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The Improvement of Implied Volatility of Black-Scholes Model: A Review

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ABSTRACT

The Black-Scholes model used in the financial industry can predict the price of the option and thus construct the hedging portfolio to avoid the risk. However, the model assumes the implied volatility to be constant, which cannot fit the curve of real market volatility, which causes inaccuracy in prediction and causes loss for practitioners. To solve the problem, mathematicians and economists add randomness to parameters of the volatility in the BS model. However, each model has its limitation. In this paper, we reviewed relative research of BS model, and two possible improvements of BS model, the local volatility function model and stochastic model, to fit the implied volatility to the real volatility surface, the volatility smile. In addition to that, we discuss the limitations of both models, where the stochastic model can predict forward volatility when the options have long maturities while behaving poorly at predicting short-term volatility. The local volatility function model has similar but opposite limitations. It can only precisely predict the short-term volatility, while the long maturity volatility surface shows flatten with a little variant. Thus, we introduce a possible improvement to consist of the strength of both the Local volatility function model and the stochastic volatility model, though it has its own limitation on the processing speed and strict requirement of the parameters.

Keywords: *Black-Scholes Model, Implied Volatility, BS model*

1. INTRODUCTION

As a financial derivative, options gradually gain more attention by investors in the financial market, where they commonly use the valuation model, such as Black and Scholes (1973) model and its extension by Merton (1973), to define the price of the European options [1-2]. Given the price of the underlying asset and the options of the asset, how to construct the portfolio to hedge the risk during the trading process is an inevitable topic discussed by the investors. To construct the hedging portfolio, the necessary step is buying or selling the options, and doing the opposite to the corresponding asset. At the maturity of the option, a perfect hedging portfolio will completely match the payoff of the options. However, how much to buy or sell, and how frequently to execute the trade process are key questions here. Fortunately, in Black-Scholes model, investors can not only calculate the price of the options but also see the possibility to construct a hedging portfolio in a rational way, which led to a concept, volatility, defining how the value of an asset may change over a certain period. Before the stock market crash in 1987, people traditionally assumed that the volatility of options of the same underlying was constant if the options had the same maturity though they may have different strike prices, and such volatility applied to Black-Scholes model is called implied volatility. However, after the crash, the investors soon realized that the volatility of certain options with lower strike price would be higher, and by observing the implied volatility surface, they discovered that the volatility is dependent on the time of the maturity and the strike price of options. Thus, the observed volatility surface is not consistent with the prediction of Black-Scholes model. This paper reviews the BS model and how scholars present improvement of the BS model with different approaches to fit the modeled volatility to the market volatility.

To get a more accurate market dynamic information of the volatility of each option, several researchers came up with alternative mathematical methods to calibrate the Black-Scholes model, such as local volatility model and stochastic volatility model, to take the facts that volatility actually varies with time and strike price of the price into account. However, these two possible calibrations on Black-Scholes model still have their own limitations, though they solve the problems to a certain extent. Multiple kinds of literature have introduced several stochastic volatility models (SV), including Heston (1993), which is the most widely used stochastic volatility model to fit the implied volatility to some long-term expiring options, but insufficiently accurate for short-term expiring options, as suggested by Gatheral (2006), while other SV models such as Hull and White (1987) with similar approaches requiring more parameters than Heston's model. Besides that, the Local Volatility model firstly proposed by Dupire (1994) found that the volatility function of each option depending on the current stock price and time can be individually determined if all prices of options with all maturity times and strike price are available. However, since the Dupire did not take the possible future variation of the stock price into account, the volatility surface shows relatively constant in the long term, although it behaves well in short time intervals, as shown in Kotzé, Oosthuizen and Pindza (2015) [3-7].

However, both SV model and LV model have their limitations when predicting the volatility of the market, so another possible improvement of calibrating the BS model was studied, called the stochastic local volatility model. This paper mainly concentrates on reviewing the past research about solving the problem that the implied volatility is not consistent with the market volatility. Based on that purpose, we will introduce the principle of defining the price of the option using BS model and discuss what is the volatility and why the implied volatility of the market is inconsistent with the volatility of the markets. After that we could study both SV model and LV model to understand how, and how far they solve the problem to fit the implied volatility to the volatility smile and introduce the possible solution to solve these two models' drawbacks.

The remainder of this paper is organized as follows. In section 2, we will briefly introduce the background principle of BS model and the volatility smile, so we can get a first impression about why volatility is important. Starting from that could introduce the related progress made by three different models each has its pros and cons. In section 4, we would present a conclusion that concludes the recent progress of calibration of the BS model, and thus anticipate the possible direction of future progress.

2. BLACK AND SCHOLES MODEL

As mentioned above, Black and Scholes Model (1973) defines explicitly the European option price by providing a valuation formula [1]:

$$C_{bs}(t, S(t)) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) \quad (1)$$

And using put-call parity, one could easily find that:

$$P_{bs}(t, S(t)) = Ke^{-r(T-t)}N(-d_2) - S(t)N(-d_1) \quad (2)$$

Where K refers to the strike price of the option, and r is the risk-free interest rate, $S(t)$ denotes the current stock price at time t, and function $N(x)$ is the standard normal distribution of x, i.e.

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

And

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} [log(S(t)/K) + (r + \sigma^2/2) * (T-t)]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Where σ is the implied volatility in Black Scholes model, which is assumed to be unique and not dependent on t, K, $S(t)$, and maturity T if the Black-Scholes price of options has no arbitrage opportunity.

Theoretically, if we know the stock price over a time, for example, 20 days, we could calculate the annual standard deviation using the sample standard deviation of the stock price over these 20 days, and thus calculate the Annual standard deviation σ to price the options. However, many scholars, such as Dupire (1994) and Heston (1993), notice that if implied volatility that matches the BS price of the option with the observed market price of the option and K is plotted, the resulting graph would look like a parabola with a minimum point of volatility appearing at a certain strike price [6, 3]. This phenomenon violates the assumption of the classical Black-Scholes model of the option prices, which assumes single volatility for all options with different strike prices. This U-shape relationship between the volatility and strike price is known as the 'volatility skew' or 'volatility smile'. Since the publication of the Black-Scholes Model, scholars and mathematicians have come up with possible explanations and calibrations on the Black-Scholes model to fit the implied volatility to the market volatility. Gatheral suggests they add randomness to the coefficient in the B-S model equation, and assume some, or all the coefficients are the functions with parameter spot price $S(t)$, and time t [5]. In the following, we would discuss some of the relatively successful models.

3. DETAILED REVIEW: DIFFERENT ASPECTS TO PREDICT VOLATILITY

3.1 Short Term Prediction: Local Volatility Models

As referred earlier, the volatility plotted using the real market price is inconsistent with the implied volatility predicted by the B-S model, and Dupire (1994) came up with local volatility models to fits the implied volatility with the actual volatility, if the price of the underlying asset follows a lognormal diffusion process equation, and start with a general deterministic local volatility model [6]:

$$\frac{dS_t}{S_t} = (r - d)dt + \sigma(S_t, t)dW_t \quad (3)$$

Dupire showed that this diffusion process is unique if the price of the stock at time T, $S(t)$, and at the beginning price $S(0)$ is fixed with some given distribution [6]. The assumption and resulting in partial derivative successfully fit the implied volatility to both the price of underlying and time and plotted the unique local volatility surface based on these assumptions.

After the local volatility surface is shown, many scholars use it to construct dynamic hedging performance, i.e Coleman, Kim, Li and Verma (2003), and compare it with the one constructed using the implied volatility that proposed by Black-Scholes model and discovered that the average hedging error using the Local Volatility function is smaller than the using implied volatility [9].

Though the behavior of local volatility function is fit for the volatility smile and are self-consistent theocratically in perfect markets, which means it allows to perfect hedging appears, such as Ayache (2004), soon find that the shape of local volatility surface for the vanilla option is not intuitive, and it could only see the present volatility smile [8]. In the long term, the volatility surface of the local volatility function is relatively flat than prediction. Kotzé, Antonie, Rudolf Oosthuizen, and Edson Pindza (2015) also find a similar pattern when they study the African index on foreign exchange options [7]. They found that the local volatility function of foreign options builds a surface with a relatively constant smile for a long time in the future. This significant defect explains that Dupire's local volatility models only considered today's prices and it makes no assumption or prediction on the behavior of the prices in the future. This situation becomes serious when some special option, such as an exotic option, is involved, since some options really depend on the future smile, rather than the present smile, though it may be perfectly fitted. Therefore, if a hedging strategy is built on the local volatility function, the volatility needs to be calibrated very frequently. In other words, to get the nearly-perfectly fitted volatility function, one has to calibrate it once the price of the stock changes as frequently as possible. Thus, the change of delta is very frequent, and involves many trades in the market, which costs a lot of transaction fees.

3.2 Long Term Prediction: Stochastic Volatility

To adjust the Local Volatility function to construct better future volatility smile to capture more complex dynamics of stock price, the stochastic volatility (SV) model is introduced. The most well-known and successful Stochastic Volatility model is Heston's model, which explains the behavior of the volatility surface when the expiration T is long enough [3]:

$$\frac{dS_t}{S_t} = \mu(t)dt + \sigma_t dW_t^S \quad (4)$$

where $d\sigma_t = \alpha(t, \sigma_t)dt + \beta(t, \sigma_t)dW_t^\sigma$, and σ_t refers to the volatility of the stock price, S_t, and dW_t^S, dW_t^σ is the Brownian motion of the stock price and its variance, respectively.

From the above equation, we could see that the Heston Stochastic Volatility model assumes that the stock price and the volatility at time t have separate dynamics and are correlated to each other, and the volatility itself is arbitrary. Compared to Black Shores Model, the arbitrary volatility solves the assumption of constant volatility, and Heston's model does not require the normal distribution of the stock prices. Thus, the SV model could produce a forward prediction of the volatility. Although many other scholars also did several research using Stochastic Volatility models, such as Hagan et al (2002) and Hull and White (1987,1988) [10, 4]. However, compared to their model, Heston's model has the advantage that it not only gives a quasi-closed solution, meaning that the answer was derived from mathematical operations under reasonable assumptions but also it requires less computational complication than other models. Gatheral (2006) examines a few Stochastic models and find that most stochastic model produces similar volatility surface, and Heston's model shows higher computational efficiency than others [5]. However, he also suggests that Heston's model could only fit the market volatility in the long term but have a poor behavior for the volatility of options with a short expiration date. Also, although compared to other SV models, Heston's SV model has higher computational efficiency, but it is still harder to calibrate the data than the local volatility model does since the parameters appear in the model need to be calibrated carefully. The harder the calibration is, the more time is needed to construct the hedging portfolios, so it is less frequent to adjust the portfolio if it is constructed in this model.

For now, we have discussed two different models for calibration of the data to fit the observed volatility. However, the SV model behaves better at predicting forward (future) volatility, with higher sensitivity and thus requiring difficult computation to calibrate, while the Local Volatility function can nearly perfect fit the observed volatility for the short term with possible frequent calibration but shows flattened volatility smile in the future which violates the observed volatility characteristics. Therefore, scholars made multiple trials to integrate both positive sides of both LV and SV models, when avoiding their drawbacks as more as possible. One of such trials is local-stochastic volatility models.

3.3 Better Prediction with Higher Calculation Difficulty: Stochastic-Local Volatility Model

Blacher (2001) first proposed the Stochastic-Local Volatility model. The general SLV model implemented from Heston's Model is [11]:

$$dS_t = \mu_1(S_t, t)dt + L(S_t, t)\sigma_1(S_t, t, V_t)dW_t^s \quad (5)$$
$$dV_t = \mu_2(V_t, t)dt + \sigma_2(t, V_t)\, dW_t^v \quad (6)$$

Where $dW_t^s * dW_t^v = \rho dt$, and $-1 < \rho < 1$.

The key factor here is the L, which is a leverage function that keeps the volatility within the reasonable range. The leverage function works like the multiplying ratio between the local volatility surface and the stochastic volatility surface, and it is determined by associated market information to make sure that the modeled volatility is fitting the observed volatility.

This topic is further studied by scholars with different calibration approaches. Lipton (2002) proposed a model that combines local volatility and stochastic volatility by adding jumps to the dynamic stock prices $S(t)$ [12]. This proposal is strongly argued by Ayache, et al (2004) that the real markets volatility smile cannot be perfectly reconstructed by mathematical models, who present a few diagrams that show the inconsistent between the modeled volatility surfaces and the real market volatility surface [8]. Tian (2013) uses both the finite difference method and Monte Carlo method on the exotic option on the foreign exchange market, giving them an implied volatility surface that looks like the market implied volatility surface [13].

The main difficulty to apply the Stochastic-Local volatility function on the real-world data is that it is hard to calibrate the leverage function, which requires a strict calibration procedure to maintain the precision of the calibration. The minor error increases the difficulty and time consumption of calibrating the data, influencing the stability of the calibration of the volatility within the different densities of spot price to a large degree. Also, the assumption to produce fast results requires a specific parametrization for SLV model, and such result was non-trivial and possibly not applicable to the market data. As a result, in the financial industry, the practitioner (trader and risk managers) usually separately calibrates the parameter of the stochastic model and the local volatility model in order to get a more plausible volatility dynamic from the market.

Therefore, although many attempts were made to fit the modeled volatility surface to the market volatility surface, each model still has its limitations, and the integration of both the stochastic volatility model and local volatility model seems provides us with a better solution but it still requires improvement. The application of using the model to really construct the hedging portfolio still requires improvement. A fast, stable, and precise pricing technique is important for the market to calibrate a large number of real data in various circumstances and provide the investors with reliable parameters and indexes to build their hedging strategies. Since the market is incomplete, leading to the information to construct the parameter required for the model may be insufficient. Thus, when the application of the model in the real-world data, the volatility surface may look different.

4. CONCLUSION

In this paper, we review the principle of the Black-Scholes model and discuss why the implied volatility of the BS model has its problem. To solve that problem, scholars used a different approach to add different parameters to the BS model and build an improved model based on BS models. In section 3, we explain how the local volatility model and stochastic volatility model fits their calculated volatility surface to the volatility smile and based on their pros and cons, we introduce the stochastic local volatility model which consists of the advantages of both SV and LV model, while having its own problems. Thus, we integrate those models to better review and suggest that the practitioners could think about their practical purpose of the volatility and choose the most suitable model to measure and predict the volatility, and the model could use a better algorithm in the program or better hardware to speed up the calibration. Next, our research would consider using the machine learning approach to automatically collect the data of volatility from the markets and calibrate it using different models, see if the AI could develop better calibration speed and precision.

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