The Basic Macaulay Duration Theories and Limitations that are Necessary for Investors to Know

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ABSTRACT
The concept of duration was proposed by Frederick Macaulay in 1938 and has become a popular tool for measuring financial instruments nowadays. This paper not only provides investors with a detailed introduction to Macaulay Duration and affirms its role but also reminds investors of Macaulay duration's limitations and offers solutions to address them. It starts from reviewing the derivation idea of Macaulay duration and explaining the rationality of its weighted average value at the time of receiving cash flow. Next, some Macaulay duration theories are detailed, including that the Macaulay duration of bonds is less than or equal to its maturity; the Macaulay duration of zero-coupon bonds is longer than that of coupon bonds with the same maturity date; and the longer the Macaulay duration, the stronger the sensitivity of bond price to interest rate changes. Based on these definitions and properties, the Macaulay duration is widely used to measure bonds' time structure and risk. However, the Macaulay duration still has some limitations. It can neither calculate the specific sensitivity of bond price to interest rate changes nor be applied to financial instruments with non-fixed cash flow, such as callable bonds. Therefore, to make up for these two shortages of Macaulay duration, economists have introduced new duration models, namely modified duration and effective duration. Investors can use this paper as a reference to determine whether the information they need is not available from the Macaulay duration or whether the bonds they invest in apply to the Macaulay duration. If the answers are no, they can then choose to meet their requirements with either modified duration or effective duration.

Keywords: *Macaulay duration, Interest rate risk, Modified duration, Callable bond, Effective duration.*

1. INTRODUCTION

1.1. Background

Frederick Macaulay introduced the concept of duration in his classic 1938 work to accurately measure the maturity of bond life [1]. Just as with many other important innovations in finance, it took a long time until the 1970s for investors to embrace the Macaulay duration as an alternative of bond maturity to measure bond metrics [2]. Since then, there has been plenty of literature on Macaulay's duration, and its equation and functions are well known by most investors. Ajlouni explains that the Macaulay duration can measure the time structure of bonds and judge the risk of bonds to a certain extent [3]. This example demonstrates the usefulness of Macaulay duration in helping investors measure bonds with different conditions. Today, the Macaulay duration is a technology that investors often use to improve investment performance and has been widely promoted as an indispensable tool for fixed-income investors [4].

However, as the bonds in the market become more and more diversified and complex, the limitations of Macaulay duration gradually emerge [5]. Some professors have described several restrictions on the Macaulay duration. For example, Richie, Mautz, and Sackley explain in their research that Macaulay duration is only applicable to cases in which the change of yield to maturity is minimal [6]. While Zheng states that Macaulay's images were only applicable to the parallel movement of interest rates in the traditional parametric model [7]. To address this conflict, this paper will detail the limitations most often involved in the practical application of the Macaulay duration and provide corresponding solutions from the new duration models derived from its limitations.

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1.2. Objectives

The second section of this paper reviews the definition and formula derivation of the Macaulay duration and explains its rationality. Then, it briefly introduces some basic characteristics of the Macaulay duration and mentions its usefulness.

However, there are still some limitations when applying the Macaulay duration. The following part of this paper will introduce two such limitations and present corresponding solutions. The third section introduces the first deficiency of the Macaulay duration, that is, it is impossible to calculate the specific sensitivity of bond price to interest rate changes. The corresponding solution is to use the correction duration to make up for it.

The fourth section of this paper discusses another limitation of the Macaulay duration, namely, it does not apply to bonds without fixed cash flow and cash flow timing. This point will be explained in detail by assessing redeemable bonds as an example finding that the corresponding solution is the effective duration.

In the final section, the core statement, including the theories of Macaulay duration, its limitations, and corresponding solutions, will be restated and summarized. Besides, suggestions for investors are also provided. When choosing their optimal investment strategies, investors should choose appropriate duration models according to their own demands and types of financial instruments.

2. MACAULAY DURATION

2.1. The Method

When considering which bonds to invest in, investors compare all aspects of different bonds, including bond price, interest rate, maturity date, and more. If all conditions of two bonds are the same except for one variable, investors can easily decide which bond to buy. For example, all other conditions of the two bonds are the same, except the maturity date. The bond maturity date measures the time at which the final principal repayment is received, which can also be regarded as the length of time that the bondholder is exposed to the risk of an interest rate rise and residual cash flow depreciation. Thus, in this case, investors will likely choose the bond with a shorter maturity in which to invest. This is because they can recover the principal and interest faster, reducing the impact of the time value of cash on them. That said, this is only the simplest example of how to choose bonds. Usually, investors face much more difficult choices in the capital markets. Bond issuers issue a variety of bonds with different conditions, and all aspects of these bonds can be different. At this time, investors can no longer determine their investment strategies through simple comparisons. Therefore, Macaulay proposed the concept of duration to help investors better measure different bonds and make wiser decisions.

In 1938, Macaulay proposed the concept of duration by combining term effect and coupon effect to describe the volatility of the bond price. It calculates the weighted average of the time when bonds generate cash flow in the future. Ajlouni describes the duration as the weight on a scale, each cash flow is a weight, and the largest weight is the repayment of bond principal [3]. Each weight represents the coupon payment (C) for time (T). Although the amount paid for each coupon before the final principal repayment is the same, the time value of C1 is greater than C2 and the time value of C2 is greater than C3. As an analogy, the earlier the coupon payment, the greater the weight should be. In other words, Macaulay duration can be regarded as the weighted average of the time it takes to generate cash flow, which measures the time structure of bonds.

It should be noted that the Macaulay duration is neither a simple calculation of the average time of receiving cash flow nor a corresponding weighted calculation of the amount paid by the coupon. Kritzman describes the reasons for this below [4]. For example, a five-year coupon bond pays £ 100 a year and £ 500 at maturity. At this time, the average time of cash flow received can be calculated. However, this simple calculation is not enough to measure the effective life of bonds, because it does not take the relative size of cash flow into account. In this bond, the principal repayment of the coupon bond is five times that of each coupon payment. Therefore, it is reasonable to give heavier weight to the time when the principal repayment is received than to the time the coupon payment is received. Then, although the time of receiving principal repayment is weighted to five times of each coupon payment, the measure of the weighted average is still flawed. This is because it still ignores the time value of money. For example, the value of £100 coupon payments received within two years may be lower than the value of $100 coupon payments received one year later due to the devaluation of the currency. Macaulay also recognizes this difference, so he believes that the time at which each cash flow is received should also be weighted. That is, instead of calculating the weighted average of the relative size of cash flow, it is the weighted average of the present value of its relative size. Therefore, the Macaulay duration is equal to the average time it takes to receive bond cash flows, where the time of receiving each cash flow is weighted by its present value as a percentage of the total cash flows' present value. Therefore, the Macaulay duration is calculated as follow:

$$D = \frac{\sum_{t=1}^{n} \frac{t \times Ct}{(1+r)^t}}{\sum_{t=1}^{n} \frac{Ct}{(1+r)^t}} \quad (1)$$

Where D is the Macaulay duration, n is the number of cash flows, t is the time to receive the cash flow, Ct is the cash flow payment, and r is the yield to maturity.

2.2. The Characteristics Of Macaulay Duration

In practical application, the Macaulay duration has several important attributes. First, the Macaulay duration is always less than or equal to the bond's maturity [3]. This is because, cash flow is given a certain weight in the early year of bond life, which helps to advance the average time of cash flow receipt. For coupon bonds, bondholders continue to receive cash flow as coupons before their maturity date. Therefore, the weighted average of the time they receive cash flows is usually less than the maturity date of the bond. In addition, Macaulay duration vary with coupons, yields, and maturity dates. For zero-coupon bonds, the Macaulay duration is equal to its maturity. This is because bondholders must wait until the end of the maturity date to receive a single cash flow of zero-coupon bonds.

Therefore, the Macaulay duration's second attribute is that, in general, the duration of zero-coupon bonds is longer than that of coupon bonds with the same maturity. For example, in the case of both zero-coupon bonds and coupon bonds with a maturity of five years, zero-coupon bonds pay £1000 on the maturity date, while coupon bonds pay £100 a year and £500 on maturity. Although both bonds enable investors to receive a cash flow of £1000 at maturity, zero-coupon bond buyers will not receive any cash flow until the fifth year of maturity, while those who buy coupon bonds will have already received £500 worth of cash flows. Thus, the weighted average time for coupon bonds to start receiving a cash flow is generally shorter than that of zero-coupon bonds with the same maturity date.

The third property of Macaulay duration is that the longer a bond's duration, the more sensitive its price is to changes in interest rates. This is because the longer the maturity of bonds, the more the cash flow will be affected by interest rate changes for a longer time [8]. Generally, the longer the duration of a bond, the greater the impact of the change in the interest rate on its price, so the greater the risk [4]. When the interest rate is reduced, the bonds with longer durations rise greatly, while when the interest rate is increased, the bonds with longer durations also fall heavily. Therefore, if the interest rate is expected to rise in the future, it is wise to select bonds with short durations. Kritzman further states that, however, this relationship is not always established, because it ignores the impact of coupon payment and the prepayment of principal [4]. The fourth part of the paper will use callable bonds as an example and explain them in detail.

3. THE FIRST LIMITATION OF MACAULAY DURATION

3.1. Interest Rate Risk

Interest rate changes in the financial markets will affect many investments, amongst which a major impact is on the value of bonds and other fixed-income financial instruments [8]. Therefore, bondholders need to observe interest rates timely and make corresponding decisions according to these changes. Interest rate risk is a potential investment loss that may be caused by the change of interest rate, and these changes in bond prices caused by changes in interest rates are called their durations [9].The original purpose of the Macaulay Duration is to measure the interest rate risk of bonds. Chen also points out that for fixed-income bonds, as interest rates rise, bond prices fall [9]. This is because when interest rates rise, the cost of missing better investments will be higher. As a result, as interest rates rise, the attractiveness of interest rates earned by bonds decreases. Therefore, to make up for their economic disadvantage, the price of these bonds must fall. When the market interest rate changes, the value of existing fixed-income securities with different maturities will change to different extents, this phenomenon is called the sensitivity of the bond and is measured by the duration of bonds [10]. Generally, the longer the maturity of a security, the higher its sensitivity and the more its price drops relative to the rise of a given interest rate.

3.2. Unavailable For Measuring Sensitivity

Although the original idea of the Macaulay duration was to measure interest rate risk, it is, in fact, not a reliable way to perform this calculation [8]. The Macaulay duration can only demonstrate that the change of interest rate is inversely proportional to the bond price. When the interest rate increases, the bond price decreases. However, it cannot reveal that the corresponding changes in bond prices are significantly different under the same level of interest rate changes. The relationship between bond price and yield presents a convex shape on the image in Figure 1. Considering a unit increase in the yield at A and B, respectively, the price decline at A is much greater than that at B. However, the Macaulay duration does not show this difference, this is the Macaulay duration's first limitation. From the original purpose, the intention of calculating the Macaulay duration was to try to understand and master the sensitivity of bond prices to interest rate changes. In other words, the original expectation of the Macaulay duration was not only to show the price change trend when the given yield

changes but also to calculate the price change percentage when the given yield changes, to understand the difference between the bond price change when the interest rate is low and the bond price change when the interest rate is high. Therefore, the Macaulay duration cannot help investors solve the problem of interest rate risk.

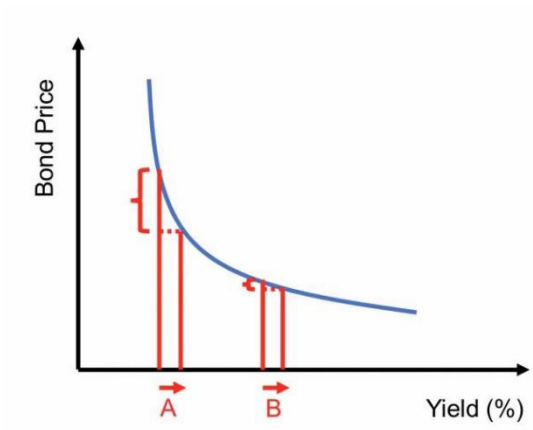

Figure 1 Price-yield curve of a (non-callable) bond.

3.3. Modified Duration

In order to make up for the limitation of the Macaulay duration, a new concept of modified duration was introduced. Different from the Macaulay duration, modified duration determines the sensitivity of bond price to interest rate change, that is, the percentage change of given yield change to bond price [11]. For example, modified duration means that if the modified maturity of a bond is 3.5% when the yield changes by 1%, the bond price will change by 3.5%. Therefore, it can calculate the specific sensitivity of the prices at A and B, mentioned above in Figure 1, to the change of yield. The formula of modified duration is as follow:

$$MD = \frac{D}{1+r} \tag{2}$$

Where MD is the Modified duration, D is the Macaulay duration, and R is the yield to maturity.

The modified duration can accurately quantify the impact of interest rate changes on bond prices, and can better help investors measure interest rate risk. The longer the modified duration, the more sensitive the bond price is to the yield change [12]. This means that the longer the modified duration, the greater the changes of bond price caused by interest rate changes. Therefore, when the modified duration is long, the bond can be seriously influenced by the risk of rising interest rate but retain strong resistance to the risk of interest rate decline. When the modified duration is short, the opposite applies. In other words, if the market interest rate level could rise, investors should choose to invest more in short-term bonds and shorten the modified duration of bonds. On the contrary, when the market interest rate is trending downward, they should increase investment in long-term bonds to prolong their modified duration.

4. THE SECOND LIMITATION OF MAC DURATION

The second limitation of the Macaulay duration is that it does not apply to bonds with variable cash flow or cash flow timing. For example, the maturity and cash flow of callable bonds are uncertain, so in this case, looking for the Macaulay duration is meaningless [13].

4.1. Callable Bonds

The first thing to understand is the concept of redeemable bonds. Callable bonds, also known as redeemable bonds, refer to the bonds that the issuer has the right to forcibly redeem from the bondholders at a certain price and a specific time, even before their maturities. Usually, callable bonds are issued when advisers expect that the future interest rate may reduce. Generally, its yield to maturity will be higher than that of non-callable bonds to compensate the buyers for the risk of early redemption [14] (François and Pardo, 2015).

To estimate when the bond issuer will choose to redeem callable bonds, imagine a callable bond with a face value of £300, issued at a coupon rate of 9% when the market interest rate is 7%. In this scenario, the issuer has to pay an interest of £9 a year. When the market interest rate drops to 5%, it is not cost-effective for the issuer to continue to pay interest at the coupon rate of 9%. At this time, the bond issuer may choose to redeem their existing bonds and instead issue new bonds with lower interest rates. To sum up, when the market interest rate falls much lower than the callable bonds' coupon rate, the issuers will redeem the bonds if they think it is more cost-effective to redeem them and reissue new bonds at a lower yield than to continue operating the existing bond.

4.2. Cannot Apply To Bonds With Non-fixed Cash Flow And Cash Flow Timing

The Macaulay duration is invalid in this case for the following reasons. First, to calculate the Macaulay duration, the bond's maturity date is needed. However, before the bonds are redeemed, it is hard to determine when the issuer will redeem them because it is jointly determined by various uncertain factors such as changes in market interest rates and the issuer's personal choice.

In addition, there is an important assumption in the study of the Macaulay duration, that is, the cash flow of bonds will not change with the fluctuation of interest rate, so it is difficult to hold for callable bonds [5]. Nawalkha, Soto, and Beliaeva claim that, from the image, the price-yield curve of callable bonds is different from that of ordinary bonds, and its trending

cannot be displayed by the Macaulay duration [15]. For example, in Figure 2, if a callable bond is redeemed, then its price-yield curve after this will become an approximate horizontal straight line, which is lower than the price interest rate curve of ordinary bonds [16]. This is because after the bond is redeemed, it will no longer run, and the issuer will no longer pay the corresponding interest. This means that the price of the bond will basically not change with the interest rate. In other words, the cash flow of callable bonds will have changed. Therefore, the Macaulay duration should not be applied to financial instruments whose cash flow is vulnerable to interest rate changes [17].

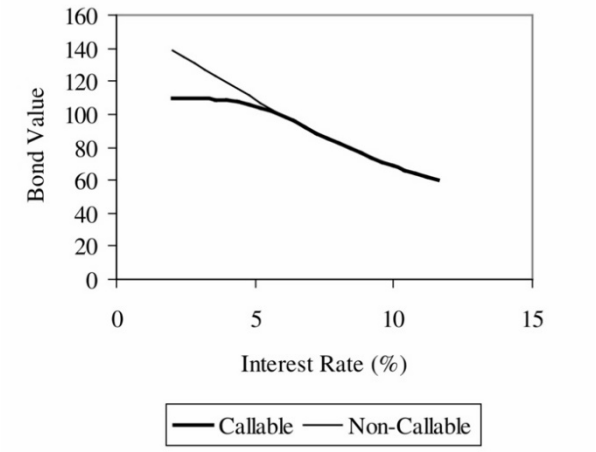

Figure 2 Price-yield curves of non-callable and callable bonds. [16]

4.3. Effective Duration

To make up for the defects of the Macaulay duration in this aspect, the effective duration is one proposed solution. Effective duration is the percentage of bond price change when the interest rate level changes. It is a direct calculation method for the price sensitivity of bonds or other financial instruments to the interest rate [18]. It uses bond prices for different yield changes, which reflect the changes in the value of implied options, in equation. The effective duration equation is as follows:

$$Effective\ duration = \frac{P_D - P_I}{2 \times P_0 \times \Delta y} \quad (3)$$

Where P_D is the price of the bond if the yield decrease by a certain percentage, P_I is the price of the bond if the yield increase by a certain percentage, P_0 is the present value of the bond, and Δy is the change in yield.

The equation demonstrates that the effective duration overcomes the second limitation of the Macaulay duration. It neither considers the change in the amount of cash flow nor the change in timing of cash flow. Therefore, the effective duration can be applied to financial instruments with an implied options. Compared with the Macaulay duration, the effective duration has a wider range of applications so investors can measure more types of bond risks and get the optimal choices [19].

5. CONCLUSION

This paper reviews the related content of Macaulay duration, including its definition, equation, and properties. The Macaulay duration is a weighted average of the time it takes a bond to generate cash flow. It measures the time structure of bonds and reflects, to some extent, the relationship between bond prices and interest rate changes. This paper also describes three of the characteristics of Macaulay duration :(1) the duration of a bond is always less than or equal to its maturity date; (2) The Macaulay duration of coupon bonds is shorter than that of zero-coupon bonds with the same maturity; (3) The longer the duration of a bond, the more sensitive its price is to changes in interest rates. Next, this paper introduces two main limitations of Macaulay duration :(1) Macaulay duration does not completely solve the interest rate risk of bonds, and cannot express the specific sensitivity of bond prices to interest rate changes; (2) Macaulay duration do not apply to financial instruments with unfixed cash flows or cash flow timing. The corresponding solutions of the two constraints are both proposed in this paper, namely modified duration and effective duration. They make up for the deficiency of Macaulay duration to a large extent. Investors can use these two new duration models to measure a wider range of financial instruments. Therefore, investors should also use and refer to other duration models to select their preferred optimal investment strategies when the Macaulay duration cannot meet their requirements or is not suitable for the types of financial instruments they want to choose.

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