{B'} R} + \overline{N^{S'} S_0}) \quad (13)$$

where

$$\overline{N^B} = R^{-1} \frac{uC(S_0m) - mC(S_0u)}{u-m}, \quad \overline{N^S} = \frac{C(S_0u) - C(S_0m)}{S_0(u-m)}$$

And

$$\overline{N^{B'}} = R^{-1} \frac{mC(S_0d) - dC(S_0m)}{m-d}, \quad \overline{N^{S'}} = \frac{C(S_0m) - C(S_0d)}{S_0(m-d)}$$

#### Proof

Apply the above procedure but consider branches  $u, m$  only. Let  $\underline{C}$  denote the payoff function for such a portfolio. If  $k \geq m$ ,

$$\overline{N^B R} + \overline{N^S S_0 u} = S_0(u - k) = C(S_0u)$$

$$\underline{C} = \frac{\overline{N^B R} + \overline{N^S S_0 m} = 0 = C(S_0m)}{u-m} \quad (14)$$

$$\left\{ \begin{array}{l} \overline{N^B R} + \overline{N^S S_0 d} = \frac{(d-m)S_0(u-k)}{u-m} < 0 = C(S_0d) \end{array} \right.$$

If  $k < m$ ,

$$\underline{C} = \left\{ \begin{array}{l} \overline{N^B R} + \overline{N^S S_0 u} = S_0(u - k) = C(S_0u) \\ \overline{N^B R} + \overline{N^S S_0 m} = S_0(m - k) = C(S_0m) \\ \overline{N^B R} + \overline{N^S S_0 d} = S_0(d - k) < 0 = C(S_0d) \end{array} \right. \quad (15)$$

In both cases,  $\underline{C}$  has less payoffs in when  $S_1 = S_0d$ , and matches the payoff when  $S_1 = S_0u$  and  $S_1 = S_0m$ , therefore we say that  $\underline{C}$  under replicates  $C$  and the price of  $C$  could at least be the price of  $\underline{C}$  [4].

Finally consider the left combination of branches

$m, d$ . Let  $\underline{C}'$  denote the payoff function for such a portfolio. If  $k \geq m$ ,

$$\underline{C}' = \left\{ \begin{array}{l} \overline{N^{B'} R} + \overline{N^{S'} S_0 u} = 0 < S_0(u - k) = C(S_0u) \\ \overline{N^{B'} R} + \overline{N^{S'} S_0 m} = 0 = C(S_0m) \\ \overline{N^{B'} R} + \overline{N^{S'} S_0 d} = 0 = C(S_0d) \end{array} \right. \quad (16)$$

If  $k < m$ ,

$$\underline{C}' = \left\{ \begin{array}{l} \overline{N^{B'} R} + \overline{N^{S'} S_0 u} = 0 < S_0(u - k) = C(S_0u) \\ \overline{N^{B'} R} + \overline{N^{S'} S_0 m} = S_0(m - k) = C(S_0m) \\ \overline{N^{B'} R} + \overline{N^{S'} S_0 d} = 0 = C(S_0d) \end{array} \right. \quad (17)$$

In both cases,  $\underline{C}'$  has less payoffs in when  $S_1 = S_0d$ , and matches the payoff when  $S_1 = S_0u$  and  $S_1 = S_0m$ , therefore we say that  $\underline{C}'$  also under replicates  $C$ , and the price of  $C$  could at least be the price of  $\underline{C}'$  [4].

Notice that if  $k \geq m$ ,  $\{N^{B'}, N^{S'}\}$  will always be 0 thus a 0 call price but not when  $k < m$ . The maximum of two possible lower bound is dependent on relations between parameters  $u, m, d, k$  and  $B_0 R^{-1}$ , which altogether indicate whether the strike price of a call option is reasonable considering both money and stock markets. This paper will not explicitly calculate this relationship, but rather simply consider the financial meaning of the under replication.

So far, we conclude that three possible combinations constructed by branches  $\{u, m, d\}$  defines pricing boundaries, and two-branches is indeed the best replication financially. Now consider the effectiveness of these boundaries in the probability aspect.

#### 4.5. Theorem

If  $d < k < u$ , the set of all possible prices  $C_0$  of a European call option under the trinomial tree model follows

$$\sup(C_0) = \overline{N^B R} + \overline{N^S S_0}, \inf(C_0) = \max(\overline{N^B R} + \overline{N^S S_0}, \overline{N^{B'} R} + \overline{N^{S'} S_0}) \quad (18)$$

**Proof** Recall Corollary 4.2 that the expected payoff of a European call option is

$$\left\{ \begin{array}{ll} \max(S_0u - K, 0) & \text{if } S_1 = S_0u \\ \max(S_0m - K, 0) & \text{if } S_1 = S_0m \\ \max(S_0d - K, 0) & \text{if } S_1 = S_0d \end{array} \right.$$

Then at time 0, the initial investment  $C_0$  required to set up the portfolio for the European call option is given by [5]

$$C_0(S_0) = E^Q[R^{-T}C(S_T)] = R^{-1}(p_u C(S_0u) + p_m C(S_0m) + p_d C(S_0d)) \quad (19)$$

If  $u > k \geq m$ ,

$$C_0(S_0) = R^{-1}q_u S_0(u - k) = R^{-1} \frac{R + (d - m)p_m - d}{u - d} S_0(u - k) \quad (20)$$

Taking the derivative with respect to  $p_m$ , we have

$$\frac{\partial C}{\partial p_m} = R^{-1} S_0 \frac{d - m}{u - d} (u - k) < 0 \quad (21)$$

If  $m > k > d$ ,

$$C_0(S_0) = R^{-1}q_u S_0(u - k)$$

$$= R^{-1} \left[ \frac{R + (d - m)p_m - d}{u - d} S_0(u - k) + p_m S_0(m - k) \right] \quad (22)$$

Taking the derivative with respect to  $p_m$ , we have

$$\frac{\partial C}{\partial p_m} = R^{-1}S_0 \left[ \frac{d-m}{u-d}(u-k) + m - k \right] < 0 \quad (23)$$

Regardless of the value of  $k$ , the price function being strictly decreasing with respect to  $p_m$ . Therefore, taking  $\max(p_m)$  and  $\min(p_m)$  grants  $\min(C_0)$  and  $\max(C_0)$  respectively. Recall *Remark 2.2* that  $\max(p_m)$  leads to two possible values of  $\{p_u, p_d\}$ , depends on the relation between  $m$  and  $R$  therefore two possible lower bounds. This finding agrees with results from *Theorem 4.3-4.4*.

## 5. CONCLUSION

This paper elucidates the incomplete market under the trinomial tree model and completing market by adding linearly independent derivatives. We also analyze the European call option pricing under the first-step trinomial tree model, using both portfolio and probability approach. In the portfolio method, we consider all possible sub-structured binomial tree model recombined from three branches. Results show one upper and two possible lower bounds, depend on market parameters. The partial derivative yields the same conclusion with different approach. The indeterminacy of the trinomial tree model is stabilized when the probability of one branch is approaching 0, where the model will eventually become binomial. One may consider other derivatives and pricing or further steps in markets under this model.

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