ATLANTIS PRESS
Advances in Social Science, Education and Humanities Research, volume 653
Proceedings of the 2022 International Conference on Social Sciences and Humanities and Arts (SSHA 2022)

Analysis of the Premium of American Option Based on both Mathematical Method and Python

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ABSTRACT
This paper mainly analyzes the influence of different factors on the price of American call option and put option and European call option and put option. This paper mainly considers the option premium from four aspects: strike price, up factor, down factor, and interest rate. First, this paper discusses the possible situation of stock prices in different periods, and then gives the corresponding market decisions and the consequences of the corresponding market decisions according to different situations. In each case, this paper shows the process of option premium change through binomial tree diagram and a lot of formula analysis, and then compares the American option with the European option. Through this paper, people can have a basic understanding of American option and European option, and people can have a deeper understanding of the operating logic behind them.

Keywords: *American calls and puts, European calls and puts, option premium*

1.INTRODUCTION

The most classic option pricing model today is the Black-Scholes pricing model, which was proposed by Black and Scholes in the 1970s. This model believes that future predictions are only related to the current stock price, and have nothing to do with historical data. The B-S pricing model proposes that option pricing is very complicated, and it will be determined by many factors, such as Time to maturity, present stock price, risk-free rate, and strike price.

In our research report, we will study American Option based on a relatively simple option pricing model, the Binomial Model. The Binomial Model is essentially a digital representation method of the Black-Scholes model, using the discrete binomial distribution as an approximate representation of the normal distribution to obtain the price of the option [1]. This model will be more intuitive and simpler, which will help people better understand the operating mode of American options. So that in- depth research can be carried out in the future. Based on the Binary model, we will assume that the stock price fluctuates only in two directions, upward or downward, and that the probability of stock price fluctuations in the upward or downward direction remains stable during the option cycle. Then, we can roughly analyze the pricing of American options. In addition, we will base on the idea of risk-neutral, which assumes that the risk preference of investors is risk-neutral (no risk premium); and the expected return of all securities is risk-free rate. As a result, we will calculate the present value of the cash flow of exercise the option at different time nodes by discounting the expected return, and use this to price the American option.

2. SECTION1: CALL OPTION

Though "an American option can be exercised at any time during its life" [2], it may be confused why investors do not exercise American call options early as it is American option characteristic to early exercise. For puts, investors exercise early in some situations. However, for calls, never. According to Kimura (2011), despite that people should buy American options because it can be exercised any time investors want, but there is little difference between the prices of American and European call options [3]. The following argument will prove why American call options will never be exercised early.

The discrete binominal tree model is the main thought of exploring values and payoffs of various options. At t_0, we assume the stock spot price is S_0, as shown in Figure 1. Then the stock price can either go up by u or go down by d. So, at t_1, the stock value is either S_0u or S_0d, at t_1 and they will become S_0u^2, S_0ud and S_0d^2 at t_M. At any time, we can exercise our options at a strike price, K. The risk-free rate is R, and we think there is always a relationship between these changing rates: $u>R\geq 1>d$. We use backward induction to determine the value at each time and compare exercise at maturity's value and exercise early value to determine whether exercise early is a better choice. This model uses risk-neutral pricing with $q=(R-d)/(u-d)$. Underlying Figure 1 is a diagram to show the two steps binominal tree.

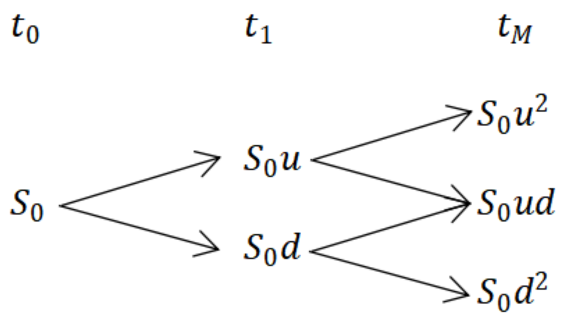

Figure 1: two steps binominal tree

2.1 Exercise at time 1:

First discuss whether exercise early in t_1, using backward induction from t_2, to t_1, getting the value of option exercise until maturity and compare it to the value of exercise immediately at t_0. After that, we will move to the circumstance at t_0.

2.1.1. Case 1: $S_0d^2>K$

Different payoff of options is below.

Exercise early at t_1: $F_1(S_0u)=S_0u-K$ $F_1(S_0d)=S_0d-K$

Exercise at t_M:
$F_2(S_0u)=R^{-1}[q(S_0u^2-K)+(1-q)(S_0ud-K)]$
$F_2(S_0d)=R^{-1}[q(S_0ud-K)+(1-q)(S_0d^2-K)$.

If the stock price goes up, compare $F_1(S_0u)$ and $F_2(S_0u)$ by using $F_1(S_0u)-F_2(S_0u)=K(R^{-1}-1)$. If the outcome bigger than 0, exercise early is a good choice because premium. If the outcome is smaller than 0 or equal, people will not be going to exercise the option early. Because $R\geq 1$, so $K(R^{-1}-1)\leq 1$, it is clearly that people would not exercise early. If the stock price goes down, we have $F_1(S_0d)-F_2(S_0d)=K(R^{-1}-1)$. The result is the same as the upper one, do not exercise early. Therefore, do not exercise early at t_1 is the better choice for people when $S_0d^2>K$.

2.1.2. Case 2: $S_0d>K$ and $S_0ud>K>S_0d^2$

Exercise early at t_1: $F_1(S_0u)=S_0u-K$ $F_1(S_0d)=S_0d-K$.

Exercise at t_M:
$F_2(S_0u)=R^{-1}[q(S_0u^2-K)+(1-q)(S_0ud-K)]$

$F_2(S_0d)=R^{-1}[q(S_0ud-K)]$.

The stock price goes up situation is same to when $S_0d^2>K$, so do not exercise early. If stock price goes down, $F_1(S_0d)-F_2(S_0d)=S_0d(1-quR^{-1})+K(q/R^{-1}-1)$. Due to $S_0d(1-quR^{-1})+K(q/R^{-1}-1)<0$, so do not exercise early. In conclusion, when $S_0d>K$, $S_0ud>K>S_0d^2$, people would not exercise early.

2.1.3. Case 3: $S_0>K>S_0d$, $S_0ud>K$

Exercise early at t_1: $F_1(S_0u)=S_0u-K$ $F_1(S_0d)=0$.

Exercise at t_M:

$F_2(S_0u)=R^{-1}[q(S_0u^2-K)+(1-q)(S_0ud-K)]$
$F_2(S_0d)=R^{-1}[q(S_0ud-K)]$.

The stock price goes up situation is same to when $S_0d^2>K$, so do not exercise early. When stock price goes down, $F_1(S_0d)-F_2(S_0d)=R^{-1}[q(K-S_0ud)]$. As $S_0ud>K$, the result is negative, so do not exercise early again.

2.1.4. Case 4: $S_0u>K>S_0d$, $S_0u^2>K>S_0ud$

Exercise early at t_1: $F_1(S_0u)=S_0u-K$ $F_1(S_0d)=0$.

Exercise at t_M: $F_2(S_0u)=R^{-1}[q(S_0u^2-K)]$ $F_2(S_0d)=0$.

If the price goes up at t_1, the different between the values of exercising the call options or not is $F_1(S_0u)-F_2(S_0u)=S_0u(1-quR^{-1})+K(qR^{-1}-1)$. This is quite like the case 2 but changes d to u. Causes $S_0u>K$, so the outcome is negative, people would not exercise their options early. If the price goes down, with the comparison the outcome is 0, investor is indifference. To conclude, exercise until t_M is a better choice.

2.1.5. Case 5: $K>S_0u$, $S_0u^2>K$

Exercise early at t_1: $F_1(S_0u)=F_1(S_0d)=0$.

Exercise at t_M: $F_2(S_0u)=R^{-1}[q(S_0u^2-K)]$ $F_2(S_0d)=0$.

If the price goes up, $F_1(S_0u)-F_2(S_0u)=R^{-1}[q(K-S_0u^2)]$. Due to $S_0u^2>K$, so the outcome is negative. If the price goes down, the two payoffs are both 0 meaning indifferent. Thus, do not exercise early is the wisdom choice.

2.1.6. Case 6: $K>S_0u^2$

In the last situation, all values of the options in different situations are 0. So, no matter what the investors do, they would not have premium.

2.1.7. Conclusion

The discussion above includes all situations in t_1. We found that the investors always do not exercise their call options early at t_1. In the next part, we will start to consider whether exercise early at t_0.

2.2 Premium in call options

To show why investors do not use American call option in advance, people can subtract the value of European call option from the value of American call option. If American call option is at a premium to European call option, investors have an incentive to use American call option. Other factors being the same, if we change the value of K, we can find that the premium of the value of the American call option minus the European call option is almost zero. This means that no matter how the K value changes, the European call option and the American call option have the same value. Investors have no incentive to use American call options.

In other factors remain unchanged, the risk-free interest rate, R goes up, we can find that when R is between 0 and 1, the American call option and the European call option have the same value, but when R is greater than 1, American call options for European call option premium begin to rise sharply, then gradually flat out. This means that when the market's risk-free rate rises sharply, the value of the American call option will be greater than the European call option. However, the index of the risk-free rate rarely exceeds 1. In other words, risk-free interest rate has no obvious effect on the price of European and American call options in real life.

2.2.1 Exercise early value at t_0:

In this part, we divide all situations into two cases. The first is $S_0>K$, $S_0u>K>S_0d$ and $S_0u^2>K>S_0ud$. Why is that? Because when $S_0>K$, S_0u must bigger than K, also S_0u^2 must bigger than K, so the backward induction have the highest value when $S_0d^2>K$ and the lowest value when $S_0u^2>K>S_0ud$ exercise until maturity, if at t_0 early exercise value smaller than the lowest exercise until maturity backward induction value, exercise early is not suit in any other situation. Similarly, in the second one, we look at $K>S_0$, with the same logic, the minimum exercise until maturity backward induction value is when $K>S_0u$ and $K>S_0u^2$.

2.2.2 Case 1

Exercise until maturity value at time 0: $F_0(S_0)=S_0-K$.

Exercise at t_2:

$F_2(S_0u^2)=S_0u^2-K \quad F_2(S_0ud)=0 \quad F_2(S_0d^2)=0$.

The value at t_1: $F_1(S_0u)=R^{-1}[q(S_0u^2-K)]$ $F_1(S_0d)=0$.

The value at t_0: $F_2(S_0)=R^{-2}[q^2(S_0u^2-K)]$.

$F_0(S_0)-F_2(S_0)= = S_0(R^2-q^2u^2)+K(q^2-R^2)<0$, the outcome is negative so people would not exercise early at t_0. As we have explained above, this is the lowest exercise until maturity backward induction value when $S_0>K$. Thus, in other circumstances, do not exercise early is the better choice.

2.2.3 Case 2

Exercise early value: $F_0(S_0)=0$

Exercise until maturity value: $F_2(S_0)=0$.

So, there are no differences between exercising the call options early or not. As the minimum value of exercise until maturity, investors are indifference, investors will not exercise early in other states.

2.3 Impact of K and R

In order to more intuitively express the impact of strike price K, risk-free rate, R on call option premium, we used Python to simulate the impact of these two factors on call option premium. Figure 2 shows the change of call premium with the strike price when all other factors are held constant. Figure 3 shows how the call premium changes with the risk-free rate R, other factors being constant. The Python model confirms our inferences and results about call options.

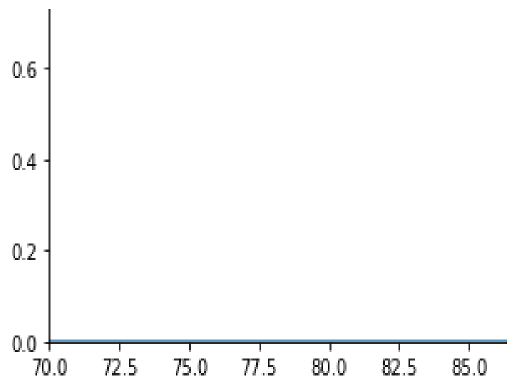

Figure 2: (The premium of American call option varies with K)

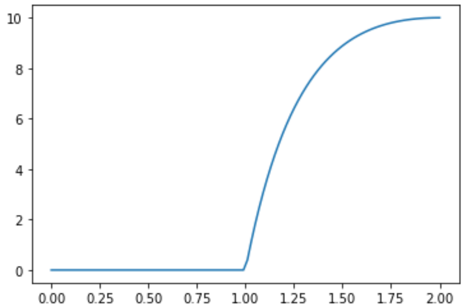

Figure 3: (The premium of American call option varies with R)

2.4 Conclusion

This whole section talks about whether to exercise the American call options early. We discuss cases separately at t_0 and t_1. The results show that, investors will never exercise the American call options early, which prove what we discuss in the very beginning.

3. SECTION 2: AMERICAN PUTS OPTION

Since the diverse kinds of the stock prices, the strike prices and rates, one of the most important parts in investigating the American puts options exercising strategies is when to exercise [4]. In this essay, we will use the binominal model to analyze these possible situations. When developing the American puts option payoff model, we will use the model as the same as we used in call options. The differences are that the payoff of the puts option at maturity is calculated as$(K\text{-}S_1,\ 0)^+$. So, the payoff at maturity time is $(K\text{-}S_0u^2, 0)^+$, $(K\text{-}S_0ud, 0)^+$ and $(K\text{-}S_0d^2, 0)^+$ respectively when the stock price at maturity time isS_0u^2, S_0ud and S_0d^2. I assumed that the strike price, K, have six different values between the six stock prices at different times (from the largest to the smallest is K_1 to K_6). In addition, because the relationship between K and the stock prices is uncertain under situation K_3 and K_4, so we must consider two more situations. These eight cases are:

(1)$K_1 \geq S_0u^2$
(2)$S_0u^2 > K_2 \geq S_0u$
(3)$S_0ud > K_3 \geq S_0ud,\ K_3 \geq S_0$
(4)$S_0ud > K_3 \geq S_0ud,\ K_3 < S_0$
(5)$S_0ud > K_4 \geq S_0d,\ K_4 \geq S_0$
(6) $S_0ud > K_4 \geq S_0d,\ K_4 < S_0$
(7)$S_0d > K_5 \geq S_0d^2$
(8)$S_0d^2 > K_6$

3.1. Case (1): $K_1 \geq S_0u^2$

European: Using the formula of payoff, we can get the value of the European options in different times (when $S_1=S_0u$, $S_1=S_0d$ and S_0 respectively):

$F(E, S_0)=\frac{K_1}{R^2}\text{-}S_0$ $F(E, S_0u)=\frac{K_1}{R^2}\text{-}S_0u$ $F(E, S_0d)=\frac{K_1}{R^2}\text{-}S_0d$

American: At t_1, if we choose to exercise the options earlier, the payoffs at S_0u and S_0d are:

$F(A, S_0u, exe)=K_1\text{-}S_0u$ $F(A, S_0d, exe)=K_1\text{-}S_0d$

If we decide not to exercise them, the payoffs will remain the same as the European's:

$F(A, S_0u, not)=F(E, S_0u)$ $F(A, S_0d, not)=F(E, S_0d)$

At $t_1(S_1=S_0u)$: $F(A, not)\text{-}F(A, exe)<0$

At $t_2(S_1=S_0d)$: $F(A, not)\text{-}F(A, exe)<0$

The better choice for us at t_1 is to exercise our American puts options.

t_0: exercise early, the payoff is $F(A, exe) =K_1\text{-}S_0$

do not exercise, the payoff is $F(A, not)=\frac{K_1}{R}\text{-}S_0$

$F(A, S_0, exercise)\text{-}F(A, S_0, not)=K_1\text{-}\frac{K_1}{R}>0.$

So, when people exercise the American puts option at t_0, they can get the largest payoff. The early exercise premium is $F(A, S_0, exe)\text{-}F(E, S_0)=\left(1\text{-}\frac{1}{R^2}\right)K_1$.

3.2. Case (2): $S_0u^2 > K_2 \geq S_0u$

European: $F(E, S_0u)=\frac{(1\text{-}q)(K_2\text{-}S_0ud)}{R}$

$F(E, S_0d)=\frac{K_2}{R}\text{-}S_0d$

$F(E, S_0)=\frac{[(1\text{-}q^2)K_2\text{-}(R^2\text{-}q^2u^2)S_0]}{R^2}$

American: If we choose to exercise the options earlier, the payoff for the stock price at S_0u and S_0d are:

$F(A, S_0u, exercise)=K_2\text{-}S_0u$, $F(A,S_0d, exe)=K_2\text{-}S_0d$

If we decide not to exercise them, the payoffs will remain the same as the European's:

$F(A, S_0u, not)=F(E, S_0u)$ $F(A, S_0d, not)=F(E, S_0d)$

At t_1 $(S_1=S_0u)$: $F(A, not)\text{-}F(A, exe)<0$

At t_2 $(S_1=S_0d)$: $F(A, not)\text{-}F(A, exe)=<0$

The better choice at t_1 is exercising the American puts options.

Then back to t_0: exercise early $F(A,S_0, exe) =K_2\text{-}S_0$.

do not exercise $F(A, S_0, not) =\frac{K_2}{R}\text{-}S_0$

$F(A,S_0, exe)\text{-}F(A,S_0, not)=K_2\text{-}\frac{K_2}{R}>0$

When people exercise the American puts option at t_0, they get the largest payoff. The early exercise premium is $F(A, S_0, exe)\text{-}F(E, S_0) =\frac{1}{R^2}[(R^2+q^2\text{-}1)K_2\text{-}q^2u^2S_0]$

3.3. Cases (3) and (4): $S_0ud>K_3\geq S_0ud$

European: $F(E,S_0u)=\frac{1}{R}(1-q)(K_3-S_0ud)$

$F(E,S_0d)=\frac{K_3}{R}-S_0d$

$F(E,S_0)=\frac{1}{R^2}[(1-q^2)K_3-(R^2-q^2u^2)S_0]$

American: If we choose to exercise the options earlier, the payoff at S_0u and S_0d is as followed:

$F(A, S_0u, exe)=0$ $F(A, S_0d, exe)=K_3-S_0d$

If we decide not to exercise them,

$F(A, S_0u, not)=F(E, S_0u)$ $F(A, S_0d, not)=F(E, S_0d)$

Compare the payoff of not exercise and exercise:

At t_1 $(S_1=S_0u)$: $F(A, not)-F(A, exe)=F(E,S_0u)>0$

At t_2 $(S_1=S_0d)$: $F(A, not)-F(A, exe)=\frac{1}{R}(1-R)K_3<0$

Thus, when stock price climbs, we should not exercise the American puts options at t_1. But if the stock price falls, we should exercise the options earlier.

Then back to t_0, if we exercise early, the payoff is:

When $S_0\geq K_3$, $F(A, S_0, exe)=K_3-S_0$

When $S_0<K_3$, $F(A, S_0, exe)=0$

If do not exercise: $F(A, S_0, not)=\frac{(1-q)(q+R)K_3-q^2u^2S_0}{R^2}-S_0$

Compared the two values in different situations:

When $S_0\geq K_3$, $F(A,S_0,exe)-F(A,S_0,not)$ is not greater or smaller than zero constantly. So, whether should people exercise their options at S_0 is depended on the specific value of u, R, d and K.

When $S_0<K_3$, $F(A,S_0,exe)-F(A,S_0,not)<0$. So, people will not exercise the options earlier.

Overall, ignoring the unstable situation when $K_3\geq S_0$, people should not exercise earlier at. The early exercise premium is $F(A,S_0,exe)-F(E,S_0)=\frac{1}{R^2}(1-q)(R-1)K_3$

3.4. Case (5) and (6): $S_0ud>K_4\geq S_0d$

European: $F(E,S_0u)=0$

$F(E,S_0d)=\frac{1}{R}(1-R)(K_4-S_0d^2)$

$F(E,S_0)=\frac{1}{R^2}(1-q^2)(K_4-S_0d^2)$

American: At t_1, if we exercise earlier: $F(A, S_0u, exe)=0$, $F(A, S_0d, exe)=K_4-S_0d$

If not: $F(A,S_0u,not)=F(E,S_0u)$, $F(A,S_0d,not)=F(E,S_0d)$

Compare the payoff of not exercise and exercise:

At $t_1(S_1=S_0u)$: $F(A, S_0u, not)-F(A, S_0u, exe)=0$

At $t_1(S_1=S_0d)$: $F(A, S_0d, not)-F(A, S_0d, exe)<0$

The better choice is exercising American puts options.

Then back to t_0, if we exercise early, the payoff is:

$K\geq S_0$: $F(A, S_0, exe)=K_4-S_0$

$K<S_0$: $F(A, S_0, exe)=0$

If we do not exercise: $F(A, S_0, not)=\frac{1}{R}(1-q)(K_4-S_0d^2)$

Compared the two values in the two situations:

$K\geq S_0$: $F(A, S_0, exe)-F(A, S_0, not)>0$

$K<S_0$: $F(A, S_0, exe)-F(A, S_0, not)<0$

So, people can exercise the options early at t_0 if the strike price is larger than the stock price. If the strike price is smaller do not exercise

In this case, the early exercise premium has two possible

Values:

$K\geq S_0$:$F(A,S_0,exe)-F(E,S_0)=\frac{Kq^2+S_0(1-q^2)d^2}{R^2}+K_4-S_0$

$K<S_0$:$F(A,S_0,not)-F(E,S_0)=\frac{(1-q)(R-q-1)(K-S_0d^2)}{R^2}$

3.5. Case (7): $S_0d>K_5\geq S_0d^2$

European:

$F(E,S_0u)=0$ $F(E,S_0d)=\frac{(1-q)(K-S_0d^2)}{R}$ $F(E,S_0)=\frac{(1-q^2)(K-S_0d^2)}{R^2}$

American: If exercise American option earlier at $S_1=S_0u$ and $S_1=S_0d$: $F(A, S_0u, exe)=F(A, S_0d, exe)=0$

If we decide not to exercise them, the payoffs will remain the same as the European's.

$F(A, S_0u, not)=F(E, S_0u)$ $F(A, S_0d, not)=F(E, S_0d)$

Compare the payoff of not exercise and exercise:

At $t_1(S_1=S_0u)$: $F(A, not)-F(A, exe)=0$

At $t_1(S_1=S_0d)$: $F(A, not)-F(A, exe)=F(E, S_0d)>0$

The better choice for us at t_1 is not to exercise our American puts options.

Then back to t_0, if we exercise early, the payoff is: $F(A, S_0, exe)=0<F(A, S_0, not)$

So, people would not exercise the options early at t_0

To conclude, people would not exercise early at any moment in this case to get the maximum payoff. So, the early exercise payoff is 0.

3.6. Case (8): $S_0d^2>K_6$

European: $F(E, S_0u)=F(E, S_0d)=F(E, S_0)=0$

American: If exercise American option earlier at $S_1=S_0u$ and $S_1=S_0d$ respectively:

$F(A, S_0u, exe)=F(A, S_0d, exe)=0$

Compare the payoff of not exercise and exercise:

At $t_1(S_1=S_0u$ or S_0d): F(A, not)-F(A, exe)=0

At t_0, $F(A, S_0, not)=F(A, S_0, exe)=0$

People would not exercise early, and the early exercise payoff is 0.

3.7. Python Works

Figure 4 to Figure 9 represent the conditions of different stock prices from S_0 to Su and Sd, the premiums of American put options under different strike prices.

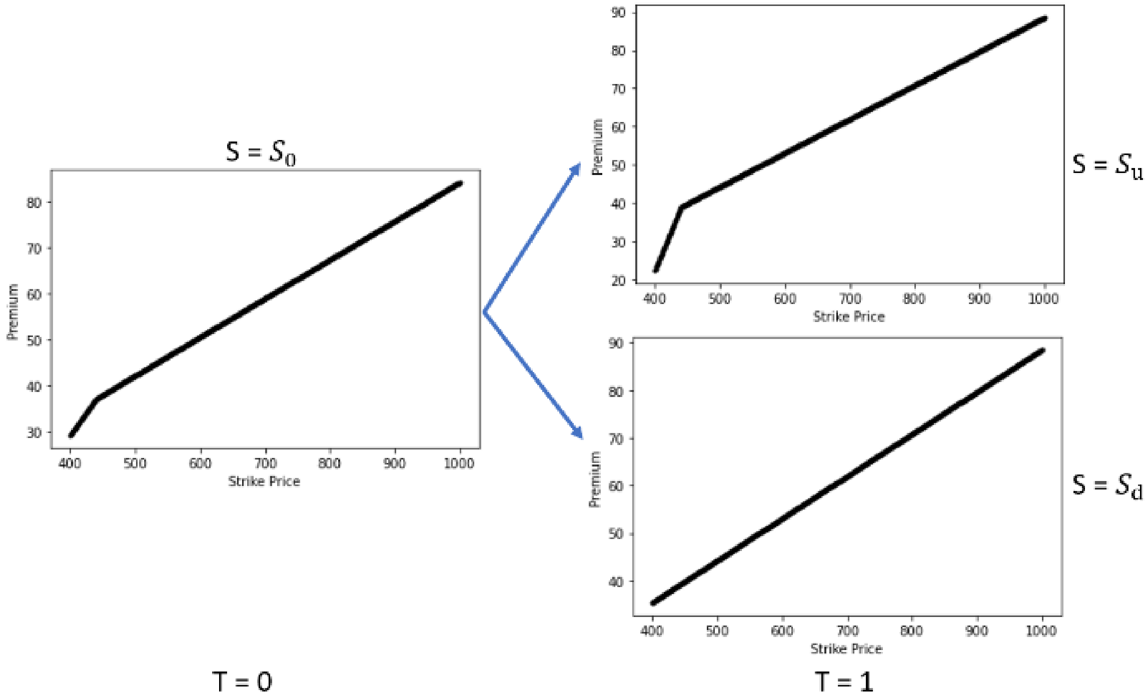

Figure 4: Python outcomes in case 1

Figure 4 shows that in case 1 the premium varies strike price between (400,1000). When strike price is between (400,440), value of European option is zero.

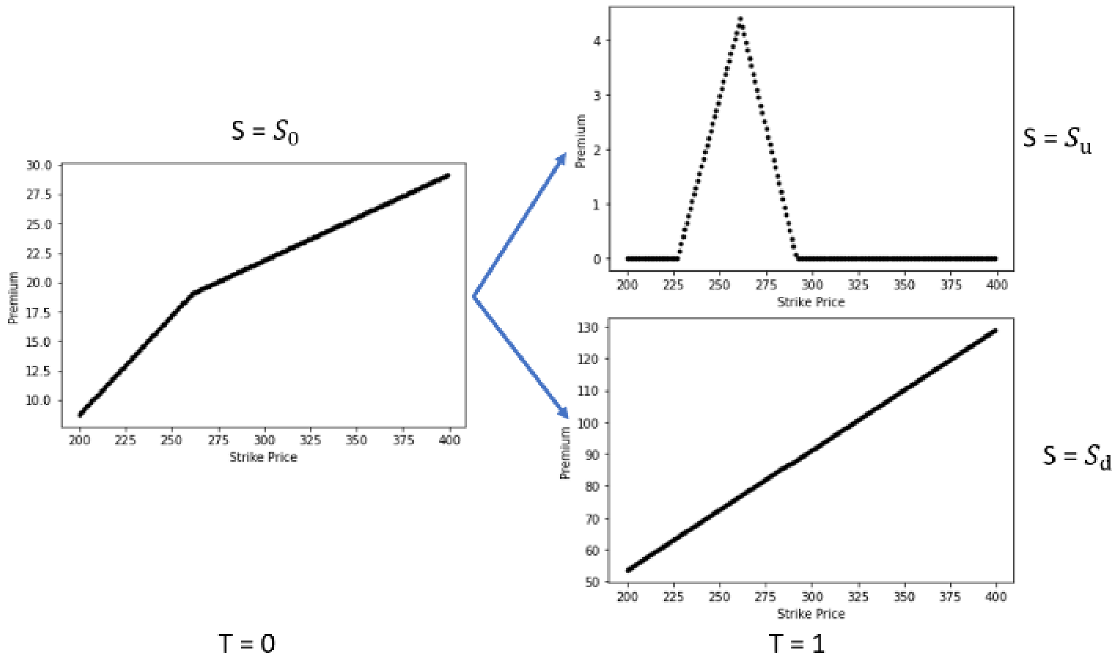

Figure 5: Python outcomes in case 2

Case 2, at T=0, value of European option is zero when strike price between (150,260); at T=1, S= S_u, value of European option is smaller than that of American option when strike price is between (225,260).

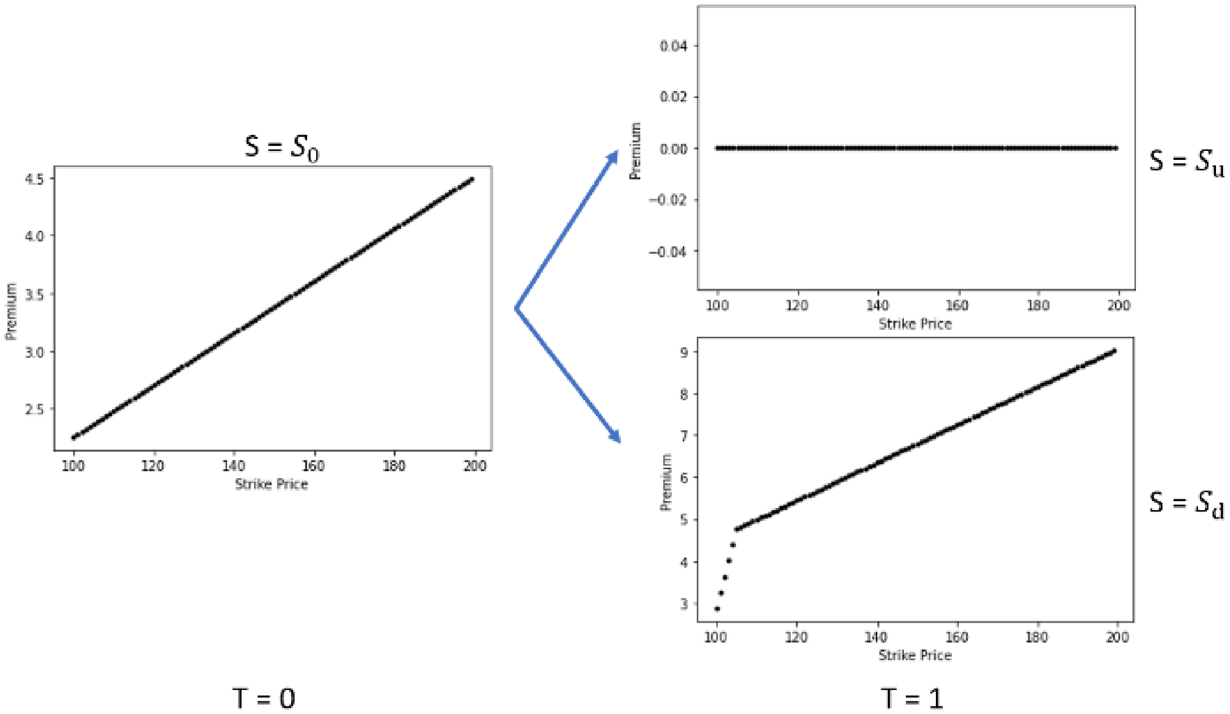

Figure 6: Python outcomes in case 3

Case 3, at T=1, value of European option decreases when strike price between (100,105).

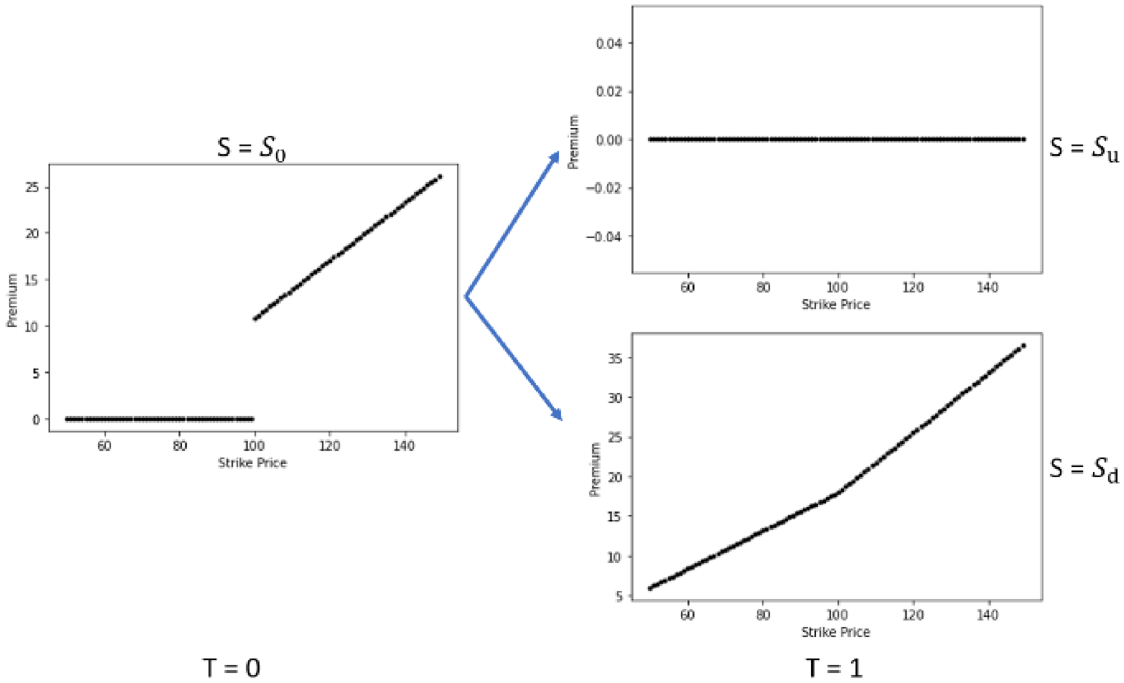

Figure 7: Python outcomes in case 4

Case 4, at T=0, value of European option fell to zero when strike price is larger than 100.

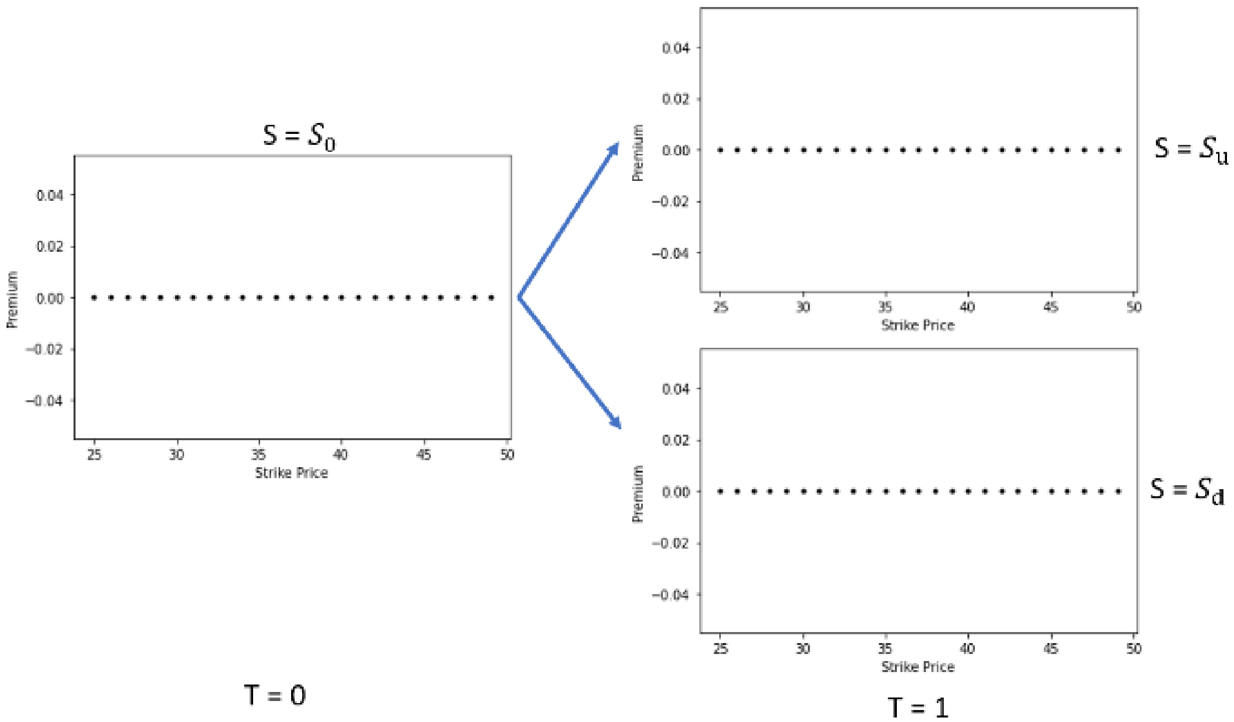

Figure 8: Python outcomes in case 5 and 6

Case 5 and 6 had the same result as showing in Figure 8. So do not exercise in case 5 and case 6.

3.8. Results on put options

Whether should people exercise their American puts options is a more complex case. Here I classify the eight situations above into three. First, when the strike price is high, higher than the rising stock price S_0u. Under this situation, people should exercise their American puts earlier at both t_0 and t_1. The second case is when the strike price is low, lower than the falling stock price S_0d. This time, which is the opposite of the previous case, people should not exercise earlier. The last case is more complicated, contains when strike price is smaller than S_0u but larger than S_0d. There are eight different choices from t_0 to t_1 as I analyzed in 2.2 and 2.3. To get a better idea of these cases, we can assume u and d is the reciprocal of each other, which means S_0 equals to S_0ud. Given this premise, at t_0, people should not exercise their American puts. At t_1, except when K is larger than S_0 and the stock price goes up, people should exercise their American pots earlier in all other situations.

4. CONCLUSION

To investigate how investors can get the largest profit in European and American option trades, we analyze the early exercise premiums for different cases under different situations by both mathematical method and python. According to the results we got, European call options always bring higher profit to people, so people prefer to not exercise their American options earlier. Situations become more complicated when it comes to put options. When the strike price is the highest comparing to increasing stock price at t1, exercising American put options can be more profitable than keeping European put options. And when the strike price is lower than the decreasing stock price at t1, people would not exercise put options earlier. What needs people to pay more attention is when the strike price is between these two stock prices, because of the irregular premium of earlier exercising, people should make their decisions independently.

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